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Complementary Formulation of the Appell Equation

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Appell equations can be expressed in two ways. The conventional one uses accelerations, i.e. second displacement derivatives, and mass, combined into an Appell function A which is sometimes called „acceleration energy". The alternative way uses force derivatives, i.e. second impulse derivatives, and compliances, combined into an alternative Appell function A^*, which might be called a „force derivative energy". The present paper is devoted to an introduction of the alternative formulation.

1 Introduction

In the conventional formulation the Appell function („acceleration energy" in J/s^2) is

$$A = \frac{1}{2}\int \ddot{\mathbf{r}}^2 dm \tag{1}$$

With generalized coordinates q, the Appell function (1) satisfies the Gaussian principle

$$\left(\frac{\partial A}{\partial \ddot{q}_i} - \Pi_i\right)\delta \ddot{q}_i = 0 \tag{2}$$

Since the $\delta \ddot{q}_i$ are independent and do not vanish, we obtain for the conventional Appell equations of motion

$$\frac{\partial A}{\partial \ddot{q}_i} - \Pi_i = 0 \tag{3}$$

where Π_i is a generalized force (Fischer and Stephan, 1972). In the conventional formulation, the system's kinematic constraints (compatibility equations) must be satisfied *before* the Appell function (1) can be established. The result, i.e. Appell's equations (3) of motion, represents a sum of forces which vanishes (i. e. dynamic equilibrium).

In the subsequent derivation, it will be shown that an alternative formulation exists, with an alternative Appell function („force derivative energy" in J/s^2) of

$$A^* = -\frac{1}{2}\int \ddot{\mathbf{I}}^2 dc \tag{4}$$

where $\mathbf{I}$ = impulse and c = compliance. With generalized impulses S (Tabarrok and Rimrott, 1994), the alternative Appell function satisfies the complementary Gaussian principle

$$\left(\frac{\partial A^*}{\partial \ddot{S}_j} - s_j\right)\delta \ddot{S}_j = 0 \tag{5}$$

Since the $\delta\ddot{S}_j$ are independent and do not vanish, we obtain for the alternative Appell equations of motion

$$\frac{\partial A^*}{\partial \ddot{S}_j} - s_j = 0 \tag{6}$$

where s_j is a generalized speed. In the alternative formulation, the system's force constraints (dynamic equilibrium equations) must be satisfied *before* the alternative Appell function (4) can be established. The result, i.e. Appell's equation (6) of motion represents a sum of speeds which vanishes (i.e. compatibility).

2 Derivation of Alternative Appell Equation

In order to obtain the alternative formulation of the Appell equation we begin with the D'Alembert equation for a mass element dm

$$\left(d\dot{\mathbf{B}} - d\mathbf{F}\right) \cdot \delta\mathbf{r} = 0 \tag{7}$$

where $d\dot{\mathbf{B}} = \ddot{\mathbf{r}}dm$.

An element of work of a force $\mathbf{F}$ can be written

$$dW = \mathbf{F}\cdot d\mathbf{r} = \frac{d\mathbf{I}}{dt}\cdot d\mathbf{r} = \frac{d\mathbf{r}}{dt}\cdot d\mathbf{I} = \dot{\mathbf{r}}\cdot d\mathbf{I} = \dot{\mathbf{u}}\cdot d\mathbf{I} \tag{8}$$

where we have chosen $\dot{\mathbf{u}}$ to identify the *extension velocity* of a force element. An element of work of the inertial force $\dot{\mathbf{B}}$ can be written

$$dW = \dot{\mathbf{B}}\cdot d\mathbf{r} = \frac{d\mathbf{B}}{dt}\cdot d\mathbf{r} = \frac{d\mathbf{r}}{dt}\cdot d\mathbf{B} = \dot{\mathbf{r}}\cdot d\mathbf{B} \tag{9}$$

Entering equations (8) and (9) into equation (7) one obtains

$$d\dot{\mathbf{r}} \cdot \delta\,\mathbf{B} - d\dot{\mathbf{u}} \cdot \delta\mathbf{I} = 0 \tag{10}$$

