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Post-buckling Axisymmetric Deflections of Thin Shells of Revolution under Axial Loading

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Large axisymmetric deflections of thin shells of revolution under axial loading are studied. Using nonlinear three-dimensional equations of the theory of elasticity which describe the axisymmetric deformations of a thin shell of revolution made of a nonlinear elastic material the approximate two-dimensional elasticity relations are obtained. The corresponding expression for the elastic potential energy of the deformed shell is also obtained. For the shell loaded axially the expression for the axial force is obtained as well.

1 Introduction

We consider the large axisymmetric deflections of thin elastic shells of revolution under axial loading. One of the possible shell equilibrium states is that for which a shell segment is close to its mirror image obtained by the reflection from the plane Q which is perpendicular to the shell axis (e.g. Figures 3 and 4). It is known (see Pogorelov, 1966; Kriegsmann and Lange, 1980; Evkin and Korovaitsev, 1992) that this state of equilibrium corresponds to relatively large shell deformations and to a relatively small axial force. That is why it is necessary to use more exact two-dimensional shell equations.

By using the nonlinear three-dimensional equations of the theory of elasticity for the axisymmetric deformation of a thin shell of revolution made of nonlinear elastic material approximate two-dimensional elasticity relations may be obtained. The corresponding expressions for the elastic potential energy of the deformed shell has also been obtained. The deformations are assumed to have the order of a small thicknesss parameter μ which is proportional to the square root of the relative shell thickness. In the elasticity relations the terms of second order with respect to the deformations are held constant and the error of the equations obtained has the order of the relative shell thickness. This error is usually small for the linear shell theory. The Kirchhoff-Love hypotheses are valid only for the first approximation.

For a shell loaded axially the expression for the axial force is obtained. This expression is more exact compared to the results reported in the papers by Pogorelov (1966), Kriegsmann and Lange (1980), and Koroteeva, Tovstik and Shuvalkin (1995).

2 The Deformed Shell Metric

Let the deformations of the neutral surface of a shell of revolution be described by the following relations (see Figure 1):

$$r_0 = r_0(s_0) \qquad \theta_0 = \theta_0(s_0) \qquad r_0' = \cos\theta_0 \qquad (\ )' \equiv \frac{d}{ds_0} \tag{1}$$

Here s_0 is the generatrix length, r_0 is the distance between a current point on the neutral surface and the axis of symmetry, and θ_0 is the angle between the shell normal and the axis.

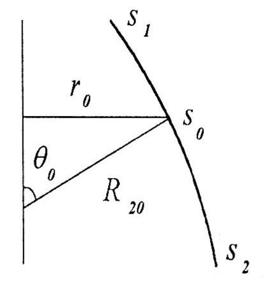

Figure 1. Shell of Revolution before Deformation

The main radii of curvature of the neutral surface before deformation, R_{10} and R_{20}, are obtained from

$$\frac{1}{R_{10}} = \theta_0' \qquad \text{and} \qquad \frac{1}{R_{20}} = \frac{\sin\theta_0}{r_0} \tag{2}$$

We denote the same variables after deformation as s, r, θ, R_1, R_2. The formulas corresponding to equations (1) and (2) are still valid. The tensile deformations of the neutral surface, ε_1 and ε_2, are given as

$$\varepsilon_1 = s' - 1 \qquad \text{and} \qquad \varepsilon_2 = \frac{r}{r_0} - 1 \qquad \text{with} \qquad r' = (1+\varepsilon_1)\cos\theta \tag{3}$$

In the shell before the deformation we introduce the orthogonal system of curvilinear coordinates $q_1 = s_0$, $q_2 = \varphi$, $q_3 = z$, where φ is the angle in a circular direction, z is the distance between a current point and the shell neutral surface. The squared distance between infinitely close shell points is

$$\left(d\mathbf{R}^0\right)^2 = H_i^2 dq_i^2 \equiv g_{ij}^0 dq_i dq_j \qquad H_1 = 1 + z\theta_0' \qquad H_2 = r_0 + z\sin\theta_0 \qquad H_3 = 1 \tag{4}$$

where H_i are the Lamé coefficients, and g_{ij}^0 are the covariant components of the metric tensor before the deformation. To describe the position of a point (s_0, φ, z) after deformation we use mobile Cartesian coordinates with the unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, which are connected to the deformed neutral surface. The position of a point (s_0, φ, z) after the deformation is described by the vector

$$\mathbf{R} = \mathbf{R}^0 + \mathbf{R}^1 \qquad \text{with} \qquad \mathbf{R}^1 = \mathbf{i}_1 u + \mathbf{i}_3(z+w) \tag{5}$$

The functions $u(s_0, z)$ and $w(s_0, z)$ describe the shear deformation and the stretching along the normal to the neutral surface respectively. If the Kirchhoff-Love hypotheses are valid then $u(s_0,z) = w(s_0,z) \equiv 0$.
The covariant components of the metric tensor after the deformation, g_{ij}, are the following:

$$g_{ij} = \mathbf{R}_i \mathbf{R}_j \qquad \text{and} \qquad \mathbf{R}_i = \frac{\partial \mathbf{R}}{\partial q_i} \tag{6}$$

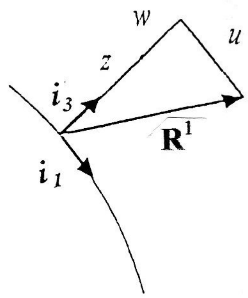

Figure 2. Shell after Deformation

The elements ε_{ij} of the Cauchy-Green deformation tensor $\mathbf{E}$, we find from

$$g_{ij} - g_{ij}^0 = 2H_iH_j\varepsilon_{ij} \tag{7}$$

For axisymmetric deformation $\varepsilon_{12} = \varepsilon_{23} = 0$.

