TECHNISCHE MECHANIK, Band 16, Heft 1, (1996), 77-88
Manuskripteingang: 14. Dezember 1995

On Fundamental Concepts of Multiphase Micropolar Materials

S. Diebels, W. Ehlers

The kinematics and the balance equations for a multiphase micropolar material of n constituents φ^α are presented. Similar to mixture theories, each constituent is assigned its own motion and, due to micropolarity, its own micromotion. The physical meaning of deformation and strain measures is discussed using the concept of convected coordinates and natural basis vectors. The balance equations of mass, linear momentum, moment of momentum and energy of the mixture are derived from Truesdell's metaphysical principles, thus allowing for an interpretation of the physical quantities of the mixture depending on the properties of the constituents. Starting from a general master relation, the local form of the balance equations are derived for the mixture φ and for all constituents φ^α by specifying the physical quantities which are balanced, their fluxes, supplies and productions. Eringen's "balance of microinertia" is included in the model as a kinematic constraint which restricts the microparticles to rigid motions.

1 Introduction

The mechanical behavior of multiphase materials is of interest in several areas of engineering, e. g. in geomechanics, soil mechanics, biomechanics, and powder metallurgy. The theoretical access to these fields is the theory of mixtures, a macroscopic theory of superimposed continua. In this theory, the properties of the constituents are averaged over a representative volume element occupied by the whole mixture. Therefore, in the resulting smeared model, material points of each constituent exist at each geometrical point. The volume fractions are introduced as scalar structural variables which describe the local composition of the mixture. This approach was discussed with respect to porous media models (i. e. mixtures including one solid constituent) by Bowen (1976, 1980), de Boer and Ehlers (1986), and Ehlers (1989). An overview concerning the macroscopic porous media approach up to 1983 is presented by Bedford and Drumheller (1983). An averaging procedure to obtain the equations on the macroscale from the microscale is discussed in detail by Hassanizadeh and Gray (1979a, 1979b).

In the framework of continuum mechanics, the material points which are assumed to be of infinitesimal size are carrying the local physical properties of the body under study. Then, the continuum deformation and strain measures are defined by use of the displacement functions of the material points. If one is interested in a more sophisticated approach to kinematics of continua, it is necessary to substitute or to identify the material points of continuum mechanics by microparticles. If these microparticles are assumed to be deformable, following Eringen and Kafadar (1976), the continuum is called micromorphic. Otherwise, i. e., if the microparticles are rigid, the continuum is called micropolar. In the micropolar case, the material point or the microparticle may undergo displacements as well as rotations. The displacements are given by the motion of the material points, while the rotation is independently described by the microrotation represented by an orthogonal tensor. The advantage of a micropolar theory of mixtures, which, in reality, are discontinua on a microscale, is that the average rotation of the microparticles is included in the macrotheory, e. g. it becomes possible to predict the local average rotation of grains of granular materials such as powder, sand, or rock. These rotations become important in areas of strain localization such as shear bands.

The aim of this work is to include micropolar properties of the constituents of a multiphase mixture into the macroscopic theory, i. e. to allow for rotations of the material points in addition to the displacements. In this case, the material points as the carriers of the physical material properties, are assumed to be rigid bodies on the microscale. If the particles rotate independently from the continuum rotation resulting from the polar decomposition of the deformation, the material under study should be a real discontinuum, e. g. a granular material or a liquid crystal. In this case, the averaging procedure together with the micropolar properties allows for the application of the usual mathematical description of the motion using differential formulations, even for media which are discontinuous on the microscale. For granular materials such as soil, rock, or other geomaterials, this approach is an elegant possibility to include the mean rotation of the grains into a macroscopic theory.

The idea to add rotational degrees of freedom to each material point stems from the work of the Cosserat brothers (1909). The interest in micropolar continua was rapidly increased in the last decades. The reader, who is interested in details to micropolar theories, is referred e. g. to Eringen (1964), Eringen and Kafadar (1976), Günther (1958), or Schaefer (1967). In the theory of shells, additional micropolar degrees of freedom were used to describe the rotation of the cross section, Ericksen and Truesdell (1958), Sansour and Bednarczyk (1995). While in the study of granular materials the inclusion of micropolar properties results from physical considerations, it was found in the study of localizations, that the inclusion of rotational degrees of freedom leads to a regularization of the problem, Tejchman and Wu (1993), Steinmann (1994).

The kinematic relations resulting from the discussed introduction of the micromotion for each constituent of the mixture as well as the resulting deformation and strain measures are discussed in section 2. The contents of section 3 are the balance equations of mass, of momentum, of moment of momentum and of energy, both for the mixture φ and for the constituents φ^α.

2 Kinematics

In mixture theories, material points of each constituent φ^α, $\alpha = 1, \ldots, n$, occupy the same spatial point in the actual configuration (superimposed continua). The volume element dv is given by the sum of the partial volume elements dv^α, which, on the microscale, are occupied by φ^α only.

$$dv = \sum_{\alpha=1}^{n} dv^\alpha \tag{1}$$

The scalar structural variables to describe the microstructure on the macroscopic scale are the volume fractions, herewith defined as the ratios of the partial volume elements dv^α in comparison to the bulk volume element dv.

$$n^\alpha = \frac{dv^\alpha}{dv} \tag{2}$$

The combination of equations (1) and (2) leads to the saturation constraint

$$\sum_\alpha n^\alpha = 1 \tag{3}$$

Starting from different reference positions $\mathbf{X}_\alpha$, particles X^α of each φ^α follow their own motion

$$\mathbf{x} = \boldsymbol{\chi}_\alpha(\mathbf{X}_\alpha,\, t) \tag{4}$$

see Figure 1.

In addition to the motion (4), each point undergoes a micromotion, which describes the total rotation of the material points. In this case, the material points are assumed to be rigid particles on the microscale. Therefore, each point has attached directors $\boldsymbol{\Xi}_\alpha$, which are rotated by the micromotion $\bar{\mathbf{R}}_\alpha$. The directors of the actual configuration are called $\boldsymbol{\xi}_\alpha$. The micromotion or the microrotation, $\bar{\mathbf{R}}_\alpha$, is represented by an orthogonal tensor with the following properties:

$$\boldsymbol{\xi}_\alpha = \bar{\mathbf{R}}_\alpha \boldsymbol{\Xi}_\alpha \qquad\qquad \bar{\mathbf{R}}_\alpha \bar{\mathbf{R}}_\alpha^T = \mathbf{I} \tag{5}$$

In this section, we discuss the deformation resulting from the motion and microrotation in terms of convected coordinates and natural basis vectors. This allows for an illustrative interpretation of the resulting deformation tensors.

