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On the Continuum Mechanics of Elastic and Inelastic Simple Materials

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The purpose of this short paper is to briefly review some basic aspects of the continuum mechanics of elastic and inelastic simple materials from a local differential-geometric point of view, both in Euclidean space and with respect to the body manifold.

1 Mathematical Preliminaries

We begin with a brief review of some mathematical concepts to be used in the continuum mechanical formulation to follow. Most, if not all, of these can be found, *e.g.*, in Noll (1972, 1973), Wang and Truesdell (1973), Bowen and Wang (1976), Marsden and Hughes (1983), or Abraham *et al.* (1988).

Let $\mathcal{W}$ and $\mathcal{Z}$ be *finite-dimensional* linear spaces, and $\mathbb{R}$ the set of all real numbers. A *linear mapping* between $\mathcal{W}$ and $\mathcal{Z}$ is a mapping $\boldsymbol{L} : \mathcal{W} \to \mathcal{Z} \mid \boldsymbol{w} \mapsto \boldsymbol{z} = \boldsymbol{L}(\boldsymbol{w})$ satisfying the condition

$$\boldsymbol{L}(a_1 \boldsymbol{w}_1 + a_2 \boldsymbol{w}_2) = a_1 \boldsymbol{L}(\boldsymbol{w}_1) + a_2 \boldsymbol{L}(\boldsymbol{w}_2) \qquad \forall\, \boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{W} \quad \text{and} \quad \forall\, a_1, a_2 \in \mathbb{R} \tag{1.1}$$

(the symbol $\forall$ stands for "for all"). For any such linear mapping, it is a common simplifying convention to write the value $\boldsymbol{L}(\boldsymbol{w})$ of $\boldsymbol{L}$ at $\boldsymbol{w}$ simply as $\boldsymbol{L}\boldsymbol{w}$, *i.e.*, to discard the parentheses. Let $\mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ represent the set of all linear mappings between $\mathcal{W}$ and $\mathcal{Z}$, itself a linear space. If $\mathcal{W}$ and $\mathcal{Z}$ have the same dimension, *i.e.*, if $\dim(\mathcal{W}) = \dim(\mathcal{Z})$ holds, then we can also talk about linear mappings between $\mathcal{W}$ and $\mathcal{Z}$ which are *invertible*. In this case, let $\mathrm{Lbj}(\mathcal{W}, \mathcal{Z}) \subset \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ represent the set of all mappings between $\mathcal{W}$ and $\mathcal{Z}$ which are both linear and invertible ("Lbj" stands for "linear bijection").

A particular case of $\mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ arises when we choose $\mathcal{Z}$ to be the set $\mathbb{R}$ of all real numbers. The corresponding linear space $\mathcal{W}^* := \mathrm{Lin}(\mathcal{W}, \mathbb{R})$ is called the *dual space of, or to,* $\mathcal{W}$, and its elements $\boldsymbol{\eta} \in \mathcal{W}^*$ are called *covectors* (the symbol " := " stands for "is defined equal to"). Note that any $\boldsymbol{z} \in \mathcal{Z}$ and $\boldsymbol{\eta} \in \mathcal{W}^*$ induce a linear mapping $(\boldsymbol{z} \otimes \boldsymbol{\eta}) \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ called the *tensor or dyadic product* of $\boldsymbol{z} \in \mathcal{Z}$ and $\boldsymbol{\eta} \in \mathcal{W}^*$, defined by $(\boldsymbol{z} \otimes \boldsymbol{\eta})\boldsymbol{w} := (\boldsymbol{\eta}\boldsymbol{w})\, \boldsymbol{z}$ for all $\boldsymbol{w} \in \mathcal{W}$. Since $\mathcal{W}^*$, like $\mathcal{W}$, is a linear space, it possesses a dual space as well, *i.e.*, $\mathcal{W}^{**} := \mathrm{Lin}(\mathcal{W}^*, \mathbb{R})$, sometimes called the second dual space of, or to, $\mathcal{W}$. Now, by definition, any $\boldsymbol{\eta} \in \mathcal{W}^*$ maps any $\boldsymbol{w} \in \mathcal{W}$ to the real number $\boldsymbol{\eta}\boldsymbol{w} \in \mathbb{R}$ (remember $\boldsymbol{\eta}\boldsymbol{w} = \boldsymbol{\eta}(\boldsymbol{w})$ by convention). On the other hand, we can also think of $\boldsymbol{w} \in \mathcal{W}$ as inducing the linear mapping $\boldsymbol{\eta} \mapsto \boldsymbol{\eta}\boldsymbol{w}$ of $\boldsymbol{\eta} \in \mathcal{W}^*$ to $\boldsymbol{\eta}\boldsymbol{w} \in \mathbb{R}$, in which case this mapping is an element of $\mathcal{W}^{**}$. Consequently, each $\boldsymbol{w} \in \mathcal{W}$ induces a unique element $\imath_{\boldsymbol{u}} \in \mathcal{W}^{**}$ of $\mathcal{W}^{**}$, defined by $\imath_{\boldsymbol{u}}\boldsymbol{\eta} := \boldsymbol{\eta}\boldsymbol{w}$ for all $\boldsymbol{\eta} \in \mathcal{W}^*$. Because of this, we can *identify* any element of $\mathcal{W}^{**}$ uniquely with one of $\mathcal{W}$; such a *natural identification* (*i.e.*, one based on the given structures of $\mathcal{W}^{**}$ and $\mathcal{W}$ alone) is signified by writing $\mathcal{W}^{**} \cong \mathcal{W}$ (*i.e.*, $\imath_{\boldsymbol{u}} \cong \boldsymbol{w}$). It is important to emphasize that this *does not mean* that $\mathcal{W}^{**}$ and $\mathcal{W}$ are *equal*, or *the same*, but rather, that we can always express mathematical relations involving the elements of one of these spaces uniquely in terms of elements of the other. Unfortunately, no such natural identification exists between $\mathcal{W}^*$ and $\mathcal{W}$ in general; with the help of additional structure (*i.e.*, an inner product), however, such an identification is induced, as discussed briefly below.

Any linear mapping $\boldsymbol{L} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ induces a corresponding linear mapping $\boldsymbol{L}^* \in \mathrm{Lin}(\mathcal{Z}^*, \mathcal{W}^*)$, called its *dual* (linear mapping), defined by

$$\underbrace{(\boldsymbol{L}^*\boldsymbol{\sigma})}_{\in \mathcal{W}^*}\, \boldsymbol{w} := \boldsymbol{\sigma}\, \underbrace{(\boldsymbol{L}\boldsymbol{w})}_{\in \mathcal{Z}} \qquad \forall\, \boldsymbol{w} \in \mathcal{W} \quad \text{and} \quad \forall\, \boldsymbol{\sigma} \in \mathcal{Z}^* \tag{1.2}$$

Let $\boldsymbol{b} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{\dim(\mathcal{W})})$ be a basis of $\mathcal{W}$, and $\boldsymbol{\beta} := (\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, \ldots, \boldsymbol{\beta}^{\dim(\mathcal{W})})$ one of $\mathcal{W}^*$. Two such bases are called *dual* if $\imath_{\boldsymbol{b}_j}\boldsymbol{\beta}^i = \boldsymbol{\beta}^i\boldsymbol{b}_j = \delta^i_j$ $(i,j = 1, \ldots, \dim(\mathcal{W}))$ holds, where δ^i_j is Kronecker's delta, *i.e.*, $\delta^i_j = 1$ for $i = j$ and $\delta^i_j = 0$ for $i \neq j$. Let these two bases be dual, $\boldsymbol{c} := (\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_{\dim(\mathcal{Z})})$ be a basis of $\mathcal{Z}$, and $\boldsymbol{\gamma} := (\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \ldots, \boldsymbol{\gamma}^{\dim(\mathcal{Z})})$ one of $\mathcal{Z}^*$ dual to $\boldsymbol{c}$. Relative to such bases, we have the component representations $\boldsymbol{L} = \sum_{m,n}[\boldsymbol{L}]^m{}_n\, \boldsymbol{c}_m \otimes \boldsymbol{\beta}^n$ and $\boldsymbol{L}^* = \sum_{k,l}[\boldsymbol{L}^*]_l{}^k\, \boldsymbol{\beta}^l \otimes \imath_{\boldsymbol{c}_k}$. Using these and (1.2) yields the relation $[\boldsymbol{L}^*]_j{}^i = \imath_{\boldsymbol{b}_j}(\boldsymbol{L}^*\boldsymbol{\gamma}^i) = (\boldsymbol{L}^*\boldsymbol{\gamma}^i)\boldsymbol{b}_j = \boldsymbol{\gamma}^i(\boldsymbol{L}\boldsymbol{b}_j) = [\boldsymbol{L}]^i{}_j$ between the components of $\boldsymbol{L}^*$ and $\boldsymbol{L}$, implying that the component matrix of $\boldsymbol{L}^*$ is the transpose of that of $\boldsymbol{L}$ (with respect to dual bases).