By making use of Newton's second law, the linear momentum $\mathbf{B}$ can be expressed in terms of the linear impulses $\mathbf{I}$, such that

$$\dot{\mathbf{r}}_k \cdot \mathbf{B}_k = \mathbf{v}_l \cdot \mathbf{I}_l \tag{11}$$

In e.g. an oscillator chain consisting of k point masses and l springs, $\dot{\mathbf{r}}$ represents the velocity of a mass, while $\mathbf{v}$ represents the velocity impressed upon a spring by the adjacent masses (see e.g. Rimrott and Tabarrok, 1993). With the help of equation (11) we can now write equation (10) as

$$(d\mathbf{v} - d\dot{\mathbf{u}})\cdot\delta\mathbf{I} = 0 \tag{12}$$

for a mass element, and

$$\int (d\mathbf{v} - d\dot{\mathbf{u}})\cdot\delta\mathbf{I} = 0 \tag{13}$$

for a mechanical system.

Now we follow Tabarrok and Rimrott (1994) and introduce the alternative form of Gauss' principle, which for the present situation would appear as

$$\int (d\mathbf{v} - d\dot{\mathbf{u}})\cdot\delta\,\ddot{\mathbf{I}} \;=\; 0 \tag{14}$$

The quantity $\dot{\mathbf{u}}$ is the extension velocity of a force element (e.g. a spring). In general *extension* = - *force* × *compliance*, or

$$\begin{aligned} \mathbf{u} &= -\dot{\mathbf{I}}c \\ \dot{\mathbf{u}} &= -\ddot{\mathbf{I}}c \\ d\dot{\mathbf{u}} &= -\ddot{\mathbf{I}}dc \end{aligned} \tag{15}$$

We now introduce the alternative Appell function (4)

$$A^* \;=\; -\frac{1}{2}\int \ddot{\mathbf{I}}^2 dc$$

Furthermore we introduce generalized impulses S and the time t, such that

$$A^* = A^*\left(S_1,\, S_2,\, \ldots,\, S_j,\, \ldots,\, S_m;\, \dot{S}_1,\, \dot{S}_2,\, \ldots,\, \dot{S}_j,\, \ldots,\, \dot{S}_m;\, \ddot{S}_1,\, \ddot{S}_2,\, \ldots,\, \ddot{S}_j, \ldots, \ddot{S}_m;\, t\right) \tag{16}$$

Then we have for the Gaussian variation

$$\delta A^* \;=\; \frac{\partial A^*}{\partial \ddot{S}_j}\delta\,\ddot{S}_j \tag{17}$$

For the second term in equation (14) we write

$$\int d\dot{\mathbf{u}}\cdot\delta\,\ddot{\mathbf{I}} = -\int \ddot{\mathbf{I}}dc\cdot\delta\,\ddot{\mathbf{I}} \;=\; -\delta\left(\frac{1}{2}\int \ddot{\mathbf{I}}^2\, dc\right) \;=\; \delta\,A^* \;=\; \frac{\partial A^*}{\partial \ddot{S}_j}\delta\,\ddot{S}_j \tag{18}$$

Furthermore we let the first term of equation (14) be

$$\int d\mathbf{v}\,\cdot\,\delta\ddot{\mathbf{I}} \;=\; s_j\delta\ddot{S}_j \tag{19}$$

where s_j is a generalized speed.

Equation (14) can now be written

$$\left(s_j - \frac{\partial A^*}{\partial \ddot{S}_j}\right)\delta\ddot{S}_j \;=\; 0 \tag{20}$$

The variations $\delta\,\ddot{S}_j$ are independent and do not vanish, hence

$$s_j \;-\; \frac{\partial A^*}{\partial \ddot{S}_j} \;=\; 0 \tag{21}$$

represents the alternative Appell equations.