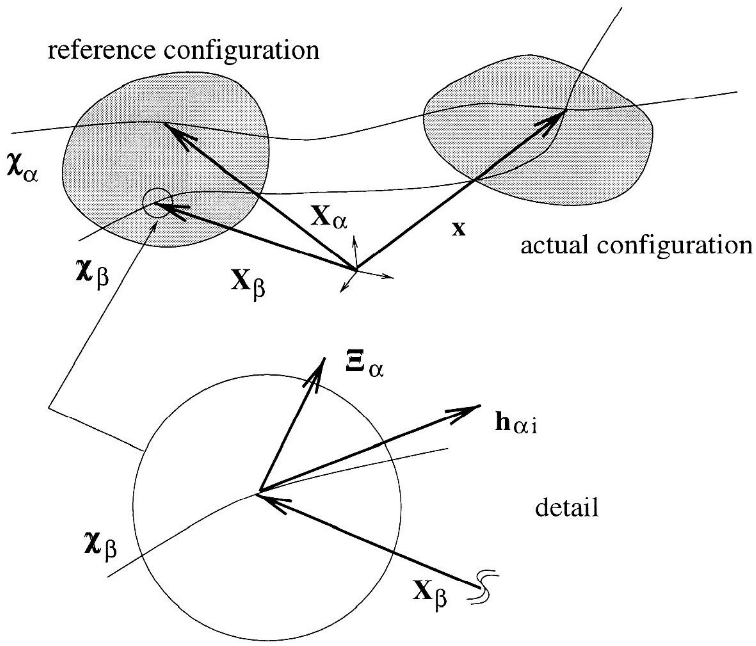

Figure 1. Reference and actual configuration of a binary mixture. Both constituents follow their own motions. In the actual configuration, the mixture is composed by superimposed continua. The detail shows natural basis vectors and directors of the reference configuration.

Introducing convective coordinates represented, for each φ^α, by parameter lines Θ^i_α $(i = 1, 2, 3)$, the reference position $\mathbf{X}_\alpha$ of the material points is given by a one-to-one mapping of the parameters Θ^i_α.

$$\mathbf{X}_\alpha = \mathbf{X}_\alpha(\Theta^i_\alpha) \Leftrightarrow \Theta^i_\alpha = \Theta^i_\alpha(\mathbf{X}_\alpha). \tag{6}$$

Note in passing, that the index α stands for φ^α while Latin indices are used as counters for vector or tensor coefficients.

The natural basis vectors $\mathbf{h}_{\alpha\,i}$ of the reference configuration and $\mathbf{a}_{\alpha\,i}$ of the actual configuration are defined as tangent vectors on the parameter lines (de Boer, 1982).

$$\mathbf{h}_{\alpha\,i} = \frac{\partial \mathbf{X}_\alpha}{\partial \Theta^i_\alpha} \qquad \mathbf{a}_{\alpha\,i} = \frac{\partial \mathbf{x}}{\partial \Theta^i_\alpha} \tag{7}$$

Defining the dual basis vectors by

$$\mathbf{h}^i_\alpha = \frac{\partial \Theta^i_\alpha}{\partial \mathbf{X}_\alpha} \qquad \mathbf{a}^i_\alpha = \frac{\partial \Theta^i_\alpha}{\partial \mathbf{x}} \tag{8}$$

the following identities hold:

$$\mathbf{h}^i_\alpha \cdot \mathbf{h}_{\alpha\,j} = \delta^i_j \qquad \mathbf{a}^i_\alpha \cdot \mathbf{a}_{\alpha\,j} = \delta^i_j \tag{9}$$

where δ^i_j is the Kronecker symbol.

As is known from classical continuum mechanics, the deformation gradient $\mathbf{F}_\alpha$ and its inverse $\mathbf{F}^{-1}_\alpha$ of each phase φ^α may be expressed by

$$\begin{aligned} \mathbf{F}_\alpha &= \frac{\partial \mathbf{x}}{\partial \mathbf{X}_\alpha} = \frac{\partial \mathbf{x}}{\partial \Theta^i_\alpha} \otimes \frac{\partial \Theta^i_\alpha}{\partial \mathbf{X}_\alpha} = \mathbf{a}_{\alpha\,i} \otimes \mathbf{h}^i_\alpha \\ \mathbf{F}^{-1}_\alpha &= \frac{\partial \mathbf{X}_\alpha}{\partial \mathbf{x}} = \frac{\partial \mathbf{X}_\alpha}{\partial \Theta^i_\alpha} \otimes \frac{\partial \Theta^i_\alpha}{\partial \mathbf{x}} = \mathbf{h}_{\alpha\,i} \otimes \mathbf{a}^i_\alpha \end{aligned} \tag{10}$$

The deformation gradient maps the basis vectors $\mathbf{h}_{\alpha\,i}$ (or line elements $d\mathbf{X}_\alpha$) from the reference configuration onto the actual configuration.

$$\mathbf{F}_\alpha \mathbf{h}_{\alpha\,i} = \left(\mathbf{a}_{\alpha\,j} \otimes \mathbf{h}^j_\alpha\right)\, \mathbf{h}_{\alpha\,i} = \mathbf{a}_{\alpha\,i} \tag{11}$$

The left and right Cauchy-Green deformation tensors are obtained from the transport mechanisms of squares of line elements from one configuration into the other. The squares of line elements ds^2_α of the actual configuration may be expressed by

$$ds^2_\alpha = d\mathbf{x}_\alpha \cdot d\mathbf{x}_\alpha = d\mathbf{X}_\alpha \cdot \mathbf{C}_\alpha d\mathbf{X}_\alpha \tag{12}$$

where $\mathbf{C}_\alpha := \mathbf{F}^T_\alpha \mathbf{F}_\alpha$. The same element dS^2_α of the reference configuration is given by

$$dS^2_\alpha = d\mathbf{X}_\alpha \cdot d\mathbf{X}_\alpha = d\mathbf{x}_\alpha \cdot \mathbf{B}^{-1}_\alpha d\mathbf{x}_\alpha \tag{13}$$

where $\mathbf{B}_\alpha := \mathbf{F}_\alpha \mathbf{F}^T_\alpha$. Written with respect to natural basis vectors, we find from the above definitions

$$\mathbf{C}_\alpha = a_{\alpha\,ij}\left(\mathbf{h}^i_\alpha \otimes \mathbf{h}^j_\alpha\right) \qquad\qquad \mathbf{B}_\alpha = h^{ij}_\alpha\left(\mathbf{a}_{\alpha\,i} \otimes \mathbf{a}_{\alpha\,j}\right) \tag{14}$$

with metric coefficients $h^{ij}_\alpha = \mathbf{h}^i_\alpha \cdot \mathbf{h}^j_\alpha$ and $a_{\alpha\,ij} = \mathbf{a}_{\alpha\,i} \cdot \mathbf{a}_{\alpha\,j}$. Therefore, the deformation is either expressed by the metric coefficients of the actual configuration and the basis vectors of the reference configuration or by the metric coefficients of the reference configuration and the basis vectors of the actual configuration.