Let $\dim(\mathcal{W}) = n$ be the dimension of $\mathcal{W}$, and $k \leq n$. A *k-multilinear mapping* of $\mathcal{W}$ into $\mathbb{R}$ is one $\mu\colon \mathcal{W}^k \to \mathbb{R} \mid (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k) \mapsto \mu(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k)$ of $\mathcal{W}^k := \mathcal{W} \times \cdots \times \mathcal{W}$ (k times) into $\mathbb{R}$ which is separately linear in each argument, *i.e.*, $\mu(\boldsymbol{w}_1, \ldots, a_1\boldsymbol{u}_i + a_2\boldsymbol{v}_i, \ldots, \boldsymbol{w}_k) = a_1\, \mu(\boldsymbol{w}_1, \ldots, \boldsymbol{u}_i, \ldots, \boldsymbol{w}_k) + a_2\, \mu(\boldsymbol{w}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{w}_k)$ holds for all $i = 1, \ldots, k$. Such a linear mapping is in addition *completely skew-symmetric* if $\mu(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_i, \ldots, \boldsymbol{w}_j, \ldots, \boldsymbol{w}_k) = \pm\, \mu(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_j, \ldots, \boldsymbol{w}_i, \ldots, \boldsymbol{w}_k)$ holds for all $i, j = 1, \ldots, k$, $i \neq j$, where $+$ obtains when the permutation is even, and $-$ when it is odd. In particular, the *non-zero* elements of the one-dimensional linear space $\mathrm{Skw}_n(\mathcal{W}^n, \mathbb{R})$ of all completely skew-symmetric n-multilinear mappings of $\mathcal{W}$ into $\mathbb{R}$ are called *volume covectors.* Perhaps the best-known example of such a volume covector is the standard Euclidean volume covector of three-dimensional Euclidean vector space (see (1.10) below). Using such volume covectors, we can define for example the *determinant*

$$\det_{\mathcal{W}}(\boldsymbol{M}) := \frac{\mu(\boldsymbol{M}\boldsymbol{w}_1, \ldots, \boldsymbol{M}\boldsymbol{w}_n)}{\mu(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)} \tag{1.3}$$

of any $\boldsymbol{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{W})$ for all linearly independent vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n \in \mathcal{W}$ and *any* volume covector $\mu \in \mathrm{Skw}_n(\mathcal{W}^n, \mathbb{R})$. As indicated by the notation, this definition is independent of the choice $\mu \in \mathrm{Skw}_n(\mathcal{W}^n, \mathbb{R})$. On the basis of this definition, one can show in particular that

$$\det_{\mathcal{W}^*}(\boldsymbol{M}^*) = \det_{\mathcal{W}}(\boldsymbol{L}^{-1}\boldsymbol{M}^*\boldsymbol{L}) = \det_{\mathcal{W}}(\boldsymbol{M}) \tag{1.4}$$

holds for all $\boldsymbol{L} \in \mathrm{Lbj}(\mathcal{W}, \mathcal{W}^*)$ and $\boldsymbol{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{W})$. Now, any two volume covectors $\mu, \omega \in \mathrm{Skw}_n(\mathcal{W}^n, \mathbb{R})$ are *equivalent* if there exists a *positive* real number $a > 0$ such that $\mu = a\omega$ holds, *i.e.*, $\mu(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n) = a\,\omega(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$ for all $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n \in \mathcal{W}$. An equivalence class $[\omega] := \{\mu \in \mathrm{Skw}_n(\mathcal{W}^n, \mathbb{R}) \mid$ there exists an $a > 0$ such that $\mu = a\omega\}$ of such volume covectors determines an *orientation* of $\mathcal{W}$. Since $\mathrm{Skw}_n(\mathcal{W}^n, \mathbb{R})$ is one-dimensional, there are two such orientations, *i.e.*, $[\omega]$ and $[-\omega]$. By convention, the chosen orientation is called *positive*, and the other *negative.* A linear space $\mathcal{W}$ endowed with a given (*i.e.*, fixed) orientation is called *oriented.* For $\mathcal{W}$ and $\mathcal{Z}$ oriented with orientations $[\omega_{\mathcal{W}}]$ and $[\omega_{\mathcal{Z}}]$, respectively, and $\dim(\mathcal{W}) = \dim(\mathcal{Z})$, any $\boldsymbol{L} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ is called *orientation-preserving* if, for any $\varpi \in [\omega_{\mathcal{Z}}]$, the corresponding volume covector $\varpi_{\boldsymbol{L}} \in \mathrm{Skw}_n(\mathcal{W}^n, \mathbb{R})$ induced by $\boldsymbol{L} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ on $\mathcal{W}$ belongs to $[\omega_{\mathcal{W}}]$, where $\varpi_{\boldsymbol{L}}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n) := \varpi(\boldsymbol{L}\boldsymbol{w}_1, \ldots, \boldsymbol{L}\boldsymbol{w}_n)$ for all linearly independent $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n \in \mathcal{W}$. Since $\varpi_{\boldsymbol{L}} \in [\omega_{\mathcal{W}}]$ can in fact hold only when $\boldsymbol{L} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ is one-to-one (*i.e.*, injective), any orientation-preserving $\boldsymbol{L} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ is also invertible (since $\dim(\mathcal{W}) = \dim(\mathcal{Z})$). Let $\mathrm{Lin}^+(\mathcal{W}, \mathcal{Z}) \subset \mathrm{Lbj}(\mathcal{W}, \mathcal{Z})$ represent the set of all such orientation-preserving, linear mappings between two oriented linear spaces $\mathcal{W}$ and $\mathcal{Z}$. In particular, elements of the set

$$\mathrm{Uni}^+(\mathcal{W}, \mathcal{W}) := \{\boldsymbol{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{W}) \mid \det_{\mathcal{W}}(\boldsymbol{M}) = +1\} \tag{1.5}$$

of unimodular linear mappings $\boldsymbol{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{W})$ with positive determinant are orientation-preserving.

A linear mapping $\boldsymbol{L} \in \mathrm{Lin}(\mathcal{W}, \mathcal{W}^*)$ is called *positive-definite* if $(\boldsymbol{L}\boldsymbol{w})\boldsymbol{w} > 0$ holds for all $\boldsymbol{w} \in \mathcal{W}$ except the zero element $\mathbf{0} \in \mathcal{W}$. A (positive-definite) *metric tensor* on a finite-dimensional linear space $\mathcal{W}$ is an element of the set

$$\mathrm{Spd}(\mathcal{W}, \mathcal{W}^*) := \{\boldsymbol{G} \in \mathrm{Lin}(\mathcal{W}, \mathcal{W}^*) \mid \boldsymbol{G}^* \cong \boldsymbol{G} \text{ and } \boldsymbol{G} \text{ positive-definite}\} \tag{1.6}$$

of all symmetric positive-definite linear mappings between $\mathcal{W}$ and $\mathcal{W}^*$. Since $\dim(\mathcal{W}) = \dim(\mathcal{W}^*)$, and any element of $\mathrm{Spd}(\mathcal{W}, \mathcal{W}^*)$ is one-to-one (*i.e.*, injective), we have $\mathrm{Spd}\,(\mathcal{W}, \mathcal{W}^*) \subset \mathrm{Lbj}(\mathcal{W}, \mathcal{W}^*)$.

Any $\boldsymbol{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{W})$ is called *symmetric (skew-symmetric) with respect to* $\boldsymbol{G} \in \mathrm{Spd}(\mathcal{W}, \mathcal{W}^*)$ when $\boldsymbol{M}^* = \boldsymbol{G}\boldsymbol{M}\boldsymbol{G}^{-1}$ $(\boldsymbol{M}^* = -\boldsymbol{G}\boldsymbol{M}\boldsymbol{G}^{-1})$ holds. In addition, any $\boldsymbol{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{W})$ is called *orthogonal with respect to* $\boldsymbol{G} \in \mathrm{Spd}(\mathcal{W}, \mathcal{W}^*)$ if $\boldsymbol{M}^*\boldsymbol{G}\boldsymbol{M} = \boldsymbol{G}$. This latter condition holds in fact only if $\boldsymbol{M}$ is one-to-one (*i.e.*, injective), in which case it is also invertible (since $\mathcal{W}$ is finite-dimensional), and so $\boldsymbol{M}^* = \boldsymbol{G}\boldsymbol{M}^{-1}\boldsymbol{G}^{-1}$ also holds. Let

$$\begin{aligned}
\mathrm{Sym}_{\boldsymbol{G}}(\mathcal{W},\mathcal{W}) &:= \{\boldsymbol{M}\in\mathrm{Lin}(\mathcal{W},\mathcal{W}) \mid \boldsymbol{M}^* = \boldsymbol{G}\boldsymbol{M}\boldsymbol{G}^{-1}\} \\
\mathrm{Spd}_{\boldsymbol{G}}(\mathcal{W},\mathcal{W}) &:= \{\boldsymbol{M}\in\mathrm{Lin}(\mathcal{W},\mathcal{W}) \mid \boldsymbol{G}\boldsymbol{M}\in\mathrm{Spd}(\mathcal{W},\mathcal{W}^*)\} \\
\mathrm{Skw}_{\boldsymbol{G}}(\mathcal{W},\mathcal{W}) &:= \{\boldsymbol{M}\in\mathrm{Lin}(\mathcal{W},\mathcal{W}) \mid \boldsymbol{M}^* = -\boldsymbol{G}\boldsymbol{M}\boldsymbol{G}^{-1}\} \\
\mathrm{Orth}^+_{\boldsymbol{G}}(\mathcal{W},\mathcal{W}) &:= \{\boldsymbol{M}\in\mathrm{Lin}^+(\mathcal{W},\mathcal{W}) \mid \boldsymbol{M}^* = \boldsymbol{G}\boldsymbol{M}^{-1}\boldsymbol{G}^{-1}\}
\end{aligned} \tag{1.7}$$