3 Examples

Example 1, Conventional Formulation

The Appell function (1) for the three-mass-two-spring system (Rimrott and Tabarrok, 1995) of Figure 1 is

$$A = \frac{1}{2}\int \ddot{\mathbf{r}}^2 dm = \frac{1}{2}m_i\ddot{q}_i^2 = \frac{1}{2}m_a\ddot{q}_a^2 + \frac{1}{2}m_b\ddot{q}_b^2 + \frac{1}{2}m_c\ddot{q}_c^2 \qquad (22)$$

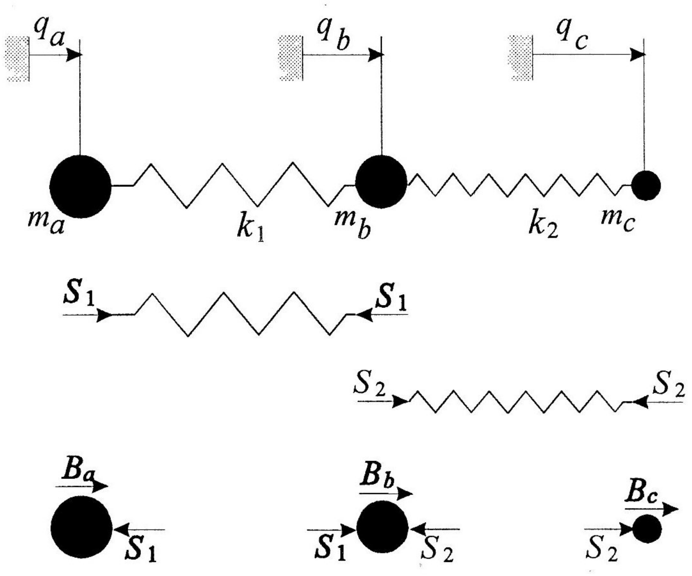

Figure 1. An Oscillator Chain

The compatibility (displacement fit) equations, which must be satisfied before the equations of motion can be established involve the extensions e of the springs and the displacements q of the masses.

$$e_1 = q_b - q_a \qquad (23a)$$

$$e_2 = q_c - q_b \qquad (23b)$$

The generalized forces Π impressed upon the masses m_a, m_b and m_c by the springs are

$$\Pi_a = -F_1 = +k_1e_1 = +k_1(q_b - q_a) \qquad (24a)$$

$$\Pi_b = F_1 - F_2 = -k_1e_1 + k_2e_2 = -k_1(q_b - q_a) + k_2(q_c - q_b) \qquad (24b)$$

$$\Pi_c = F_2 = -k_2e_2 = -k_2(q_c - q_b) \qquad (24c)$$

The Appell equations (3) of motion are

$$\frac{\partial A}{\partial \ddot{q}_i} - \Pi_i = 0$$

With $i = a$, we obtain

$$m\ddot{q}_a - k_1(q_b - q_a) = 0 \qquad (25a)$$

With $i = b$, we obtain

$$m\ddot{q}_b + k_1(q_b - q_a) - k_2(q_c - q_b) = 0 \quad (25b)$$

And with $i = c$, we obtain

$$m\ddot{q}_c + k_2(q_c - q_b) = 0 \quad (25c)$$

Written in matrix form

$$\begin{bmatrix} m_a & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & m_c \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_b \\ \ddot{q}_c \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} q_a \\ q_b \\ q_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (26)$$

The first matrix is the *mass matrix*, the second is the *stiffness matrix*. The system has 3 degrees of freedom, represented by the three variables q_a, q_b and q_c. It also has 3 equilibrium (force fit) equations (26). It has 2 compatibility (displacement fit) equations (23). Displacement fit equations and force fit equations together amount to 5 for the present problem, corresponding to the 5 elements, i.e. 3 masses and 2 springs.

Example 1, Alternative Formulation

The alternative Appell function (4) for the system of Figure 1 is

$$A^* = -\frac{1}{2}\int \ddot{\mathbf{I}}^2 dc = -\frac{1}{2}\frac{\ddot{S}_1^2}{k_1} - \frac{1}{2}\frac{\ddot{S}_2^2}{k_2} \quad (27)$$

since $I_1 = S_1$ and $I_2 = S_2$; i.e. the impulses I and the generalized impulses S are identical in the present problem. For the compliances we have $c_1 = \frac{1}{k_1}$ and $c_2 = \frac{1}{k_2}$.