The Green strain tensor $\mathbf{E}_\alpha$ and the Almansi strain tensor $\mathbf{A}_\alpha$ are defined as the difference of the actual and the reference metric coefficients related to the reference basis and the actual basis, respectively. They describe the difference of squares of line elements $ds^2_\alpha - dS^2_\alpha$ related to the tensor basis of the reference configuration or the actual configuration, respectively

$$\mathbf{E}_\alpha = \left(a_{\alpha\,ij} - h_{\alpha\,ij}\right)\mathbf{h}^i_\alpha \otimes \mathbf{h}^j_\alpha \qquad\qquad \mathbf{A}_\alpha = \left(a_{\alpha\,ij} - h_{\alpha\,ij}\right)\mathbf{a}^i_\alpha \otimes \mathbf{a}^j_\alpha\,. \tag{15}$$

In addition, the microrotation $\bar{\mathbf{R}}_\alpha$ maps the director Ξ_α from the reference configuration onto the director $\boldsymbol{\xi}_\alpha$ of the actual configuration. Therefore, it is a two-field tensor like the deformation gradient, and may be expressed by

$$\bar{\mathbf{R}}_\alpha = (\bar{R}_\alpha)^j{}_k\left(\mathbf{a}_{\alpha\,j} \otimes \mathbf{h}^k_\alpha\right) =: \bar{\mathbf{a}}_{\alpha\,k} \otimes \mathbf{h}^k_\alpha =: \mathbf{a}_{\alpha\,j} \otimes \bar{\mathbf{h}}^j_\alpha \tag{16}$$

From the representation (16) of the microrotation, it is concluded that

$$\boldsymbol{\xi}_\alpha = \bar{\mathbf{R}}_\alpha \Xi_\alpha = (\bar{R}_\alpha)^j{}_k\left(\mathbf{a}_{\alpha\,j} \otimes \mathbf{h}^k_\alpha\right)\, \Xi^i_\alpha \mathbf{h}_{\alpha i} = \xi^j_\alpha \mathbf{a}_{\alpha j} \tag{17}$$

Note in passing that equation (16) allows for the introduction of two new sets of basis vectors, which either result from the summation over k or over j.

$$\bar{\mathbf{a}}_{\alpha\,k} = (\bar{R}_\alpha)^j{}_k \mathbf{h}^k_\alpha \qquad\qquad \bar{\mathbf{h}}^j_\alpha = (\bar{R}_\alpha)^j{}_k \mathbf{a}_{\alpha\,j} \tag{18}$$

For what follows, we identify the basis vectors of the actual configuration with the directors without loss of generality, i. e. $\boldsymbol{\xi}_\alpha = \mathbf{a}_{\alpha\,i}$. In this case, the micromotion rotates the new basis vectors $\bar{\mathbf{h}}_{\alpha\,i}$ onto the basis vectors $\mathbf{a}_{\alpha\,i} = \boldsymbol{\xi}_\alpha$. The micropolar deformation $\bar{\mathbf{U}}_\alpha$ maps the basis vectors $\mathbf{h}_{\alpha\,i}$ onto the reference directors $\Xi_\alpha = \bar{\mathbf{h}}_{\alpha\,i}$, see Figure 2. On the other hand, the new basis vector $\bar{\mathbf{a}}_{\alpha\,i}$ is mapped onto the basis vector $\mathbf{a}_{\alpha\,i}$ by the micropolar deformation $\bar{\mathbf{V}}_\alpha$, while the reference basis vector $\mathbf{h}_{\alpha\,i}$ is interpreted as director Ξ_α, and is rotated by the micromotion onto the new basis vector $\bar{\mathbf{a}}_{\alpha\,i}$.

Depending either on the first or on the second choice, the micropolar or the so-called first Cosserat deformation tensors $\bar{\mathbf{U}}_\alpha$ and $\bar{\mathbf{V}}_\alpha$ operate either on the reference or on the actual geometry, respectively. Since $\bar{\mathbf{R}}_\alpha$ is an orthogonal tensor, the metric coefficients $\bar{h}_{\alpha\, ij}$ and $a_{\alpha\, ij}$ coincide as well as the coefficients $\bar{a}_{\alpha\, ij}$ and $h_{\alpha\, ij}$, see Figure 2.

The definition of the new basis vectors $\bar{\mathbf{a}}_{\alpha\, i}$ and $\bar{\mathbf{h}}_{\alpha i}$ leads to the introduction of two new configurations. The transport mechanism between the reference configuration, the new configurations, and the actual configuration are either defined by the Cosserat deformation $\bar{\mathbf{U}}_\alpha$ and the microrotation $\bar{\mathbf{R}}_\alpha$ or by the microrotation $\bar{\mathbf{R}}_\alpha$ and the Cosserat deformation $\bar{\mathbf{V}}_\alpha$. According to Figure 2, the following decompositions of the deformation gradient $\mathbf{F}_\alpha$ in analogy to the polar decomposition are possible:

$$\mathbf{F}_\alpha = \bar{\mathbf{R}}_\alpha \bar{\mathbf{U}}_\alpha = \bar{\mathbf{V}}_\alpha \bar{\mathbf{R}}_\alpha \tag{19}$$

Note that the microrotation $\bar{\mathbf{R}}_\alpha$ is different from the orthogonal tensor $\mathbf{R}_\alpha$ resulting from the polar decomposition of the deformation gradient. Therefore, the Cosserat deformation tensors $\bar{\mathbf{U}}_\alpha$ and $\bar{\mathbf{V}}_\alpha$ are in general non-symmetric. They describe the change of the scalar product between the directors and the line elements from the reference configuration to the actual configuration, as can directly be seen form the following scalar products:

$$\boldsymbol{\xi}_\alpha \cdot d\mathbf{x}_\alpha = \boldsymbol{\Xi}_\alpha \cdot \bar{\mathbf{U}}_\alpha d\mathbf{X}_\alpha \qquad \boldsymbol{\Xi}_\alpha \cdot d\mathbf{X}_\alpha = \boldsymbol{\xi}_\alpha \cdot \bar{\mathbf{V}}_\alpha^{-1} d\mathbf{x}_\alpha \tag{20}$$

The Cosserat deformation measures are connected to the left and right Cauchy-Green tensors of classical continuum mechanics by

$$\bar{\mathbf{U}}_\alpha^T \bar{\mathbf{U}}_\alpha = \mathbf{C}_\alpha \qquad \bar{\mathbf{V}}_\alpha \bar{\mathbf{V}}_\alpha^T = \mathbf{B}_\alpha \tag{21}$$

The Cosserat deformations may be expressed as two-field tensors composed either by $\mathbf{h}_{\alpha\, k}$ and $\bar{\mathbf{h}}_\alpha^k$ or by $\bar{\mathbf{a}}_\alpha^k$ and $\mathbf{a}_{\alpha\, k}$, which follows from the representation in natural basis vectors

$$\bar{\mathbf{U}}_\alpha = \bar{\mathbf{R}}_\alpha^T \mathbf{F}_\alpha = a_{\alpha\, jk} \left(\bar{\mathbf{h}}_\alpha^j \otimes \mathbf{h}_\alpha^k\right) \qquad \bar{\mathbf{V}}_\alpha = \mathbf{F}_\alpha \bar{\mathbf{R}}_\alpha^T = h_\alpha^{jk} \left(\mathbf{a}_{\alpha\, j} \otimes \bar{\mathbf{a}}_{\alpha\, k}\right) \tag{22}$$

The second Cosserat deformation measure is related to the curvature, i. e. to the gradient of the micro-motion. Starting form the identity $\operatorname{Grad}_\alpha \left(\bar{\mathbf{R}}_\alpha^T \bar{\mathbf{R}}_\alpha\right) = \overset{3}{\mathbf{0}}$, it can be found that the curvature tensor