denote the sets of all symmetric, symmetric positive-definite, skew-symmetric, and orientation-preserving orthogonal, linear mappings of $\mathcal{W}$ onto itself, respectively, again with respect to $\boldsymbol{G}\in\mathrm{Spd}(\mathcal{W},\mathcal{W}^*)$. In particular, note that $\det_{\mathcal{W}}(\boldsymbol{G}^{-1}\boldsymbol{M}^*\boldsymbol{G}) = \det_{\mathcal{W}}(\boldsymbol{M}) = \det_{\mathcal{W}}(\boldsymbol{M}^{-1}) = 1/\det_{\mathcal{W}}(\boldsymbol{M})$ for any $\boldsymbol{M}\in\mathrm{Orth}^+_{\boldsymbol{G}}(\mathcal{W},\mathcal{W})$ from (1.4), *i.e.*, $\det_{\mathcal{W}}(\boldsymbol{M}) = 1$.
If we (can) single out a *particular* (positive-definite) metric $\boldsymbol{G}\in\mathrm{Spd}(\mathcal{V},\mathcal{V}^*)$ on a finite-dimensional linear space $\mathcal{V}$, this metric induces the structure of a (positive-definite) *inner product space* on $\mathcal{V}$. In particular, $\boldsymbol{G}$ induces the inner product $\boldsymbol{v}_1\cdot\boldsymbol{v}_2 := (\boldsymbol{G}\boldsymbol{v}_1)\boldsymbol{v}_2$ of any two $\boldsymbol{v}_1,\boldsymbol{v}_2\in\mathcal{V}$. Being linear and invertible, $\boldsymbol{G}$ maps any basis $\boldsymbol{b}$ of $\mathcal{V}$ to one $\boldsymbol{G}\boldsymbol{b} := (\boldsymbol{G}\boldsymbol{b}_1,\boldsymbol{G}\boldsymbol{b}_2,\ldots)$ of $\mathcal{V}^*$, representing the *reciprocal basis of* $\boldsymbol{b}$. Since the components $[\boldsymbol{G}]_{ij} := (\boldsymbol{G}\boldsymbol{b}_i)\boldsymbol{b}_j$ of $\boldsymbol{G}$ with respect to $\boldsymbol{b}$ are in general not equal to δ_{ij}, $\boldsymbol{G}\boldsymbol{b}$ is in general not a dual basis of $\boldsymbol{b}$; but if this is in fact the case, *i.e.*, if $[\boldsymbol{G}]_{ij} = \delta_{ij}$ holds, then $\boldsymbol{b}$ is called *orthogonal.* In addition, we have the *transpose*

$$\boldsymbol{L}^{\mathrm{T}} := \boldsymbol{G}^{-1}\boldsymbol{L}^*\boldsymbol{G} \tag{1.8}$$

of any $\boldsymbol{L}\in\mathrm{Lin}(\mathcal{V},\mathcal{V})$. With respect to a basis $\boldsymbol{b}$ of $\mathcal{V}$, and one $\boldsymbol{\beta}$ of $\mathcal{V}^*$ dual to $\boldsymbol{b}$, we have $[\boldsymbol{L}^{\mathrm{T}}]^i{}_j = [\boldsymbol{G}^{-1}]^{ik}[\boldsymbol{L}^*]_k{}^m[\boldsymbol{G}]_{mj} = [\boldsymbol{G}^{-1}]^{ik}[\boldsymbol{L}]^m{}_k[\boldsymbol{G}]_{mj}$, and in particular $[\boldsymbol{L}^{\mathrm{T}}]^i{}_j = [\boldsymbol{L}]^j{}_i$ if $\boldsymbol{\beta} = \boldsymbol{G}\boldsymbol{b}$, *i.e.*, if $\boldsymbol{b}$ is orthogonal. On the basis of (1.7) and (1.8), we have the usual forms

$$\begin{aligned}
\mathrm{Sym}(\mathcal{V},\mathcal{V}) &:= \{\boldsymbol{L}\in\mathrm{Lin}(\mathcal{V},\mathcal{V}) \mid \boldsymbol{L}^{\mathrm{T}} = \boldsymbol{L}\} \\
\mathrm{Spd}(\mathcal{V},\mathcal{V}) &:= \{\boldsymbol{L}\in\mathrm{Sym}(\mathcal{V},\mathcal{V}) \mid \boldsymbol{L}\boldsymbol{v}\cdot\boldsymbol{v} > 0 \quad \forall\boldsymbol{v}\neq\boldsymbol{0}\} \\
\mathrm{Skw}(\mathcal{V},\mathcal{V}) &:= \{\boldsymbol{L}\in\mathrm{Lin}(\mathcal{V},\mathcal{V}) \mid \boldsymbol{L}^{\mathrm{T}} = -\boldsymbol{L}\} \\
\mathrm{Orth}^+(\mathcal{V},\mathcal{V}) &:= \{\boldsymbol{L}\in\mathrm{Lin}^+(\mathcal{V},\mathcal{V}) \mid \boldsymbol{L}^{\mathrm{T}} = \boldsymbol{L}^{-1}\}
\end{aligned} \tag{1.9}$$

for the sets of all symmetric, skew-symmetric, and orientation-preserving orthogonal, linear mappings of the inner product space $\mathcal{V}$ onto itself. Since the metric is fixed in the case of an inner product space, it does not appear in the notation for these sets in (1.9), in contrast to the more general case defined in (1.7).
Since $\boldsymbol{G}$ is linear and invertible, it induces the natural identification $\mathcal{V}^* \cong \mathcal{V}$ ($\boldsymbol{\nu} \cong \boldsymbol{G}^{-1}\boldsymbol{\nu}$), *i.e.*, any covector $\boldsymbol{\nu}\in\mathcal{V}^*$ can be identified or associated with a unique vector $\boldsymbol{G}^{-1}\boldsymbol{\nu}\in\mathcal{V}$. Via linearity and duality, this identification can be extended to tensors as well. In particular, $\boldsymbol{G}$ itself can be identified or associated with the identity linear mapping $\boldsymbol{I}\in\mathrm{Lin}(\mathcal{V},\mathcal{V})$ on $\mathcal{V}$, *i.e.*, $\boldsymbol{G}\cong\boldsymbol{I}$, induced via $\boldsymbol{I} = \boldsymbol{G}\boldsymbol{G}^{-1}$, for example; we emphasize that *this does not mean* that $\boldsymbol{G}$ and $\boldsymbol{I}$ are *equal*, or *the same*, something that would make no mathematical sense. To avoid the confusion on this point sometimes found in the literature, we dispense here with the identification $\mathcal{V}^*\cong\mathcal{V}$, and deal instead *directly and explicitly* with $\boldsymbol{G}$ in the formulation.

Perhaps the most well-known example of such an inner product space is the vector (translation) space of three-dimensional Euclidean space, *i.e.*, the mathematical model for "space" in classical physics, and so continuum mechanics. To be more precise, the Euclidean space of classical physics is a three-dimensional affine space $(E,\mathcal{V})$, where E is the point space, and $\mathcal{V}$ the corresponding translation (vector) space endowed with the Euclidean metric tensor $\boldsymbol{G}$ and compatible Euclidean norm $|\boldsymbol{v}| := \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$ for all $\boldsymbol{v}\in\mathcal{V}$. The fact that $(E,\mathcal{V})$ is an affine space means, roughly speaking, that any two points $p,q\in E$ of E can be "connected" by an element $\boldsymbol{a}\in\mathcal{V}$ of $\mathcal{V}$, in which case one writes $q = p+\boldsymbol{a}$ or $\boldsymbol{a} = q-p$. Among other things, $\boldsymbol{G}$ induces the standard (compatible) Euclidean volume covector $\omega_{\boldsymbol{G}}\in\mathrm{Skw}_3(\mathcal{V}^3,\mathbb{R})$, which in turn induces the cross product $\boldsymbol{v}_1\times\boldsymbol{v}_2$ of any two $\boldsymbol{v}_1,\boldsymbol{v}_2\in\mathcal{V}$, such that

$$\omega_{\boldsymbol{G}}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3) = (\boldsymbol{v}_1\times\boldsymbol{v}_2)\cdot\boldsymbol{v}_3 \tag{1.10}$$

holds for all $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\in\mathcal{V}$.

2 Material Body and Euclidean Space

Any material is idealized in continuum mechanics via the notion of a material body. The kinematics of such a material body is represented in E with the help of regions $B_\kappa \subset E$ of E which the body may occupy, for example, during its motion and corresponding time-dependent deformation in E. The mathematical properties of such regions depends on the type of material body being considered. In the case of a *classical* material body, *e.g.*, one not containing cracks or other macroscopic discontinuities, Noll and Virga (1988) showed that the corresponding *classical* regions can be modeled by the so-called *fit regions* of E, *i.e.*, subsets P of E that (1) are bounded, (2) are regularly open (*i.e.*, each equal to the interior of its closure), (3) have finite perimeter, and (4) have boundary of volume measure zero (*i.e.*, zero volume). As shown by Del Piero and Owen (1993), fit regions are no longer adequate as mathematical models for the regions occupied by a material body when the body in question contains "two-dimensional" macroscopic discontinuities (*e.g.*, unopened cracks or fractures), and must be replaced by *piecewise* fit regions, representing *finite unions* of fit regions, the individual fit regions of the union being separated, roughly speaking, by the two-dimensional discontinuities in question.

On the basis of fit regions, one can formulate the *classical* notion of *deformation* (relative to these). Indeed, a deformation from one fit region $P \subset E$ into a second fit region $Q \subset E$ takes the form of a mapping

$$\xi : P \longrightarrow Q \quad | \quad p \longmapsto q = \xi(p) \tag{2.1}$$

representing then (mathematically) a *morphism* of such regions in E, *i.e.*, a C^1 diffeomorphism of E restricted to such regions. This being the case, it is still useful to briefly discuss the physical constraints lying behind such a mathematical model for classical deformations. For example, we have the physical notion that a given part of a classical material body usually cannot merge with or pass through another part of the same body in a deformation, representing the impenetrability of matter. As a restriction on ξ, this takes the form

$$\xi(p_1) = \xi(p_2) \implies p_1 = p_2 \qquad \forall p_1, p_2 \in P \tag{2.2}$$

in terms of material points, *i.e.*, ξ can neither "create" nor "destroy" material points. Since the opening or closing of a macroscopic crack or other discontinuity does just that, *i.e.*, makes two material points out of one, or makes one out of two, respectively, ξ would not satisfy (2.2) in this case. Mathematically, (2.2) requires ξ to be one-to-one, or *injective*. Beyond impenetrability, we have the notion that a deformation of a classical material body simply changes the *shape* of the body, *i.e.*, no material is gained by, or lost from, the body in a deformation. The corresponding restriction on ξ is given by

$$\xi[P] = Q \tag{2.3}$$

such that ξ is onto, or *surjective*. Together, (2.2) and (2.3) require ξ to be one-to-one and onto, or *bijective*, *i.e.*, invertible. Finally, in the context of the theory of simple materials (*e.g.*, Noll, 1972), ξ is required to be continuously differentiable at each $p \in P$, *i.e.*,

$$(D_p\xi)\boldsymbol{v} := \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\xi(p + \epsilon\boldsymbol{v}) - \xi(p)\right] \in \mathcal{V} \quad \text{exists } \forall \boldsymbol{v} \in \mathcal{V} \text{ at each } p \in P \tag{2.4}$$

yielding the Fréchet derivative $(D_p\xi) \in \mathrm{Lbj}(\mathcal{V}, \mathcal{V})$ of ξ at each $p \in P$, *i.e.*, *the deformation gradient*. Together, (2.2)-(2.4) require ξ to be a C^1 diffeomorphism of P onto Q.