The linear momenta B of the masses and the linear impulses S of the springs are related by integrals of Newton's second law (without integration constants). These dynamic equilibrium relationships must be satified before the equations of motion can be established.

$$B_a = -S_1 \quad (28a)$$

$$B_b = S_1 - S_2 \quad (28b)$$

$$B_c = S_2 - S_3 \quad (28c)$$

The generalized speeds s_j impressed upon the springs by the adjacent point masses are

$$s_1 = v_b - v_a = \frac{B_b}{m_b} - \frac{B_a}{m_a} = \frac{S_1 - S_2}{m_b} + \frac{S_1}{m_a} \quad (29a)$$

$$s_2 = v_c - v_b = \frac{B_c}{m_c} - \frac{B_b}{m_b} = \frac{S_2 - S_3}{m_c} - \frac{S_1 - S_2}{m_b} \quad (29b)$$

The alternative Appell equations (6) of motion are

$$\frac{\partial A}{\partial \ddot{S}_j} - s_j = 0$$

With j =1, we obtain

$$-\frac{\ddot{S}_1}{k_1} - \frac{S_1 - S_2}{m_b} - \frac{S_1}{m_a} = 0 \tag{30a}$$

For j = 2, we obtain

$$-\frac{\ddot{S}_2}{k_2} - \frac{S_2 - S_3}{m_c} - \frac{S_1 - S_2}{m_b} = 0 \tag{30b}$$

Written in matrix form

$$\begin{bmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{bmatrix} \begin{bmatrix} \ddot{S}_1 \\ \ddot{S}_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{m_a} + \frac{1}{m_b} & -\frac{1}{m_b} \\ -\frac{1}{m_b} & \frac{1}{m_b} + \frac{1}{m_c} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{31}$$

where the first matrix is the *compliance matrix* and the second is the *susceptance matrix.* The system has 2 degrees of freedom, represented by the variables S_1 and S_2. Equations (31) also represents the system's 2 equations of compatibility (speed fit), since e.g. $\ddot{S}_2 / k_2$ represents a speed. The system also has 3 equations of equilibrium (impulse fit) represented by equations (28). The sum of impulse fits and speed fits is 5 for the present problem.

In comparing the two formulations it is seen, that, for the present problem, the alternative formulation (6) results in *two* differential equations of motion, while the conventional formulation (3) resulted in *three* differential equations of motion.

Example 2, Conventional Formulation

Consider the nonlinear oscillator shown in Figure 2. Its Appell function (1) is

$$A = \frac{1}{2}\int \ddot{\mathbf{r}}^2 dm = \frac{1}{2} m\ddot{q}^2 \tag{32}$$