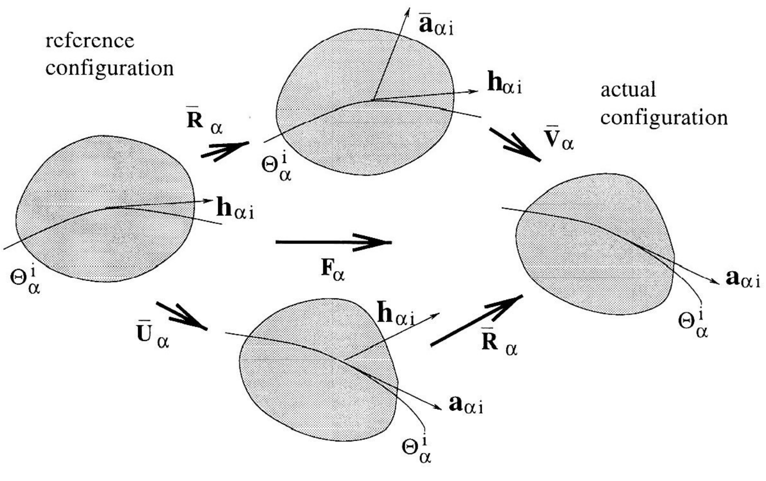

Figure 2. Introduction of two new configurations with basis vectors $\bar{\mathbf{a}}_{\alpha\, i}$ and $\bar{\mathbf{h}}_{\alpha\, i}$. The new basis vectors $\bar{\mathbf{h}}_{\alpha\, i}$ are transformed into the basis vectors of the actual configuration by the microrotation. On the other hand, the microrotation maps the basis vectors $\mathbf{h}_{\alpha\, i}$ onto the new basis vectors $\bar{\mathbf{a}}_{\alpha\, i}$ of the new configuration.

$$\overset{3}{\mathbf{K}}_\alpha := \bar{\mathbf{R}}_\alpha^T \mathrm{Grad}_\alpha \bar{\mathbf{R}}_\alpha \tag{23}$$

is skew-symmetric with respect to the first two basis vectors. If $\overset{3}{\mathbf{K}}_\alpha$ is represented in terms of the basis vectors we find

$$\begin{aligned} \overset{3}{\mathbf{K}}_\alpha &= a_{\alpha\, jr} \left(\gamma_{\alpha\, lk}{}^r - \bar{\Gamma}_{\alpha\, lk}{}^r\right) \left(\bar{\mathbf{h}}_\alpha^j \otimes \bar{\mathbf{h}}_\alpha^l \otimes \mathbf{h}_\alpha^k\right) \\ &= \left(\gamma_{\alpha\, lk}{}^r - \bar{\Gamma}_{\alpha\, lk}{}^r\right) \left(\bar{\mathbf{h}}_{\alpha\, r} \otimes \bar{\mathbf{h}}_\alpha^l \otimes \mathbf{h}_\alpha^k\right) \end{aligned} \tag{24}$$

In equation (24) the Christoffel symbols are defined in the usual way (de Boer, 1982).

$$\frac{\partial \mathbf{a}_{\alpha\, i}}{\partial \Theta_\alpha^k} = \gamma_{\alpha\, ik}{}^r\, \mathbf{a}_{\alpha\, r} \qquad\qquad \frac{\partial \bar{\mathbf{h}}_\alpha^i}{\partial \Theta_\alpha^k} = -\bar{\Gamma}_{\alpha\, rk}{}^i\, \bar{\mathbf{h}}_\alpha^r \tag{25}$$

Due to skew symmetry, the third order curvature tensor $\overset{3}{\mathbf{K}}_\alpha$ may be reduced to a second order tensor $\bar{\mathbf{K}}_\alpha$ by multiplication with the Ricci tensor $\overset{3}{\mathbf{E}}$.

$$\bar{\mathbf{K}}_\alpha = -\frac{1}{2}(\overset{3}{\mathbf{E}}\overset{3}{\mathbf{K}}_\alpha)^{\underline{2}} \tag{26}$$

The superscript $(\cdots)^{\underline{2}}$ defines the association (26) to yield a second-order tensor.

In the same way as was shown for the curvature, it can be shown that the gyration tensor

$$\bar{\mathbf{\Omega}}_\alpha = \bar{\mathbf{R}}_\alpha \left(\bar{\mathbf{R}}_\alpha^T\right)'_\alpha \tag{27}$$

is skew-symmetric. The symbol $(\cdots)'_\alpha = d_\alpha(\cdots)/dt$ denotes the material time derivative following the motion of φ^α. The angular velocity of the directors is given via the axial vector of the gyration tensor

$$\bar{\boldsymbol{\omega}}_\alpha = \frac{1}{2}\overset{3}{\mathbf{E}}\bar{\mathbf{\Omega}}_\alpha^T \tag{28}$$

If we assume that the material is micropolar, then the micromotion is a pure rotation. In this case, the particles behave like rigid bodies on the microscopic scale. The distribution of the partial density ρ^α relative to the center of the microparticles is described by the microinertia $\mathbf{\Theta}^\alpha$. However, Eringen's (1976) balance of microinertia is not a balance in the sense of the balance of mass or momentum but it is rather a kinematic constraint, which follows from the assumption of rigid microparticles. In this case, the microinertia of the particles may be rotated back to the reference position by the inverse micromotion. This back rotated microinertia must then be materially constant. Note in passing that a result of this type is known from rigid body dynamics. It follows from $\left(\bar{\mathbf{R}}_\alpha^T \mathbf{\Theta}^\alpha \bar{\mathbf{R}}_\alpha\right)'_\alpha = \mathbf{0}$ that

$$(\mathbf{\Theta}^\alpha)'_\alpha = 2 \text{ sym} \left[\bar{\mathbf{\Omega}}_\alpha \mathbf{\Theta}^\alpha\right] \tag{29}$$

The operator "sym" yields the symmetric part of its argument.

Anticipating the results from the balance of mass (39), the microinertia constraint ("balance of microinertia") for φ^α reads

$$(\rho^\alpha \mathbf{\Theta}^\alpha)'_\alpha + \rho^\alpha \mathbf{\Theta}^\alpha \mathrm{div}\, \mathbf{x}'_\alpha = 2 \text{ sym} \left[\rho^\alpha \bar{\mathbf{\Omega}}_\alpha \mathbf{\Theta}^\alpha\right] + \hat{\rho}^\alpha \mathbf{\Theta}^\alpha \tag{30}$$

where the operator "div" is the divergence corresponding to "grad", and $\mathbf{x}'_\alpha$ is the velocity of φ^α. Let $\dot{\mathbf{x}}$ be the barycentric velocity, i. e. the mean velocity of the whole mixture, and let $\mathbf{d}_\alpha$ be the diffusion velocity of φ^α defined by

$$\mathbf{d}_\alpha = \mathbf{x}'_\alpha - \dot{\mathbf{x}} \tag{31}$$

Then, summing up the balances of all φ^α leads to

$$\begin{aligned}\sum_\alpha (\rho^\alpha \boldsymbol{\Theta}^\alpha)^\bullet + \sum_\alpha (\rho^\alpha \boldsymbol{\Theta}^\alpha) \operatorname{div} \dot{\mathbf{x}} = \\ \operatorname{div} \sum_\alpha (-\rho^\alpha \boldsymbol{\Theta}^\alpha \otimes \mathbf{d}_\alpha) + 2 \operatorname{sym} \sum_\alpha \left[\rho^\alpha \bar{\boldsymbol{\Omega}}_\alpha \boldsymbol{\Theta}^\alpha\right] + \sum_\alpha \hat{\rho}^\alpha \boldsymbol{\Theta}^\alpha\end{aligned} \tag{32}$$