To obtain yet a further restriction on ξ, and in particular on $(D_p\xi) \in \mathrm{Lin}(\mathcal{V}, \mathcal{V})$, consider three linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathcal{V}$ which are positively-oriented (*i.e.*, $\omega_{\boldsymbol{G}}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) > 0$). These can be though of as being "attached" to $p \in P$ (in a way that will be made precise below). On the basis of this association, such vectors span a "linear" (infinitesimal) neighborhood of $p \in P$ with volume $\omega_{\boldsymbol{G}}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$. Now, such vectors are deformed into vectors $(D_p\xi)\boldsymbol{v}_1, (D_p\xi)\boldsymbol{v}_2, (D_p\xi)\boldsymbol{v}_3 \in \mathcal{V}$ by ξ. These vectors will analogously span a linear neighborhood of $\xi(p) \in Q$ if they are linearly independent, which is the case only if $(D_p\xi)$ is invertible, *i.e.*, only if $(D_p\xi) \in \mathrm{Lbj}(\mathcal{V}, \mathcal{V})$. In this case, the corresponding volume $\omega_{\boldsymbol{G}}((D_p\xi)\boldsymbol{v}_1, (D_p\xi)\boldsymbol{v}_2, (D_p\xi)\boldsymbol{v}_3)$, or equivalently, $\det_{\mathcal{V}}(D_p\xi)$ via (1.3), is non-zero. Furthermore, since volume as a physical concept only makes sense when it is represented as a *positive* quantity, it would seem reasonable to require $\omega_{\boldsymbol{G}}((D_p\xi)\boldsymbol{v}_1, (D_p\xi)\boldsymbol{v}_2, (D_p\xi)\boldsymbol{v}_3) > 0$, or equivalently $\det_{\mathcal{V}}(D_p\xi) > 0$, again via (1.3), for each $p \in P$, *i.e.*,

$$(D_p\xi) \in \mathrm{Lin}^+(\mathcal{V}, \mathcal{V}) \quad \text{at each } p \in P \tag{2.5}$$

Conditions (2.2) and (2.3) require ξ to be *bijective*, while (2.2)–(2.5) imply that ξ is an *orientation-preserving, C^1 diffeomorphism* of P onto Q.

Up to now, we have dealt primarily with the mathematical representation for the regions occupied by a classical material body in E, as well as that for a deformation between these, subject to certain physical requirements such as the impenetrability of matter. The material (body) itself, however, can also be represented mathematically, *i.e.*, as a set B which is connected with the regions it occupies in E via mappings

$$\kappa : B \longrightarrow B_\kappa \subset E \quad | \quad b \longmapsto p = \kappa(b) \tag{2.6}$$

called *placements* of B into E. The corresponding image $B_\kappa := \kappa[B]$ represents a region in E occupied by B via κ. In general, B_κ need not be classical, *i.e.*, a fit region of E. Equivalence classes of such placements induce all mathematical structure(s) of physical relevance on B from E. Conceptually speaking, note that such placements are not "coordinate charts" on B, but rather mappings between manifolds. To this author's knowledge, the term "placement" was first introduced by Noll (1972) for the concept which had been referred to previously as "configuration," which he restricted to equivalence classes of such placements inducing the same metric structure on B. Using an alternative approach, such equivalence classes can also be introduced locally, as shown in §4 below.

Let $\mathrm{Pla}(B, E)$ represent the set of all placements of B into E. Any two placements $\kappa, \gamma \in \mathrm{Pla}(B, E)$ are said to be (C^1) *compatible at* $b \in B$ if the induced mapping $\xi : B_\kappa \to B_\gamma$ (with $\gamma = \xi \circ \kappa$) between B_κ and B_γ is an orientation-preserving, C^1 diffeomorphism *of some neighborhood of* $\kappa(b) \in B_\kappa$ *onto one of* $\gamma(b) \in B_\gamma$. In particular, these neighborhoods could be fit regions of E. Compatibility at $b \in B$ induces an equivalence relation between elements of $\mathrm{Pla}(B, E)$ at $b \in B$ via the corresponding diffeomorphisms; let $[\kappa]_b \subset \mathrm{Pla}(B, E)$ represent the corresponding equivalence class with respect to $\kappa \in \mathrm{Pla}(B, E)$. A stronger form of compatibility between any two $\kappa, \gamma \in \mathrm{Pla}(B, E)$ arises when they are compatible at each $b \in U$ in some subset $U \subset B$ of B, something one could refer to as *compatibility in* $U \subset B$. In particular, if $U = B$, we could call this simply *local* compatibility. The strongest form of compatibility between any two $\kappa, \gamma \in \mathrm{Pla}(B, E)$ arises when the corresponding deformation $\xi : B_\kappa \to B_\gamma$ is an orientation-preserving, C^1 diffeomorphism of B_κ onto B_γ; in particular, this arises when B_κ and B_γ are fit regions of E, as discussed above. Such compatibility could be called *global* or *classical* compatibility. As with compatibility at a point, both compatibility in any subset of B, and global compatibility, induce corresponding equivalence relations between the elements of $\mathrm{Pla}(B, E)$. Clearly, all globally compatible placements are both compatible in any $U \subset B$ and compatible at each $b \in B$, while all placements compatible in any $U \subset B$ are also compatible at each $b \in U$, but the converses are in general not true. In this context, a *simple deformation*, as introduced by Del Piero and Owen (1993), connects two elements of $\kappa, \gamma \in \mathrm{Pla}(B, E)$ compatible in some *proper* subset $U \subset B$ of B when in addition $\kappa[U] \subset B_\kappa$ and $\gamma[U] \subset B_\gamma$ are piecewise fit regions of E.

Generally speaking, compatibility of placements is an issue relevant to the material body as a whole, having to do for example with whether or not the material body is *inhomogeneous* (*i.e.*, contains dislocations, cracks, glide systems, and so on; see Noll, 1967, or Del Piero and Owen, 1993). In what follows, however, we will be interested in the material behaviour of a single material point $b \in B$, and as such in the weakest form of compatibility between placements of B, *i.e.*, an equivalence class $[\kappa]_b$ of compatible placements of B at b.

3 Curves and Deformation at a Point in E

To investigate the notion of a classical deformation $\xi : P \to Q$ further, and in particular that of the deformation gradient, consider a curve at a point $p \in P$, *i.e.*, a C^1 map $c_p : I_0 \to P$ with $c_p(0) = p$ for $0 \in I_0 \subset \mathbb{R}$, with I_0 a time interval. Such a curve is deformed to one

$$c_{\xi(p)} := \xi \circ c_p : I_0 \to Q \tag{3.1}$$

at $\xi(p)$ in Q. The derivative of this last relation yields

$$\dot{c}_{\xi(p)}(0) = (D_p\xi)\, \dot{c}_p(0) \tag{3.2}$$

at $p = c_p(0)$, with $\dot{c}_p(0) := \lim_{s \to 0} \frac{1}{s}\,[c_p(s) - c_p(0)] \in \mathcal{V}$ the tangent vector to c_p at $p = c_p(0)$. On the basis of (3.2), one sees that the classical deformation gradient $(D_p\xi) \in \mathrm{Lin}^+(\mathcal{V}, \mathcal{V})$ at $p \in P$ represent the deformation of tangent vectors to curves there; in fact, to equivalence classes of curves, *i.e.*, two curves c_p and k_p at $p \in P$ are equivalent if $\dot{k}_p(0) = \dot{c}_p(0)$ hold. The corresponding equivalence classes $[c_p] := \{k_p \mid k_p \sim c_p\}$ all possess the same tangent vector $\boldsymbol{v} = \dot{c}_p(0) \in \mathcal{V}$. On this basis, we have the natural identification

$$[c_p] \quad \cong \quad (p, \boldsymbol{v}) \in \{p\} \times \mathcal{V} \tag{3.3}$$

at $p \in P$. Note that each equivalence class also induces a classical line element $\boldsymbol{dX} := \dot{c}_p(0)\, ds \in \mathcal{V}$ at $p \in P$ with respect to any one-dimensional line element $ds \in \mathbb{R}$. In affine spaces, the pair $\boldsymbol{v}_p := (p, \boldsymbol{v})$ is also refered to as a *tangent vector* (in the differential-geometric sense), and the set of all these is the *tangent space*

$$T_p E := \{p\} \times \mathcal{V} \tag{3.4}$$

to E at $p \in P$. Note that (3.3) implies that each $[c_p]$ is represented uniquely by an element $\boldsymbol{v}_p \in T_p E$ of $T_p E$. The definition (3.4) suggests the interpretation of $T_p E$ as $\mathcal{V}$ "attached to" $p \in E$. In this context, it is common to describe the elements of $\mathcal{V}$ as being "free" vectors, and those of $T_p E$ as being "bound" vectors. On this basis, we can rewrite (3.2) in the classical form