The force Π (Figure 3) is

$$\Pi = -hq^3 \tag{33}$$

The Appell equation (3) of motion is

$$\frac{\partial A}{\partial \ddot{q}} - \Pi = 0$$

resulting in

$$m\ddot{q} + hq^3 = 0 \tag{34}$$

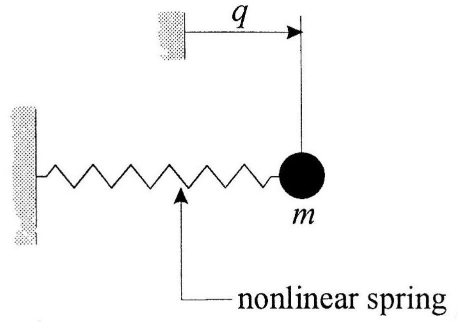

Figure 2. A Nonlinear System

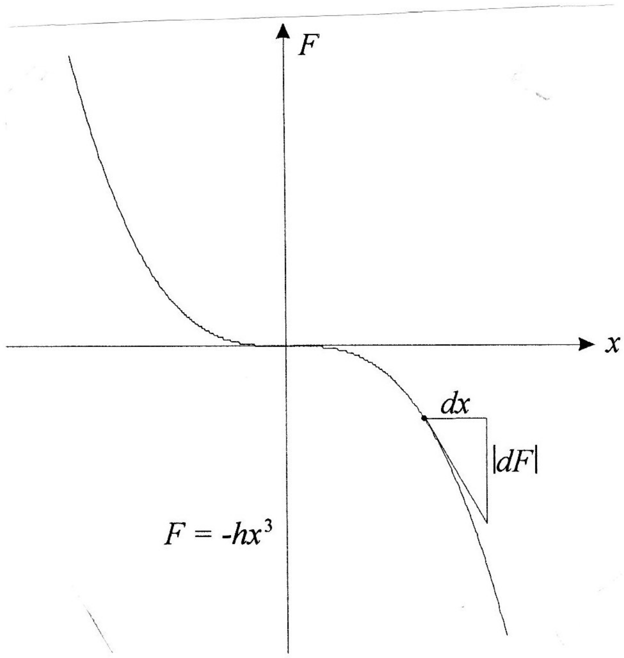

Figure 3. Nonlinear Spring

Example 2, Alternative Formulation

We must be careful to interpret the complementary Appell function (4)

$$A^* = -\frac{1}{2}\int \ddot{\mathbf{I}}^2 dc$$

properly. Since we are dealing with one spring only

$$A^* = -\frac{1}{2}\ddot{I}^2 c \tag{35}$$

The spring is nonlinear, i.e.

$$F = -hx^3 \tag{36}$$

For the tangent spring stiffness (Figure 3)

$$k = \frac{\partial F}{\partial x} = -3hx^2 \tag{37}$$

and for the associated compliance

$$c = \frac{1}{k} = -\frac{1}{3hx^2} = -\frac{1}{3h^{1/3}F^{2/3}} \tag{38}$$

We introduce a generalized impulse $S = I$ and $\dot{S} = F$ and obtain for the alternative Appell function (35)

$$A^* = -\frac{1}{2}\ddot{S}^2\frac{1}{3h^{1/3}\dot{S}^{2/3}} \tag{39}$$

We also require the extension speed of the spring

$$s = v = \frac{B}{m} = \frac{S}{m} \tag{40}$$

The alternative Appell equation (6) of motion

$$\frac{\partial A^*}{\partial \ddot{S}} - s = 0$$

consequently leads to

$$\frac{\ddot{S}}{3h^{1/3}\dot{S}^{2/3}} + \frac{S}{m} = 0 \tag{41}$$

Example 3, Conventional Formulation

The angular displacement θ of the two-mass-one-spring system of Figure 4, can be used to represent the linear displacements of the masses, with

$$x_1 = l\cos\theta \tag{42a}$$

$$y_2 = l\sin\theta \tag{42b}$$

The conventional Appell function (1) is then

$$A = \frac{1}{2}\int \ddot{\mathbf{r}} dm = \frac{1}{2}m\ddot{x}_1^2 + \frac{1}{2}m\ddot{y}_2^2 = \frac{1}{2}ml^2\left(\ddot{\theta}^2 + \dot{\theta}^4\right) \tag{43}$$