The symbol $(\cdots)^\bullet = d(\cdots)/dt$ represents the time derivative following the barycentric velocity $\dot{\mathbf{x}}$. With the definitions of the microinertia of the mixture, $\rho\, \boldsymbol{\Theta} := \sum_\alpha \rho^\alpha \boldsymbol{\Theta}^\alpha$, the inertia flux $\overset{3}{\mathbf{Q}} := -\sum_\alpha \rho^\alpha \boldsymbol{\Theta}^\alpha \otimes \mathbf{d}_\alpha$, the inertia supply $\boldsymbol{\Pi} := 2 \sum_\alpha \operatorname{sym} \left[\rho^\alpha \bar{\boldsymbol{\Omega}}_\alpha \boldsymbol{\Theta}^\alpha\right]$ and the inertia production $\hat{\boldsymbol{\Theta}} := \sum_\alpha \hat{\rho}^\alpha \boldsymbol{\Theta}^\alpha$, the "balance of microinertia" of the whole mixture reads

$$(\rho\, \boldsymbol{\Theta})^\bullet + \rho\, \boldsymbol{\Theta} \operatorname{div} \dot{\mathbf{x}} = \operatorname{div} \overset{3}{\mathbf{Q}} + \boldsymbol{\Pi} + \hat{\boldsymbol{\Theta}} \tag{33}$$

The flux of microinertia of the mixture results from diffusion, which changes the compound of the local mixture volume element, while the inertia supply $\boldsymbol{\Pi}$ and the inertia production $\hat{\boldsymbol{\Theta}}$ directly results from the sum of the corresponding quantities of φ^α, see equation (30).

3 Balance equations

In this section, the balance equations of mass, linear momentum, moment of momentum and energy are presented for the constituents φ^α and for the mixture φ. Following Truesdell's metaphysical principles (Truesdell, 1984), the balances of the mixture follow from the partial balances by summation, since the mixture is the sum of its parts. In addition, the mixture does not know whether it is a mixture or not, and therefore, the mixture balance equations must have the same form as is known from classical continuum mechanics.

In a master balance, changes of a scalar physical quantity Ψ^α are balanced by fluxes $\boldsymbol{\phi}^\alpha$, supplies σ^α and productions $\hat{\Psi}^\alpha$. According to Haupt (1993), the master balance may be written for a body $\mathcal{B}$ with volume element dv, boundary $\partial\mathcal{B}$, and surface element $d\mathbf{a}$ as

$$\frac{d_\alpha}{dt} \int_{\mathcal{B}} \Psi^\alpha dv = \int_{\partial\mathcal{B}} \boldsymbol{\phi}^\alpha \cdot d\mathbf{a} + \int_{\mathcal{B}} \sigma^\alpha dv + \int_{\mathcal{B}} \hat{\Psi}^\alpha dv \tag{34}$$

Application of the divergence theorem leads to the local balance equation for Ψ^α in the following form:

$$(\Psi^\alpha)'_\alpha + \Psi^\alpha \operatorname{div} \mathbf{x}'_\alpha = \operatorname{div} \boldsymbol{\phi}^\alpha + \sigma^\alpha + \hat{\Psi}^\alpha \tag{35}$$

The balance of the mixture has the same structure, only the time derivatives are related to the barycentric velocity $\dot{\mathbf{x}}$. Therefore, the local form of the mixture balance reads

$$\dot{\Psi} + \Psi \operatorname{div} \dot{\mathbf{x}} = \operatorname{div} \boldsymbol{\phi} + \sigma + \hat{\Psi} \tag{36}$$

If the balance of the mixture and the sum of the balances of the constituents have to be identical, the following restrictions hold:

$$\begin{aligned}\Psi &= \textstyle\sum_\alpha \Psi^\alpha \\ \boldsymbol{\phi} &= \textstyle\sum_\alpha (\boldsymbol{\phi}^\alpha - \Psi^\alpha \mathbf{d}_\alpha) \\ \sigma &= \textstyle\sum_\alpha \sigma^\alpha \\ \hat{\Psi} &= \textstyle\sum_\alpha \hat{\Psi}^\alpha\end{aligned} \tag{37}$$

balance of	Ψ, $\boldsymbol{\Psi}$	ϕ, $\boldsymbol{\Phi}$	σ, $\boldsymbol{\sigma}$	$\hat{\Psi}$, $\hat{\boldsymbol{\Psi}}$
mass	ρ	$\mathbf{0}$	0	0
momentum	$\rho\dot{\mathbf{x}}$	$\mathbf{T}$	$\rho\mathbf{b}$	$\mathbf{0}$
moment of momentum	$\mathbf{x}\times\rho\dot{\mathbf{x}}+\rho\boldsymbol{\Theta}\bar{\boldsymbol{\omega}}$	$\mathbf{x}\times\mathbf{T}+\mathbf{M}$	$\mathbf{x}\times\rho\mathbf{b}+\rho\mathbf{m}$	$\mathbf{0}$
energy	$\rho\varepsilon+\frac{1}{2}\rho\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}+$ $+\frac{1}{2}\bar{\boldsymbol{\omega}}\cdot\rho\boldsymbol{\Theta}\bar{\boldsymbol{\omega}}$	$\mathbf{T}^T\dot{\mathbf{x}}+\mathbf{M}^T\bar{\boldsymbol{\omega}}-\mathbf{q}$	$\rho\mathbf{b}\cdot\dot{\mathbf{x}}+\rho\mathbf{m}\cdot\bar{\boldsymbol{\omega}}+\rho\, r$	0
balance of	Ψ^α, $\boldsymbol{\Psi}^\alpha$	ϕ^α, $\boldsymbol{\Phi}^\alpha$	σ^α, $\boldsymbol{\sigma}^\alpha$	$\hat{\Psi}^\alpha$, $\hat{\boldsymbol{\Psi}}^\alpha$
mass	ρ^α	$\mathbf{0}$	0	$\hat{\rho}^\alpha$
momentum	$\rho^\alpha\mathbf{x}'_\alpha$	$\mathbf{T}^\alpha$	$\rho^\alpha\mathbf{b}^\alpha$	$\hat{\mathbf{s}}^\alpha$
moment of momentum	$\mathbf{x}\times\rho^\alpha\mathbf{x}'_\alpha+\rho^\alpha\bar{\boldsymbol{\omega}}$	$\mathbf{x}\times\mathbf{T}+\mathbf{M}$	$\mathbf{x}\times\rho^\alpha\mathbf{b}^\alpha+\rho^\alpha\mathbf{m}^\alpha$	$\hat{\mathbf{m}}^\alpha$
energy	$\rho^\alpha\varepsilon^\alpha+\frac{1}{2}\rho^\alpha\mathbf{x}'_\alpha\cdot\mathbf{x}'_\alpha+$ $+\bar{\boldsymbol{\omega}}_\alpha\cdot\rho^\alpha\boldsymbol{\Theta}^\alpha\bar{\boldsymbol{\omega}}_\alpha$	$\mathbf{T}^{\alpha\,T}\mathbf{x}'_\alpha+\mathbf{M}^{\alpha\,T}\bar{\boldsymbol{\omega}}_\alpha-\mathbf{q}^\alpha$	$\rho^\alpha\mathbf{b}^\alpha\cdot\mathbf{x}'_\alpha+\rho^\alpha\mathbf{m}^\alpha\cdot\bar{\boldsymbol{\omega}}_\alpha+\rho^\alpha r^\alpha$	$\hat{e}^\alpha$

Table 1. Balance equations, corresponding physical quantities, fluxes, supplies and productions.