$$\boldsymbol{u} = (D_p \xi)\, \boldsymbol{v} \tag{3.5}$$

relative to $\boldsymbol{v} = \dot{c}_p(0)$ and $\boldsymbol{u} = \dot{c}_{\xi(p)}(0)$, or in the modern form

$$\boldsymbol{u}_{\xi(p)} = (T_p \xi)\, \boldsymbol{v}_p \tag{3.6}$$

relative to $\boldsymbol{v}_p := (p, \boldsymbol{v})$ and $\boldsymbol{u}_{\xi(p)} := (\xi(p), \boldsymbol{u})$, where the tangent linear form $(T_p \xi) \in \mathrm{Lin}^+\,(T_p E,\, T_{\xi(p)} E)$ of ξ at $p \in P$ is related to the corresponding Fréchet derivative $(D_p \xi) \in \mathrm{Lin}^+\,(\mathcal{V}, \mathcal{V})$ of ξ via the natural orientation-preserving linear map

$$\boldsymbol{\Lambda}_q \in \mathrm{Lin}^+(\mathcal{V}, T_q E) \quad \mid \quad \boldsymbol{a} \longmapsto \boldsymbol{a}_q = (q, \boldsymbol{a}) = \boldsymbol{\Lambda}_q \boldsymbol{a} \tag{3.7}$$

between $\mathcal{V}$ and $T_q E$ at each $q \in E$, *i.e.*,

$$\begin{tikzcd} T_p E \arrow[r, "T_p \xi"] \arrow[d, "\boldsymbol{\Lambda}_p^{-1}"'] & T_{\xi(p)} E \\ \mathcal{V} \arrow[r, "D_p \xi"'] \arrow[ur, phantom, "(T_p \xi) = \boldsymbol{\Lambda}_{\xi(p)} (D_p \xi)\, \boldsymbol{\Lambda}_p^{-1}"] & \mathcal{V} \arrow[u, "\boldsymbol{\Lambda}_{\xi(p)}"'] \end{tikzcd} \tag{3.8}$$

Note that (3.7) induces the *Euclidean parallelism*

$$\boldsymbol{\Lambda}_{pq} := \boldsymbol{\Lambda}_p \boldsymbol{\Lambda}_q^{-1} \in \mathrm{Lin}^+(T_q E, T_p E) \quad \mid \quad \boldsymbol{a}_q \longmapsto \boldsymbol{a}_p = \boldsymbol{\Lambda}_{pq} \boldsymbol{a}_q \tag{3.9}$$

for all $p, q \in E$, often refered to as the "shifter" (see, *e.g.*, Marsden and Hughes, 1983).

4 Body Element at a Point in B

The same consideration using curves at any point $b \in B$ of the body manifold can be carried out with the help of the placements $\kappa \in \mathrm{Pla}(B, E)$ of B. Because there exists no intrinsic differential structure on B analogous to that on E, however, this can be done only with the help of the placements of B into E. Indeed, two curves $c_b, k_b : I_0 \to B$ at $b \in B$ are equivalent (at b) if there exists a placement $\kappa \in \mathrm{Pla}(B, E)$ such that $c_{\kappa(b)} \sim k_{\kappa(b)}$ at $\kappa(b) \in B_\kappa$ as defined above, where $c_{\kappa(b)} := \kappa \circ c_b : I_0 \to B_\kappa$. In fact, if $c_{\kappa(b)} \sim k_{\kappa(b)}$ for $\kappa \in \mathrm{Pla}(B, E)$, then $c_{\gamma(b)} \sim k_{\gamma(b)}$ for all $\gamma \in [\kappa]_b$ via the definition of $[\kappa]_b$ and the chain rule. Indeed, with $\gamma = \xi \circ \kappa$, and so $c_{\gamma(b)} = \xi \circ c_{\kappa(b)}$, we have $\dot{c}_{\gamma(b)}(0) = (D_{\kappa(b)} \xi) \dot{c}_{\kappa(b)}(0)$, and so $\dot{c}_{\gamma(b)}(0) = \dot{k}_{\gamma(b)}(0) \Longrightarrow (D_{\kappa(b)} \xi) \dot{c}_{\kappa(b)}(0) = (D_{\kappa(b)} \xi) \dot{k}_{\kappa(b)}(0) \Longrightarrow \dot{c}_\kappa(0) = \dot{k}_{\kappa(b)}(0)$ since $(D_{\kappa(b)} \xi) \in \mathrm{Lin}^+(\mathcal{V}, \mathcal{V})$ is invertible. Since the equivalence relation is then independent of the choice of the element of $[\kappa]_b$, the corresponding equivalence class $[c_b]$ of curves at $b \in B$ depends only on $[\kappa]_b$. Since $[\kappa]_b$ is arbitrary, however, $[c_b]$ depends in fact only on the *existence* of such a class (which always does on a differentiable manifold). Consequently, we can "forget" which $[\kappa]_b$ we used to define it, and just work with $[c_b]$ as "given." By analogy with the Euclidean case above, then, we can then interpret each $[c_b]$ as a tangent vector $v_b = [c_b]$ at $b \in B$; in contrast to the Euclidean case, however, each such tangent vector does not split naturally into b plus a vector part. Indeed, this can be done only with respect to some $\gamma \in \mathrm{Pla}(B, E)$ (see below). The set of all such vectors, *i.e.*, equivalence classes of curves, at $b \in B$, is, analogous to the Euclidean case above, the tangent space $T_b B$ to B at $b \in B$, which Noll (1972) calls the *body element at* $b \in B$ (the tangent space at a point of a manifold can also be based on other kinds of equivalence classes; see, *e.g.*, Abraham *et al.*, 1988). Now, by definition of $[c_b]$, any $\gamma \in [\kappa]_b$ induces a mapping of $[c_b]$ to $[c_{\gamma(b)}]$, and so of $v_b \in T_b B$ to $\boldsymbol{v}_{\gamma(b)} \in T_{\gamma(b)} E$, *i.e.*,

$$(T_b \gamma) : T_b B \longrightarrow T_{\gamma(b)} E \quad | \quad v_b \longmapsto (\gamma(b), \boldsymbol{v}) = \boldsymbol{v}_{\gamma(b)} = (T_b \gamma) v_b \tag{4.1}$$

representing the so-called *tangent linear map* to γ at b. In a general manifold setting, it is this tangent map that represents the notion of derivative, and the one we would need to work with to formulate a generalized concept of "deformation gradient" in this case. Euclidean space, however, possesses much more (additional) structure than a normal manifold; in particular, we have the natural orientation-preserving, invertible linear mapping (3.7), which transforms the tangent map $(T_b \gamma)$ of any $\gamma \in [\kappa]_b$ into the *differential*

$$(d_b \gamma) := \boldsymbol{A}_{\gamma(b)}^{-1} (T_b \gamma) : T_b B \longrightarrow \mathcal{V} \quad | \quad v_b \longmapsto \boldsymbol{v} = (d_b \gamma) v_b \tag{4.2}$$

of γ at $b \in B$, mapping each v_b to its vector part $\boldsymbol{v} \in \mathcal{V}$ relative to γ. Since, via the Euclidean parallelism (3.9), this vector part can be "shifted" or "parallel-transported" to any other point in E, it is in fact this vector part, and not the corresponding tangent vector, that is central in a Euclidean space setting (something recognized a long time ago, *e.g.*, by Noll, 1972, 1973).

Now, for all $\gamma \in [\kappa]_b$, $(d_b \gamma) = (D_{\kappa(b)} \xi)(d_b \kappa)$ holds via $\gamma = \xi \circ \kappa$ and the chain rule, with $(D_{\kappa(b)} \xi) \in \mathrm{Lin}^+(\mathcal{V}, \mathcal{V})$. In particular, we could have $(D_{\kappa(b)} \xi) = \boldsymbol{I}$, *i.e.*, $(d_b \gamma) = (d_b \kappa)$, representing a second equivalence relation between γ and κ subordinate to the one defining $[\kappa]_b$. Let $\langle \kappa \rangle_b \subset [\kappa]_b$ represent the corresponding equivalence class. Any such $\langle \kappa \rangle_b$ induces the operations of vector addition $u_b + v_b := (d_b \kappa)^{-1}(\boldsymbol{u} + \boldsymbol{v})$ and scalar multiplication $a v_b := (d_b \kappa)^{-1}(a \boldsymbol{v})$ on $T_b B$ via the corresponding operations on $\mathcal{V}$, and so endows $T_b B$ with the structure of a three-dimensional linear (vector) space. As such, we can orient this space, and require that the differential $(d_b \gamma)$ of any $\gamma \in [\kappa]_b$ be an orientation-preserving, linear invertible mapping, *i.e.*, $(d_b \gamma) \in \mathrm{Lin}^+(T_b B, \mathcal{V})$. Analogous to placements $\kappa \in \mathrm{Pla}(B, E)$ of B into E, the elements of $\mathrm{Lin}^+(T_b B, \mathcal{V})$ can be thought of as placements of $T_b B$ into $\mathcal{V}$. By definition of $T_b B$ (*i.e.*, via any $[\kappa]_b$ and equivalent curves at b), as well as the equivalence classes $\langle \kappa \rangle_b$, *any* $\boldsymbol{K} \in \mathrm{Lin}^+(T_b B, \mathcal{V})$ is induced by some $\langle \kappa \rangle_b$ via $\boldsymbol{K} = (d_b \kappa)$.