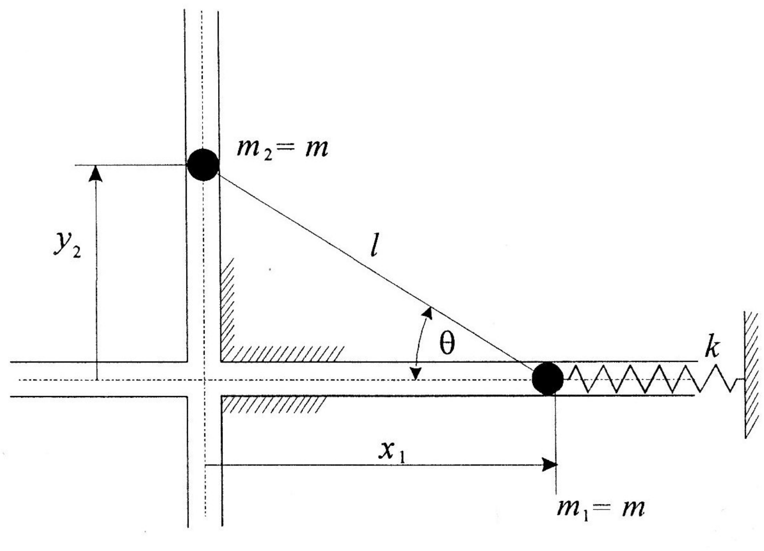

Figure 4. Two-Mass-One-Spring System

The deformation of the spring is

$$e = x_1 = l\cos\theta \tag{44}$$

The generalized force Π for the present problem is the torque M. Since the potential energy of the spring is

$$V = \frac{1}{2}kx_1^2 = \frac{1}{2}kl^2\cos^2\theta \tag{45}$$

we find the torque to be

$$M = -\frac{\partial V}{\partial \theta} = kl^2 \sin\theta \cos\theta = \frac{1}{2}kl^2 \sin 2\theta \tag{46}$$

The generalized force is consequently

$$\Pi = \frac{1}{2}kl^2 \sin 2\theta \tag{47}$$

The Appell equation (3) of motion is

$$\frac{\partial A}{\partial \ddot{\theta}} - \Pi = 0$$

With equations (43) and (47) we thus have

$$ml^2\ddot{\theta} - \frac{1}{2}kl^2 \sin 2\theta = 0 \tag{48}$$

Obviously a single degree-of-freedom system, it has one compatibility (displacement fit) equation (44) which is satisfied a priori, and one equilibrium (force fit) equation (48), altogether two fit equations, corresponding to the two elements, one inertial element ml^2 and one force element k.

Example 3, Alternative Formulation

We write for the alternative Appell function (4)

$$A^* = -\frac{1}{2}\int \ddot{\mathbf{I}}^2 dc = -\frac{1}{2}\ddot{S}_\theta^2 c_\theta \tag{49}$$

with

$$S_\theta = H = ml^2\dot{\theta} \tag{50}$$

where H is the angular momentum. The problem here is to find the correct expression for the compliance c. We begin with equation (46) and form the tangent stiffness

$$k_\theta = \frac{\partial M}{\partial \theta} = kl^2 \cos 2\theta \tag{51}$$

whose inverse is the compliance sought, i.e.

$$c_\theta = \frac{1}{k_\theta} = \frac{\partial \theta}{\partial M} = \frac{1}{kl^2 \cos 2\theta} \tag{52}$$

The time derivative $\dot{S}_\theta$ of the generalized impulse S_θ is the torque M. From equation (46) we thus have

$$\dot{S}_\theta = M = \frac{1}{2}kl^2 \sin 2\theta \tag{53}$$

from which

$$\cos 2\theta = \sqrt{1-\left(\frac{2\dot{S}_\theta}{kl^2}\right)^2} \tag{54}$$

From equations (49), (52) and (54) we obtain the alternative Appell function as

$$A^* = -\frac{1}{2}\frac{\ddot{S}_\theta^2}{\sqrt{k^2 l^4 - 4\dot{S}^2}} \tag{55}$$

For the generalized speed we use

$$s_\theta = \dot{\theta} = \frac{H}{ml^2} = \frac{S_\theta^2}{ml^2} \tag{56}$$

such that the alternative Appell equation (6) gives us

$$\frac{\ddot{S}_\theta}{\sqrt{k^2 l^4 - 4\dot{S}^2}} + \frac{S_\theta}{ml^2} = 0 \tag{57}$$

In its alternative formulation the system has one equilibrium (impulse fit) equation (50), which must be satisfied a priori, i.e. before the alternativeAppell function can be formulated, and one compatibility (speed fit) equation (57) which is represented by the alternative Appell equation.

4 Conclusion

It has been shown, that an alternative form of the Appell equation of motion can be defined. By means of three simple examples, the application of the conventional and the alternative formulations has been illustrated.

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