For vector-valued physical quantities, similar results may be obtained.

The balances of mass, momentum, moment of momentum and energy are obtained by specifying Ψ, ϕ, σ and $\hat{\Psi}$ for the mixture φ and the corresponding quantities for each constituent φ^α, see Table 1.

Substituting the density $\rho = \Psi$ into the master balance leads to the balance of mass

$$\dot{\rho}+\rho\ \mathrm{div}\ \dot{\mathbf{x}}=0 \tag{38}$$

for the mixture. On the other hand, on obtains for φ^α, with $\Psi^\alpha=\rho^\alpha$, $\hat{\Psi}^\alpha=\hat{\rho}^\alpha$,

$$(\rho^\alpha)'_\alpha+\rho^\alpha\ \mathrm{div}\ \mathbf{x}'_\alpha=\hat{\rho}^\alpha \tag{39}$$

The mass exchange from one constituent to another due to phase changes is modeled by the mass production $\hat{\rho}^\alpha$. According to the restrictions (37), the following relations hold:

$$\begin{aligned} \rho &= \textstyle\sum_\alpha \rho^\alpha \\ \mathbf{0} &= \textstyle\sum_\alpha \rho^\alpha \mathbf{d}_\alpha \\ 0 &= \textstyle\sum_\alpha \hat{\rho}^\alpha \end{aligned} \tag{40}$$

Thus, the density of the mixture ρ, is the sum of the partial densities ρ^α, the sum of the diffusion mass fluxes is zero, and the total mass of the mixture is conserved.

The balance of linear momentum of the mixture φ is

$$(\rho\dot{\mathbf{x}})^{\cdot}+\rho\dot{\mathbf{x}}\ \mathrm{div}\ \dot{\mathbf{x}}=\mathrm{div}\ \mathbf{T}+\rho\mathbf{b} \tag{41}$$

with Cauchy stress tensor $\mathbf{T}$ and body forces $\rho\mathbf{b}$, while for each constituent φ^α, we obtain

$$(\rho^\alpha\mathbf{x}'_\alpha)'_\alpha+\rho^\alpha\mathbf{x}'_\alpha\mathrm{div}\ \mathbf{x}'_\alpha=\mathrm{div}\ \mathbf{T}^\alpha+\rho^\alpha\mathbf{b}^\alpha+\hat{\mathbf{s}}^\alpha. \tag{42}$$

The required restrictions are

$$
\begin{aligned}
\rho\dot{\mathbf{x}} &= \textstyle\sum_\alpha \rho^\alpha \mathbf{x}'_\alpha \\
\mathbf{T} &= \textstyle\sum_\alpha (\mathbf{T}^\alpha - \rho^\alpha \mathbf{x}'_\alpha \otimes \mathbf{d}_\alpha) \\
\rho\mathbf{b} &= \textstyle\sum_\alpha \rho^\alpha \mathbf{b}^\alpha \\
\mathbf{0} &= \textstyle\sum_\alpha \hat{\mathbf{s}}^\alpha
\end{aligned}
\tag{43}
$$

including the definition of the barycentric velocity $\dot{\mathbf{x}}$, the total stress tensor $\mathbf{T}$ as the sum of the partial stresses minus the diffusion of momentum (comparable to the Reynolds stresses in turbulence theories) and the body forces of the mixture. The last restriction, $(43)_4$, guarantees that the momentum of the mixture is conserved.

If the results of the balance of mass (38) and (39) are taken into account, we find

$$
\rho\ddot{\mathbf{x}} = \operatorname{div}\, \mathbf{T} + \rho\mathbf{b} \tag{44}
$$

and

$$
\rho^\alpha \mathbf{x}''_\alpha = \operatorname{div}\, \mathbf{T}^\alpha + \rho^\alpha \mathbf{b}^\alpha + \hat{\mathbf{s}}^\alpha - \hat{\rho}^\alpha \mathbf{x}'_\alpha \tag{45}
$$

For the balance of moment of momentum, we find from Table 1,

$$
(\mathbf{x} \times \rho\dot{\mathbf{x}} + \rho\, \boldsymbol{\Theta}\bar{\boldsymbol{\omega}})^{\cdot} + (\mathbf{x} \times \rho\dot{\mathbf{x}} + \rho\, \boldsymbol{\Theta}\bar{\boldsymbol{\omega}})\operatorname{div}\, \dot{\mathbf{x}} = \operatorname{div}\, (\mathbf{x} \times \mathbf{T} + \mathbf{M}) + \mathbf{x} \times \rho\mathbf{b} + \rho\, \mathbf{m} \tag{46}
$$

and for φ^α

$$
\begin{aligned}
&(\mathbf{x} \times \rho^\alpha \mathbf{x}'_\alpha + \rho^\alpha \boldsymbol{\Theta}^\alpha \bar{\boldsymbol{\omega}}_\alpha)'_\alpha + (\mathbf{x} \times \rho^\alpha \mathbf{x}'_\alpha + \rho^\alpha \boldsymbol{\Theta}^\alpha \bar{\boldsymbol{\omega}}_\alpha)\operatorname{div}\, \mathbf{x}'_\alpha \\
&\qquad = \operatorname{div}\, (\mathbf{x} \times \mathbf{T}^\alpha + \mathbf{M}^\alpha) + \mathbf{x} \times \rho^\alpha \mathbf{b}^\alpha + \rho^\alpha \mathbf{m}^\alpha + \hat{\mathbf{m}}^\alpha
\end{aligned}
\tag{47}
$$

leading to the restrictions

$$
\begin{aligned}
\rho\, \boldsymbol{\Theta}\bar{\boldsymbol{\omega}} &= \textstyle\sum_\alpha \rho^\alpha \boldsymbol{\Theta}^\alpha \bar{\boldsymbol{\omega}}_\alpha \\
\mathbf{M} &= \textstyle\sum_\alpha (\mathbf{M}^\alpha - \rho^\alpha \boldsymbol{\Theta}^\alpha \bar{\boldsymbol{\omega}}_\alpha \otimes \mathbf{d}_\alpha) \\
\rho\, \mathbf{m} &= \textstyle\sum_\alpha \rho^\alpha \mathbf{m}^\alpha \\
\mathbf{0} &= \textstyle\sum_\alpha \hat{\mathbf{m}}^\alpha
\end{aligned}
\tag{48}
$$

The spin of the mixture $\rho\, \boldsymbol{\Theta}\, \bar{\boldsymbol{\omega}}$, results from the sum of the partial spins. The couple stresses $\mathbf{M}$ can be derived in the same way as was done for the stresses $\mathbf{T}$, namely, as the sum of the partial couple stresses minus a correction according to the diffusion. The total spin of the mixture is conserved due to the fact that the sum of the production terms is zero.