Note that $T_b B$ inherits *only* the *linear* structure of $\mathcal{V}$ in the above fashion, *i.e.*, it does not inherit the inner product structure of $\mathcal{V}$ in this way. Unlike $\mathcal{V}$, then, $T_b B$ is *not* an inner product space, *i.e.*, there exists no special or "canonical" element of $\mathrm{Spd}(T_b B, T_b^* B)$ analogous to the Euclidean metric tensor $\boldsymbol{G}$ of $\mathcal{V}$. On the other hand, each $\boldsymbol{K} \in \mathrm{Lin}^+(T_b B, \mathcal{V})$ does induce a metric tensor

$$G_{\boldsymbol{K}} = \boldsymbol{K}^* \boldsymbol{G} \boldsymbol{K} \in \mathrm{Spd}(T_b B, T_b^* B) \tag{4.3}$$

and volume covector

$$\omega_{\boldsymbol{K}} \in \mathrm{Skw}_3((T_b B)^3, \mathbb{R}) \quad | \quad (v_1, v_2, v_3) \longmapsto \omega_{\boldsymbol{G}}(\boldsymbol{K} v_1, \boldsymbol{K} v_2, \boldsymbol{K} v_3) =: \omega_{\boldsymbol{K}}(v_1, v_2, v_3) \tag{4.4}$$

on $T_b B$. As discussed by Noll (1972, §3), elements of $\mathrm{Spd}(T_b B, T_b^* B)$ can be interpreted as *configurations* of $T_b B$. From the point of view of these configurations, two placements $\boldsymbol{J}, \boldsymbol{K} \in \mathrm{Lin}^+(T_b B, \mathcal{V})$ of $T_b B$ are equivalent if they induce the same configuration of $T_b B$, *i.e.*,

$$\boldsymbol{J} \sim \boldsymbol{K} \quad :\Longleftrightarrow \quad G_{\boldsymbol{J}} = G_{\boldsymbol{K}} \quad \Longleftrightarrow \quad \boldsymbol{J} \boldsymbol{K}^{-1} \in \mathrm{Orth}^+(\mathcal{V}, \mathcal{V}) \tag{4.5}$$

Each corresponding equivalence class

$$[\boldsymbol{K}] := \{ \boldsymbol{J} \in \mathrm{Lin}^+(T_b B, \mathcal{V}) \mid \boldsymbol{J} \sim \boldsymbol{K} \} \subset \mathrm{Lin}^+(T_b B, \mathcal{V}) \tag{4.6}$$

induces a *unique* configuration

$$G_{[\boldsymbol{K}]} := G_{\boldsymbol{J}} \qquad \forall \boldsymbol{J} \in [\boldsymbol{K}] \tag{4.7}$$

and volume covector

$$\omega_{[\boldsymbol{K}]} := \omega_{\boldsymbol{J}} \qquad \forall \boldsymbol{J} \in [\boldsymbol{K}] \tag{4.8}$$

of T_bB. The former holds by definition of the equivalence class $[\boldsymbol{K}]$, while the latter follows from the result $\omega_{\boldsymbol{J}} = \det_{\mathcal{V}}(\boldsymbol{J}\boldsymbol{K}^{-1})\,\omega_{\boldsymbol{K}}$ for all $\boldsymbol{J},\boldsymbol{K} \in \mathrm{Lin}^+(T_bB,\mathcal{V})$, and the fact that $\det_{\mathcal{V}}(\boldsymbol{J}\boldsymbol{K}^{-1}) = 1$, *i.e.*, $\boldsymbol{J}\boldsymbol{K}^{-1} \in \mathrm{Orth}^+(\mathcal{V},\mathcal{V})$, for all $\boldsymbol{J} \in [\boldsymbol{K}]$.

5 Simple Kinematics of a Material Point

A *classical* motion of B with respect to, or in, E, during the time interval $I \subset \mathbb{R}$, can be represented as a curve in $\mathrm{Pla}(B,E)$, *i.e.*,

$$\zeta : I \longrightarrow \mathrm{Pla}(B,E) \quad | \quad t \longmapsto \zeta_t := \zeta(t) \tag{5.1}$$

such that $\zeta_t[B] \subset E$ a fit region for all $t \in I$. With the help of the differential of ζ at each $t \in I$ (see (4.2)), the corresponding *dynamic* (*i.e.*, time-dependent) *deformation gradient at* $b \in B$ takes the form of a curve

$$\boldsymbol{F} : I \longrightarrow \mathrm{Lin}^+(T_bB,\mathcal{V}) \quad | \quad t \longmapsto (d_b\zeta_t) =: \boldsymbol{F}(t) \tag{5.2}$$

in $\mathrm{Lin}^+(T_bB,\mathcal{V})$. In the context of the interpretation of the element of $\mathrm{Lin}^+(T_bB,\mathcal{V})$ as placements of T_bB in $\mathcal{V}$, a comparison of (5.1) and (5.2) implies that $\boldsymbol{F}$ can be interpreted as a *motion* of T_bB in $\mathcal{V}$. The standard or usual form of the deformation gradient is in the current context one relative to some reference placement $\boldsymbol{K} \in \mathrm{Lin}^+(T_bB,\mathcal{V})$, *i.e.*,

$$\boldsymbol{F_K} : I \longrightarrow \mathrm{Lin}^+(\mathcal{V},\mathcal{V}) \quad | \quad t \longmapsto \boldsymbol{F}(t)\boldsymbol{K}^{-1} =: \boldsymbol{F_K}(t) \tag{5.3}$$

In contrast to this "referential" form of the dynamic deformation gradient, we might call (5.2) its "material" form, meaning that it is independent of any $\boldsymbol{K} \in \mathrm{Lin}^+(T_bB,\mathcal{V})$. In any case, it is this form that most clearly represents the "two-point" tensor character of the deformation gradient.

Beyond the various restrictions placed by the impenetrability of matter on the concept of deformation discussed in §2, there is one further with respect to $\boldsymbol{F_K} : I \to \mathrm{Lin}^+(\mathcal{V},\mathcal{V})$ worth mentioning here. Assume for the moment that $\det_{\mathcal{V}}(\boldsymbol{F_K}) : I \to \mathbb{R}$ starts out positive, but could become negative during the motion of the body. Since $\zeta : I \to \mathrm{Pla}(B,E)$ is assumed continuous, and $\det_{\mathcal{V}} : \mathrm{Lin}(\mathcal{V},\mathcal{V}) \to \mathbb{R}$ is continuous, the only way this could happen is if $\det_{\mathcal{V}}(\boldsymbol{F_K}(s)) = 0$ for some $s \in I$. As discussed in §2 with respect to the concept of deformation, this would mean in the current context that the volume of the linear neighborhood around $\zeta_t(b)$ would go to zero, in which case two distinct material points could become one during ζ, and so violate (2.2). Consequently, the impenetrability of matter in fact requires $\det_{\mathcal{V}}(\boldsymbol{F_K}) : I \to \mathbb{R}^+$, *i.e.*, $\boldsymbol{F_K} : I \to \mathrm{Lin}^+(\mathcal{V},\mathcal{V})$, in the "dynamic" case, consistent with the "static" case discussed in §2.

6 Simple Elastic Materials

In the theory of simple materials (Noll, 1972), one assumes the dependence of the material behaviour of any $b \in B$ on deformation is confined to the deformation of an infinitesimal or "linear" neighborhood of $b \in B$, *i.e.*, to that of T_bB, as represented by the deformation gradient (5.2). In this case, the dependent constitutive fields such as the Cauchy stress $\boldsymbol{T}(t,\zeta_t(b)) \in \mathrm{Sym}(\mathcal{V},\mathcal{V})$ (for non-polar materials) associated with the motion of $b \in B$ are assumed to depend in general only on the *histories* of $\boldsymbol{F}$ and temperature at $b \in B$. In the simplest case, *i.e.*, that of purely *thermoelastic* behaviour, this reduces to a dependence on the *current* values of these alone, *i.e.*,

$$\boldsymbol{T} = \mathcal{E}_{\boldsymbol{K}}(\boldsymbol{F_K}) \tag{6.1}$$

with respect to some local placement $\boldsymbol{K} \in \mathrm{Lin}^+(T_bB,\mathcal{V})$ of T_bB, where $\mathcal{E}$ stands for "elastic." As done in this last relation, we leave temperature out of the notation in what follows for simplicity.

To insure that constitutive relations such as $\mathcal{E}_{\boldsymbol{K}}$, and so the material behaviour, do not depend on the choice of (Euclidean) observer, they are subject to the requirement of material frame-indifference (*e.g.*, Truesdell and Noll, 1965; Wang and Truesdell, 1973; Marsden and Hughes, 1983; Truesdell, 1993). In particular, this requirement reduces $\mathcal{E}_{\boldsymbol{K}}$ to the form

$$\mathcal{E}_{\boldsymbol{K}}(\boldsymbol{F_K}) = \boldsymbol{F_K}\,\mathcal{R}_{\boldsymbol{K}}(\boldsymbol{C_K})\,\boldsymbol{F_K}^{\mathrm{T}} \tag{6.2}$$

(*e.g.*, Truesdell and Noll, 1965), where

$$\boldsymbol{C_K} := \boldsymbol{F_K}^{\mathrm{T}}\boldsymbol{F_K} : I \longrightarrow \mathrm{Spd}(\mathcal{V},\mathcal{V}) \tag{6.3}$$

represents the right or referential Cauchy-Green tensor relative to $\boldsymbol{K}$, and $\mathcal{R}$ stands for "reduced." On the basis of (6.2), material frame-indifference clearly restricts the general dependence of $\mathcal{E}_{\boldsymbol{K}}$ on $\boldsymbol{F}_{\boldsymbol{K}}$ to a particular form in which the constitutive function $\mathcal{R}_{\boldsymbol{K}}$ of $\boldsymbol{C}_{\boldsymbol{K}}$ determines the material behaviour.

With the help of the relation $\boldsymbol{F}_{\boldsymbol{K}}^{\mathrm{T}} = \boldsymbol{G}^{-1}\boldsymbol{F}_{\boldsymbol{K}}^{*}\boldsymbol{G}$ from (1.8), we obtain the form

$$\boldsymbol{C}_{\boldsymbol{K}} = \boldsymbol{G}^{-1}\boldsymbol{F}_{\boldsymbol{K}}^{*}\boldsymbol{G}\boldsymbol{F}_{\boldsymbol{K}} = \boldsymbol{G}^{-1}\boldsymbol{K}^{-*}\boldsymbol{F}^{*}\,\boldsymbol{G}\boldsymbol{F}\boldsymbol{K}^{-1} = \boldsymbol{K}\,G_{[\boldsymbol{K}]}^{-1}G_{\boldsymbol{F}}\boldsymbol{K}^{-1} \tag{6.4}$$

for $\boldsymbol{C}_{\boldsymbol{K}}$ from (4.3), (4.7), (5.3) and (6.3), where

$$G_{\boldsymbol{F}} := \boldsymbol{F}^{*}\boldsymbol{G}\boldsymbol{F} : I \longrightarrow \mathrm{Spd}(T_bB, T_b^{*}B) \tag{6.5}$$

represents the "deformation process" determined by $\boldsymbol{F}$ (in the terminology of Noll, 1972, §5). Since $G^{-1}G_{\boldsymbol{F}}\colon I \to \mathrm{Spd}_G(T_bB, T_bB)$ for any $G \in \mathrm{Spd}(T_bB, T_b^{*}B)$, $G_{[\boldsymbol{K}]}^{-1}G_{\boldsymbol{F}}$ is directly analogous to the right or referential Cauchy-Green tensor $\boldsymbol{C}_{\boldsymbol{K}}\colon I \to \mathrm{Spd}(\mathcal{V},\mathcal{V})$. Using the form (6.4) of $\boldsymbol{C}_{\boldsymbol{K}}$, (6.2) becomes

$$\mathcal{E}_{\boldsymbol{K}}(\boldsymbol{F}_{\boldsymbol{K}})\boldsymbol{G}^{-1} = \boldsymbol{F}[\boldsymbol{K}^{-1}\,\mathcal{R}_{\boldsymbol{K}}(\boldsymbol{K}\;G_{[\boldsymbol{K}]}^{-1}G_{\boldsymbol{F}}\boldsymbol{K}^{-1})\,\boldsymbol{K}\;G_{[\boldsymbol{K}]}^{-1}]\boldsymbol{F}^{*} = \boldsymbol{F}[\mathcal{R}(G_{[\boldsymbol{K}]}^{-1}G_{\boldsymbol{F}})\;G_{[\boldsymbol{K}]}^{-1}]\boldsymbol{F}^{*} \tag{6.6}$$

via (1.8), (4.7), (5.3), (6.2), (6.4), and the "material" or "intrinsic" form $\mathcal{R}$ of $\mathcal{R}_{\boldsymbol{K}}$, where

$$\mathcal{R}_{\boldsymbol{J}}(\boldsymbol{H}) = \boldsymbol{J}\,\mathcal{R}(\boldsymbol{J}^{-1}\boldsymbol{H}\boldsymbol{J})\,\boldsymbol{J}^{-1} \tag{6.7}$$

holds for all $\boldsymbol{H} \in \mathrm{Lin}(\mathcal{V},\mathcal{V})$ and all referential placements $\boldsymbol{J} \in \mathrm{Lin}^{+}(T_bB, \mathcal{V})$. In the terminology of Noll (1972), an "intrinsic" quantity is one independent of the "frame space" $\mathcal{V}$. Since $[\boldsymbol{K}]$, and so $G_{[\boldsymbol{K}]}$, is arbitrary, we may replace $G_{[\boldsymbol{K}]}$ by an arbitrary reference configuration G of T_bB on the right-hand side of (6.6), obtaining finally the reduced form

$$\boldsymbol{T}\boldsymbol{G}^{-1} = \mathcal{E}_{\boldsymbol{K}}(\boldsymbol{F}_{\boldsymbol{K}})\,\boldsymbol{G}^{-1} = \mathcal{E}_G(\boldsymbol{F}) \tag{6.8}$$

of the elastic constitutive relative (6.1) relative to $G \in \mathrm{Spd}(T_bB, T_b^{*}B)$, with

$$\mathcal{E}_G(\boldsymbol{J}) := \boldsymbol{J}\,\mathcal{M}_G(G_{\boldsymbol{J}})\,\boldsymbol{J}^{*} \tag{6.9}$$

and

$$\mathcal{M}_G(G_{\boldsymbol{J}}) := \mathcal{R}(G^{-1}G_{\boldsymbol{J}})\,G^{-1} \tag{6.10}$$

for all $\boldsymbol{J} \in \mathrm{Lin}^{+}(T_bB, \mathcal{V})$, where $\mathcal{M}$ stands for "material." Since, from a physical point of view, we have merely "rewritten" (6.2) in the material form (6.8) via (6.4)–(6.7), (6.9) and (6.10), the reduced "elastic" *form* (6.8) for $\boldsymbol{T}\boldsymbol{G}^{-1}$, as well the dependence of the *constitutive part* $\mathcal{M}_G$ of this on a (time-dependent) *metric*, *i.e.*, on $G_{\boldsymbol{F}}$, follow directly from the requirement of material frame indifference. In addition, note that the dependence of the reduced constitutive form $\mathcal{R}_{\boldsymbol{K}}$ on an arbitrary local reference placement $\boldsymbol{K} \in \mathrm{Lin}^{+}(T_bB, \mathcal{V})$ of T_bB is transformed to one on an arbitrary reference configuration $G \in \mathrm{Spd}(T_bB, T_b^{*}B)$ of T_bB in the material case. Lastly, note that (6.8) can be written in the purely material form

$$S_{\boldsymbol{F}} = \mathcal{M}_G(G_{\boldsymbol{F}}) \tag{6.11}$$

via (6.9), where

$$S_{\boldsymbol{J}} := \boldsymbol{J}^{-1}(\boldsymbol{T}\boldsymbol{G}^{-1})\boldsymbol{J}^{-*} \in \mathrm{Sym}(T_b^{*}B, T_bB) \tag{6.12}$$

represents the material or intrinsic stress tensor of Noll (1972, §6) with respect to $\boldsymbol{J} \in \mathrm{Lin}^{+}(T_bB, \mathcal{V})$.

In many materials (*e.g.*, metals, alloys, fiber-reinforced materials), one often observes that the material response depends on the direction in which the material is loaded; in this case, one says that the material behaves *anisotropically*. On the other hand, if the material response is independent of direction, the material is said to behave *isotropically*. If anisotropic material behaviour is present, it must be accounted for in the *form* of the constitutive relations in question, *e.g.*, in the form of $\mathcal{M}_G$. Abstractly, changes of (material) direction are effected by by elements of the set $\mathrm{Lin}^{+}(T_bB, T_bB)$ of all orientation-preserving, linear invertible transformations or mappings of T_bB onto itself, representing the connected subgroup of the general linear group $\mathrm{Lbj}(T_bB, T_bB)$ on T_bB. In particular, all changes of direction $H \in \mathrm{Lin}^{+}(T_bB, T_bB)$ leaving the *form* of $\mathcal{M}_G$, *i.e.*, the material behaviour, unchanged, belong to the *material symmetry group*

$$\mathfrak{G}(\mathcal{M}_G) := \{H \in \mathrm{Lin}^{+}(T_bB, T_bB) \mid a_H\mathcal{M}_G = \mathcal{M}_G\} \tag{6.13}$$

of $\mathcal{M}_G$, where

$$(a_H \mathcal{M}_G)(G_J) := H\,\mathcal{M}_G(H^* G_J H)\,H^* \tag{6.14}$$

represents the *action* of $H \in \mathrm{Lin}^+(T_b B, T_b B)$ (*i.e.*, of $\mathrm{Lin}^+(T_b B, T_b B)$) on $\mathcal{M}_G$ for all $\boldsymbol{J} \in \mathrm{Lin}^+(T_b B, \mathcal{V})$. Classically, a material behaves as a *simple elastic fluid* if $\mathfrak{G}(\mathcal{M}_G) = \mathrm{Uni}^+(T_b B, T_b B)$ for some, and hence any, G, as a *simple anisotropic elastic solid with respect to* G if there exists a G such that $\mathfrak{G}(\mathcal{M}_G)$ is a proper subgroup of $\mathrm{Orth}_G^+(T_b B, T_b B)$, and as a *simple isotropic elastic solid with respect to* G if there exists a G such that $\mathfrak{G}(\mathcal{M}_G) = \mathrm{Orth}_G^+(T_b B, T_b B)$. In this latter case, for example, $\mathcal{M}_G$ takes the simple form

$$\mathcal{M}_G(G_{\boldsymbol{F}}) = c_1\, G^{-1} + c_2\,(G^{-1} G_{\boldsymbol{F}})\, G^{-1} + c_3\,(G^{-1} G_{\boldsymbol{F}})^2 G^{-1} \tag{6.15}$$

where $c_{1,2,3}$ are all isotropic functions of $G^{-1} G_{\boldsymbol{F}}$, *i.e.*, of the invariants of $G^{-1} G_{\boldsymbol{F}}$. With $G = G_{[\boldsymbol{K}]}$ for some local placement $\boldsymbol{K}$, the material form (6.15) of an isotropic elastic constitutive relation can be transformed into the equivalent referential (*i.e.*, classical) form

$$\mathcal{R}_{\boldsymbol{K}}(\boldsymbol{C}_{\boldsymbol{K}}) = c_1\,\boldsymbol{I} + c_2\,\boldsymbol{C}_{\boldsymbol{K}} + c_3\,\boldsymbol{C}_{\boldsymbol{K}}^2 \tag{6.16}$$

via (6.7), (6.10) and (6.15) relative to $\boldsymbol{C}_{\boldsymbol{K}}$, where now the coefficients $c_{1,2,3}$ are all isotropic functions of $\boldsymbol{C}_{\boldsymbol{K}}$, *i.e.*, of the invariants of $\boldsymbol{C}_{\boldsymbol{K}}$. Substituting the referential form (6.16) into (6.2) yields the equivalent spatial form