Taking into account the balances of mass (38), of momentum (44), and the "balance of microinertia" (33), the above equation (46) reduces to

$$
\rho\, \boldsymbol{\Theta}\, \dot{\bar{\boldsymbol{\omega}}} = \mathbf{I} \times \mathbf{T} + \operatorname{div}\, \mathbf{M} + \rho\, \mathbf{m} - \left(\operatorname{div}\, \overset{3}{\mathbf{Q}} + \boldsymbol{\Pi} + \hat{\boldsymbol{\Theta}}\right) \bar{\boldsymbol{\omega}} \tag{49}
$$

for the mixture. Equivalently, with equations (39), (45), and (30), we obtain for φ^α

$$\begin{aligned}\rho^\alpha\Theta^\alpha(\bar\omega_\alpha)'_\alpha = \mathbf{I}\times\mathbf{T}^\alpha + \text{div }\mathbf{M}^\alpha + \rho^\alpha\mathbf{m}^\alpha - 2\text{ sym}\left[\rho^\alpha\bar{\Omega}_\alpha\Theta^\alpha\right]\bar\omega_\alpha \\ +\hat{\mathbf{m}}^\alpha - \hat\rho^\alpha\Theta^\alpha\bar\omega_\alpha - \mathbf{x}\times\hat{\mathbf{s}}^\alpha\end{aligned} \tag{50}$$

The Cauchy stress tensor $\mathbf{T}$ as well as the partial stress tensors $\mathbf{T}^\alpha$ are non-symmetric as is not known in classical continuum mechanics context. In the static case, the skew-symmetric parts expressed by the axial vectors $\mathbf{I}\times\mathbf{T}$ are balanced by the couple stresses $\mathbf{M}$ and the volume moments $\rho\,\mathbf{m}$ ($\mathbf{M}^\alpha$, $\rho^\alpha\mathbf{m}^\alpha$, and $\hat{\mathbf{m}}^\alpha$ for φ^α).

Finally, we obtain the balance of energy for the mixture φ

$$\begin{aligned}&\left(\rho\,\varepsilon + \tfrac{1}{2}\rho\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + \tfrac{1}{2}\bar\omega\cdot\rho\,\Theta\,\bar\omega\right)^{\cdot} + \left(\rho\varepsilon + \tfrac{1}{2}\rho\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + \tfrac{1}{2}\bar\omega\cdot\rho\,\Theta\,\bar\omega\right)\text{div }\dot{\mathbf{x}} \\ &\quad = \text{div }\left(\mathbf{T}^T\dot{\mathbf{x}} + \mathbf{M}^T\bar\omega - \mathbf{q}\right) + \rho\mathbf{b}\cdot\dot{\mathbf{x}} + \rho\,\mathbf{m}\cdot\bar\omega + \rho\,r\end{aligned} \tag{51}$$

and for the constituents φ^α

$$\begin{aligned}&\left(\rho^\alpha\varepsilon^\alpha + \tfrac{1}{2}\rho^\alpha\mathbf{x}'_\alpha\cdot\mathbf{x}'_\alpha + \tfrac{1}{2}\bar\omega_\alpha\cdot\rho^\alpha\Theta^\alpha\bar\omega_\alpha\right)'_\alpha \\ &\quad + \left(\rho^\alpha\varepsilon^\alpha + \tfrac{1}{2}\rho^\alpha\mathbf{x}'_\alpha\cdot\mathbf{x}'_\alpha + \tfrac{1}{2}\bar\omega_\alpha\cdot\rho^\alpha\Theta^\alpha\bar\omega_\alpha\right)\text{div }\mathbf{x}'_\alpha \\ &\quad = \text{div }\left(\mathbf{T}^{\alpha\,T}\mathbf{x}'_\alpha + \mathbf{M}^{\alpha\,T}\bar\omega_\alpha - \mathbf{q}^\alpha\right) + \rho^\alpha\mathbf{b}^\alpha\cdot\mathbf{x}'_\alpha + \rho^\alpha\mathbf{m}^\alpha\cdot\bar\omega_\alpha + \rho^\alpha r^\alpha + \hat{e}^\alpha\end{aligned} \tag{52}$$

The restrictions require that

$$\begin{aligned}\rho\,\varepsilon &= \textstyle\sum_\alpha\left(\rho^\alpha\varepsilon^\alpha + \frac{1}{2}\rho^\alpha\mathbf{d}_\alpha\cdot\mathbf{d}_\alpha + \frac{1}{2}\nu_\alpha\cdot\rho^\alpha\Theta^\alpha\nu_\alpha\right) \\ \mathbf{q} - \frac{1}{2}\left(\overset{3}{\mathbf{Q}}{}^{\overset{13}{T}}\bar\omega\right)\bar\omega &= \textstyle\sum_\alpha\Big(\mathbf{q}^\alpha + \rho^\alpha\varepsilon^\alpha\mathbf{d}_\alpha + \frac{1}{2}\rho^\alpha(\mathbf{d}_\alpha\cdot\mathbf{d}_\alpha)\mathbf{d}_\alpha \\ &\quad + \frac{1}{2}(\nu_\alpha\cdot\rho^\alpha\Theta^\alpha\nu_\alpha)\mathbf{d}_\alpha - \mathbf{T}^{\alpha\,T}\mathbf{d}_\alpha - \mathbf{M}^{\alpha\,T}\nu_\alpha\Big) \\ \rho\,r &= \textstyle\sum_\alpha\left(\rho^\alpha r^\alpha + \rho^\alpha\mathbf{b}^\alpha\cdot\mathbf{d}_\alpha + \rho^\alpha\mathbf{m}^\alpha\cdot\nu_\alpha\right) \\ 0 &= \textstyle\sum_\alpha\hat{e}^\alpha\end{aligned} \tag{53}$$

where $\nu_\alpha = \bar\omega_\alpha - \bar\omega$ is the angular diffusion velocity. It can be seen, that the internal energy of the mixture is not only the sum of the internal energies of the constituents, but contains kinetic energies of the diffusive motion, which may be interpreted in the same manner as Brown's molecular motion. The heat flux of the mixture is modified by a term resulting from the inertia flux of the mixture, $\overset{3}{\mathbf{Q}}$. Note that the microinertia flux $\overset{3}{\mathbf{Q}}$ vanishes according to equation $(40)_2$, if all constituents have the same microinertia, i. e. $\Theta^\alpha = \Theta \quad \forall\alpha$. The total heat flux of the mixture results from the heat fluxes of the constituents, the diffusion of the internal energy of φ^α, the diffusion of the kinetic diffusion energies, and the diffusion power of the stresses and couple stresses. The energy supply of the mixture, i. e. the radiation, also contains terms resulting from the power of the diffusive motion.

These equations can be reduced by the balance of mass (38), (39), microinertia (33), (30), momentum (44), (45), and moment of momentum (49), (50). We obtain for the mixture

$$\rho\,\dot\varepsilon = \mathbf{T}\cdot\mathbf{L} + \mathbf{M}\cdot\bar{\mathbf{W}} - \bar\omega\cdot(\mathbf{I}\times\mathbf{T}) - \text{div }\mathbf{q} + \rho\,r + \frac{1}{2}\bar\omega\cdot\left((\text{div }\overset{3}{\mathbf{Q}}) + \mathbf{\Pi} + \hat\Theta\right)\bar\omega \tag{54}$$

with the spatial velocity gradient, $\mathbf{L} = \text{grad }\dot{\mathbf{x}}$, and the spatial gradient of the angular velocity, $\bar{\mathbf{W}} = \text{grad }\bar\omega$. The result for φ^α reads

$$
\begin{aligned}
\rho^\alpha(\varepsilon^\alpha)'_\alpha = &\ \mathbf{T}^\alpha \cdot \mathbf{L}_\alpha + \mathbf{M}^\alpha \cdot \bar{\mathbf{W}}_\alpha - \bar{\omega}_\alpha \cdot (\mathbf{I} \times \mathbf{T}^\alpha) - \operatorname{div} \mathbf{q}^\alpha + \rho^\alpha r^\alpha \\
&+ \bar{\omega}_\alpha \cdot \operatorname{sym}\left[\rho^\alpha \bar{\mathbf{\Omega}}_\alpha \mathbf{\Theta}^\alpha\right] \bar{\omega}_\alpha + \hat{e}^\alpha - \hat{\rho}^\alpha \varepsilon^\alpha - \hat{\mathbf{s}}^\alpha \cdot \mathbf{x}'_\alpha + \tfrac{1}{2}\hat{\rho}^\alpha \mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha \\
&- \hat{\mathbf{m}}^\alpha \cdot \bar{\omega}_\alpha + (\mathbf{x} \times \hat{\mathbf{s}}^\alpha) \cdot \bar{\omega}_\alpha + \tfrac{1}{2}\bar{\omega}_\alpha \cdot \hat{\rho}^\alpha \mathbf{\Theta}^\alpha \bar{\omega}_\alpha
\end{aligned} \tag{55}
$$