$$\mathcal{E}_{\boldsymbol{K}}(\boldsymbol{F}_{\boldsymbol{K}}) = [c_1\,\boldsymbol{I} + c_2\,\boldsymbol{B}_{\boldsymbol{K}} + c_3\,\boldsymbol{B}_{\boldsymbol{K}}^2]\boldsymbol{B}_{\boldsymbol{K}} \tag{6.17}$$

of the isotropic elastic constitutive relation with respect to the spatial or left Cauchy-Green tensor

$$\boldsymbol{B}_{\boldsymbol{K}} := \boldsymbol{F}_{\boldsymbol{K}} \boldsymbol{F}_{\boldsymbol{K}}^{\mathrm{T}} : I \longrightarrow \mathrm{Spd}(\mathcal{V}, \mathcal{V}) \tag{6.18}$$

relative to $\boldsymbol{K}$ (actually, both $\boldsymbol{C}_{\boldsymbol{K}}$ and $\boldsymbol{B}_{\boldsymbol{K}}$ depend only on $[\boldsymbol{K}]$). Since the invariants of $G^{-1} G_{\boldsymbol{F}}$, $\boldsymbol{C}_{\boldsymbol{K}}$ and $\boldsymbol{B}_{\boldsymbol{K}}$ are the same, the coefficients $c_{1,2,3}$ remain unchanged. As such, (6.15)–(6.17) are simply different *representations* of the same constitutive relation, *i.e.*, the same material behaviour.

If anisotropic material behaviour is present, one can represent it explicitly in the constitutive relation with the help of so-called "structure" tensors (*e.g.*, Boehler, 1978; Liu, 1982; Boehler, 1987; Zhang and Ryschlewski, 1990; Svendsen, 1994), *i.e.*,

$$\mathcal{M}_G(G_{\boldsymbol{F}}) = \mathcal{S}_G(G_{\boldsymbol{F}}, \mathfrak{S}) \tag{6.19}$$

where $\mathfrak{S}$ is a material tensor (*i.e.*, one on $T_b B$), or a set of such tensors, representing the anisotropy involved, and $\mathcal{S}_G$ is an *isotropic* function of its arguments with respect to G. For example, in the case of transverse isotropy, the material response is isotropic in a given plane of the material, and otherwise anisotropic. The corresponding structure tensor is given by the perpendicular to this plane, *i.e.*, its orientation, represented mathematically by the tensor product of the unit normal $G n \in T_b^* B$ (with respect to G) to this plane at b with $n \in T_b B$, representing an element $N := n \otimes (G n) \in \mathrm{Uni}^+(T_b B, T_b B)$ of $\mathrm{Uni}^+(T_b B, T_b B)$, *i.e.*, $\mathrm{tr}_{T_b B}(N) = (G n) n = 1$. As such, (6.19) takes the form

$$\begin{aligned} \mathcal{S}_G(G_{\boldsymbol{F}}, N) &= c_1\, G^{-1} + c_2\,(G^{-1} G_{\boldsymbol{F}})\, G^{-1} + c_3\,(G^{-1} G_{\boldsymbol{F}})^2 G^{-1} \\ &+ c_4\, N G^{-1} \\ &+ c_5\,[(G^{-1} G_{\boldsymbol{F}}) N + N (G^{-1} G_{\boldsymbol{F}})]\, G^{-1} \\ &+ c_6\,[(G^{-1} G_{\boldsymbol{F}})^2 N + N (G^{-1} G_{\boldsymbol{F}})^2]\, G^{-1} \end{aligned} \tag{6.20}$$

where the coefficients c_{1-6} are now isotropic functions of $G^{-1} G_{\boldsymbol{F}}$ and N.

7 Inelastic Simple Materials and Elastic Material Isomorphism

Additional variables appear in constitutive relations such as (6.11) when the material response is affected by processes other than elastic deformation and temperature changes, *e.g.*, inelastic processes, or damage. Many such processes can be described by so-called "internal variables," which, being "internal," represent *material* fields or quantities. Consider for example the case of inelastic processes. Here, the internal variable in question at $b \in B$ is the *inelastic transformation* $P : I \rightarrow \mathrm{Lin}^+(T_b B, T_b B)$. When inelastic processes occur, the material response, and so the constitutive fields in question, such as the Cauchy stress T, could depend *in general* on the total deformation of the material (as usual) *as well as* the inelastic processes, *i.e.*,

$$TG^{-1} = \mathcal{C}_G(F, P) \tag{7.1}$$

Analogous to the elastic case (6.8)-(6.9), the requirement of material frame-indifference reduces $\mathcal{C}_G$ to the form

$$\mathcal{C}_G(F, P) = F\,\mathcal{I}_G(G_F, P)\,F^* \tag{7.2}$$

where the constitutive part $\mathcal{I}_G$ of $\mathcal{C}_G$ is analogous to $\mathcal{M}_G$ in the elastic case. A further restriction on the form of $\mathcal{I}_G$ arises on the basis of observations, which imply that inelastic processes do not markedly influence the purely *elastic* behaviour of the material, meaning here that P does not alter the *form* of the material elastic constitutive relation $\mathcal{M}_G$. This will in fact be the case when $\mathcal{I}_G$ takes the special form

$$\mathcal{I}_G(G_F, P) = (a_{P^{-1}}\mathcal{M}_G)(G_F) = P^{-1}\,\mathcal{M}_G(P^{-*}G_F P^{-1})\,P^{-*} \tag{7.3}$$

via (6.14), in which $\mathcal{I}_G$ depends on P only the *action* of its inverse on $\mathcal{M}_G$, which indeed preserves the form of $\mathcal{M}_G$ (this action is expressed relative to P^{-1} rather than P merely in order to simplify comparison of the current formulation with the standard one in what follows). In this case, P represents a *material isomorphism of* $\mathcal{M}_G$ (*e.g.*, Noll, 1972; Bertram and Kraska, 1995). In terms of the "elastic part"

$$E := FP^{-1} : I \longrightarrow \mathrm{Lin}^+(T_bB, \mathcal{V}) \tag{7.4}$$

of F, (7.2) and (7.3) take the simpler forms

$$\mathcal{C}_G(F, P) = E\,\mathcal{M}_G(G_E)\,E^* = \mathcal{E}_G(E) \tag{7.5}$$

and

$$\mathcal{I}_G(G_F, P) = (a_{P^{-1}}\mathcal{M}_G)(G_F) = P^{-1}\,\mathcal{M}_G(G_E)\,P^{-*} \tag{7.6}$$

respectively, via (6.10) and (6.5), respectively, with $G_E := E^*GE$ the deformation process induced by E. Although the interpretation of E as the elastic part of F can also be motivated on micromechanical grounds (*e.g.*, in crystal plasticity), it is the result $(7.5)_2$ that motivates this interpretation of E in the current continuum mechanical context.

From the definition (6.13) of $\mathfrak{G}(\mathcal{M}_G)$, as well as (6.14), we see from (7.3) that, when P is in fact an elastic material isomorphism, it is not unique with respect to $\mathfrak{G}(\mathcal{M}_G)$, *i.e.*, $a_{(HP)^{-1}}\mathcal{M}_G = a_{P^{-1}}\mathcal{M}_G$ for all $H \in \mathfrak{G}(\mathcal{M}_G)$. This non-uniqueness is usually expressed with respect to $E = FP^{-1}$, *i.e.*, $F = EP = (EH^{-1})(HP)$ for all $H \in \mathrm{Lin}^+(T_bB, T_bB)$, and so for all $H \in \mathfrak{G}(\mathcal{M}_G)$. In particular, if the material is isotropic, *i.e.*, if $\mathfrak{G}(\mathcal{M}_G) = \mathrm{Orth}^+_G(T_bB, T_bB)$, then (7.3) actually depends only on the symmetric part U of P, *i.e.*,

$$\mathcal{I}_G(G_F, P) = (a_{U^{-1}}\mathcal{M}_G)(G_F) = U^{-1}\,\mathcal{M}_G(U^{-*}G_F U^{-1})\,U^{-*} \tag{7.7}$$

via the polar decomposition $P = RU$ of P with respect to G, where $R : I \to \mathrm{Orth}^+_G(T_bB,\ T_bB)$ and $U : I \to \mathrm{Spd}_G(T_bB, T_bB)$.

The inelastic form $F = EP$ for F from (7.4) is reminiscent of the standard elastoplastic multiplicative decomposition $F_K = F^{\mathrm{E}}_K F^{\mathrm{P}}_K$ of the classic deformation gradient F_K (*e.g.*, Lee and Liu, 1967) relative to $K \in \mathrm{Lin}^+(T_bB, \mathcal{V})$. In fact, the relation between F_K and F given in (5.3), as well as the definition (7.4) of E, imply the correspondences $F^{\mathrm{E}}_K = E_K = EK^{-1}$ and $F^{\mathrm{P}}_K = P_K = KPK^{-1}$.

As usual, the material model is completed by the specification of an evolution constitutive relation for P (or just its symmetric part U in the case of elastic isotropy), *i.e.*, a so-called flow rule, which takes the general form

$$\dot{P} = \mathcal{F}(P, G_F, \dot{G}_F) \tag{7.8}$$

As usual, in the rate-independent, *i.e.*, elastoplastic, case, $\mathcal{F}$ is a homogeneous function of $\dot{G}_F$, and independent of $\dot{G}_F$ in the rate-dependent, *i.e.*, elastoviscoplastic, case. For an example of such a material or intrinsic flow rule appropriate for crystal plasticity, see Bertram and Kraska (1995).

Finally, we note that not all cases of "internal variables" involve material isomorphisms. For example, in contrast to the case of inelastic deformation, processes such as damage or cracking are observed to influence the elastic behaviour of the material, and so could not in general be described by such material isomorphisms.

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