4 Conclusions

The kinematic relations and the balance equations for a mixture of n micropolar constituents are presented. The deformation measures are based on squares of line elements which can be expressed as scalar products of natural basis vectors. In addition to the constituent's motion $\mathbf{x} = \chi_\alpha(\mathbf{X}_\alpha,\ t)$, each constituent φ^α is assigned a micromotion $\bar{\mathbf{R}}_\alpha$ which rotates a director from the reference configuration into the actual configuration. The micromotion $\bar{\mathbf{R}}_\alpha$ may be written as two-field tensor which allows to introduce two new configurations, resulting either from the transport of the reference basis vectors by the micromotion or of the actual basis vectors by the inverse micromotion. The Cosserat deformation tensors are found to map these new configurations onto the reference or the actual configuration. The deformation gradient may be decomposed similarly to the polar decomposition into the non-symmetric Cosserat deformation tensors and the micromotion, $\mathbf{F}_\alpha = \bar{\mathbf{R}}_\alpha \bar{\mathbf{U}}_\alpha$, or, vice versa, $\mathbf{F}_\alpha = \bar{\mathbf{V}}_\alpha \bar{\mathbf{R}}_\alpha$. In terms of the natural basis vectors, the curvature, which represents the second Cosserat deformation measure, may be interpreted as generalized Christoffel symbols. Therefore, it represents the spatial variation of the basis vectors introduced with the new configurations.

The balance equations of mass, momentum, moment of momentum and energy are derived from a master balance by specifying the physical quantity, as well as its flux, supply, and production terms. From Truesdell's metaphysical principles it is concluded that the sum of the balances of the constituents gives the corresponding balance of the mixture. These balances must have the same structure as have the balances of classical continuum mechanics. This restriction allows to compute the physical quantities, their fluxes, supplies and productions of the mixture from the quantities of the constituents. Including rotational degrees of freedom into the theory leads to changes in the balance of moment of momentum and of energy while the balance of mass and momentum remain unchanged compared with the non-polar theory. In general, the partial stress tensors may be non-symmetric as a result of the balance of moment of momentum, the balance of energy contains additional terms resulting from the angular velocity corresponding to the micromotion in combination with the microinertia. Eringen's balance of microinertia is interpreted as kinematic constraint, which results from the assumption of rigid microparticles in the framework of micropolar theories.

The balance equations presented in this article must be completed by constitutive laws which connect the motion and the micromotion of each constituent with the stress tensor and the couple stress tensor. In addition, the heat flux and the production terms must be given by constitutive assumptions. These constitutive assumptions must not violate the second law of thermodynamics. In a next step, an appropriate form of the second law of thermodynamics must be developed from which the necessary restrictions may be derived.

Literature

1. Bedford, A.; Drumheller, D. S.: Theories of immiscible and structured mixtures. Int. J. Engng. Sci., 21, (1983), 863 – 960.
2. de Boer, R.: Vektor- und Tensorrechnug für Ingenieure. Springer-Verlag, Berlin, (1982).
3. de Boer, R.; Ehlers, W.: Theorie der Mehrkomponentenkontinua mit Anwendung auf bodenmechanische Probleme, Teil I. Forschungsberichte aus dem Fachbereich Bauwesen, Heft 40, Universität-GH-Essen, (1986).
4. Bowen, R. M.: Theory of mixtures. In A. C. Eringen (ed.): Continuum Physics, Vol. III, pp. 1 – 127, Academic Press, New York, (1976).
5. Bowen, R. M.: Incompressible porous media models by use of the theory of mixtures. Int. J. Engng. Sci., 18, (1980), 1129 – 1148.
6. Cosserat, E.; Cosserat, F.: Théorie des Corps Déformable. A. Herman, Paris, (1909).

7. Ehlers, W.: Poröse Medien – ein kontinuumsmechanisches Modell auf der Basis der Mischungstheorie. Forschungsberichte aus dem Fachbereich Bauwesen, Heft 47, Universität-GH-Essen, (1989).
8. Ericksen, J. L.; Truesdell, C.: Exact theory of stress and strain in rods and shells. Arch. Rational Mech. Anal., 1, (1958), 295 – 323.
9. Eringen, A. C.: Simple Microfluids. Int. J. Engng. Sci., 2, (1964), 205 – 217.
10. Eringen, A. C.; Kafadar, C. B.: Polar field theories. In A. C. Eringen (ed.): Continuum Physics, Vol. IV, pp. 1 – 73, Academic Press, New York, (1976).
11. Günther, W.: Zur Statik und Kinematik des Cosseratschen Kontinuums. Abh. Braunschweig. Wiss. Ges., 10, (1958), 195 – 213.
12. Hassanizadeh, S. M.; Gray, W. G.: General conservation equations for multi-phase-systems: 1. Averaging procedure. Adv. Water Resources, 2, (1979a), 131 – 144.
13. Hassanizadeh, S. M.; Gray, W. G.: General conservation equations for multi-phase-systems: 2. Mass, momenta, energy, and entropy equations. Adv. Water Resources, 2, (1979b), 191 – 203.
14. Haupt, P.: Foundations of continuum mechanics. In K. Hutter (ed.): Continuum Mechanics in Environmental Sciences and Geophysics, CISM Courses and Lectures No. 337, pp. 1 – 77, Springer-Verlag, Wien, (1993).
15. Sansour, C.; Bednarczyk, H.: The Cosserat surface as a shell model, theory and finite-element formulation. Comput. Methods Appl. Mech. Engrg., 120, (1995), 1 – 32.
16. Schaefer, H.: Das Cosserat-Kontinuum. ZAMM, 47, (1967), 485 – 498.
17. Steinmann, P.: A micropolar theory of finite deformation and finite rotation multiplicative elastoplasticity. Int. J. Solids Struc., 31, (1994), 1063 – 1084.
18. Tejchman, J.; Wu, W.: Numerical study on patterning of shear bands in a Cosserat continuum. Acta Mech., 99, (1993), 61 – 74.
19. Truesdell, C.: Rational Thermodynamics (2nd ed.), p. 221. Springer-Verlag, New York, (1984).

Addresses: Dr.-Ing. Stefan Diebels and Prof. Dr.-Ing. Wolfgang Ehlers, Institut für Mechanik (Bauwesen), Universität Stuttgart, D-70550 Stuttgart