The derivatives $\partial \mathbf{R}_i / \partial q_j$ are expressed through the Christoffel symbols Γ_{ij}^k

$$\frac{\partial \mathbf{R}_i}{\partial q_j} = \Gamma_{ij}^k \mathbf{R}_k \quad \text{with} \quad \Gamma_{ij}^k = g^{kl}\Gamma_{ij,l} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial q_j} + \frac{\partial g_{il}}{\partial q_i} - \frac{\partial g_{ij}}{\partial q_l}\right) \tag{8}$$

where the g^{kl} are the contravariant components of the metric tensor. Below we expand the variables ε_{ij} and Γ_{ij}^k into powers of the small thickness parameter μ. For the deformed state which we consider below, the following estimations are valid:

$$\begin{array}{lllll} z \sim \mu^2 & \{\varepsilon_i, \varepsilon_{ii}\} \sim \mu & \varepsilon_{13} \sim \mu^2 & w \sim \mu^3 & u \sim \mu^4 \\ \dfrac{\partial y}{\partial s_0} \sim \dfrac{y}{\mu} & y = \{\varepsilon_i, \varepsilon_{ij}, \theta, u, w\} & & & \\ \dfrac{\partial y}{\partial z} \sim \dfrac{y}{\mu^2} & y = \{\varepsilon_i, u, w\} & \mu^4 = \dfrac{h^2}{12(1-\nu^2)R^2} & & \end{array} \tag{9}$$

where h is the shell thickness, R is the shell typical size, ν is Poisson's ratio. Further the auxiliary parameter $\mu_0 = 1$ denotes the terms order, namely the term with multiplier μ_o^k has the order μ^k. Due to formulas (6) to (8) we get

$$\begin{aligned} \varepsilon_{11} &= \mu_0(\varepsilon_1 + z\theta') + \mu_0^2\left((\varepsilon_1 + z\theta')^2/2 + w\theta' - z\theta_0'\right) + O(\mu^3) \\ \varepsilon_{13} &= \mu_0^2(u_z + w')/2 + O(\mu^3) \\ \varepsilon_{22} &= \mu_0\varepsilon_2 + \mu_0^2\left(\varepsilon_2^2/2 + z\kappa_2^0\right) + O(\mu^3) \qquad \kappa_2^0 = (\sin\theta - \sin\theta_0)/r_0 \\ \varepsilon_{33} &= \mu_0 w_z + \mu_0^2 w_z^2/2 + O(\mu^3) \end{aligned} \tag{10}$$

3 The Equilibrium Equations and the Elasticity Relations

In the axisymmetric case the equilibrium equations (see Novozhilov, 1958; Lurje, 1970) are given as

$$\begin{aligned} \frac{\partial \sigma_{11}^o}{\partial s_0} + \frac{\partial \sigma_{13}^o}{\partial z} + \Gamma_{11}^1\sigma_{11}^o + 2\Gamma_{13}^1\sigma_{13}^o + \Gamma_{22}^1\sigma_{22}^o + \Gamma_{33}^1\sigma_{33}^o &= 0 \\ \frac{\partial \sigma_{13}^o}{\partial s_0} + \frac{\partial \sigma_{33}^o}{\partial z} + \Gamma_{11}^3\sigma_{11}^o + 2\Gamma_{13}^3\sigma_{13}^o + \Gamma_{22}^3\sigma_{22}^o + \Gamma_{33}^3\sigma_{33}^o &= 0 \end{aligned} \tag{11}$$

where

$$\begin{array}{lll} \sigma_{ij}^o = V\dfrac{\sigma_{ij}^*}{H_iH_j} & \sigma_{ij} = \dfrac{S_i(1+E_j)}{S_i^*}\sigma_{ij}^* & \dfrac{S_1^*}{S_1} = \sqrt{(1+2\varepsilon_{22})(1+2\varepsilon_{33}) - \varepsilon_{23}^2} \quad \text{(cycl)} \\ V = H_1H_2H_3 & E_j = \sqrt{1+2\varepsilon_{jj}} - 1 & \end{array} \tag{12}$$

Here $\sigma_i = \sigma_{ij}\mathbf{k}_j$ are the actual shell stresses after deformation, and σ^*_{ij} are the energy stress tensor components. The boundary conditions on the lateral surfaces $z = \pm h_2 = h/2$ are the following:

$$\sigma^*_{3j} = \frac{S^*_3}{S_3(1+E_j)} p^\pm_j \qquad j = 1, 3 \qquad \text{at} \qquad z = \pm h_2 \tag{13}$$

where $\mathbf{p}^\pm$ is the pressure on the lateral surfaces. The shell material is assumed to be elastic and isotropic. Let the potential $\Phi(I_1, I_2, I_3)$ be given as a function of the invariants of the deformation tensor **E**. We use the following invariants

$$I_1 = \varepsilon_{ii} \qquad I_2 = \varepsilon_{ij}\varepsilon_{ji} \qquad I_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} \tag{14}$$

Then the stresses σ^*_{ij} are equal to

$$\sigma^*_{ij} = \frac{\partial\Phi}{\partial\varepsilon_{ij}} = A_1\delta_{ij} + A_2\varepsilon_{ij} + A_3\varepsilon_{ik}\varepsilon_{kj} \qquad A_k = k\frac{\partial\Phi}{\partial I_k} \qquad k = 1, 2, 3 \tag{15}$$

where δ_{ij} is the Kronecker delta.

For the 5-constants elasticity theory

$$\Phi = \frac{1}{2}\lambda I_1^2 + GI_2 + \alpha_1 I_1^3 + \alpha_2 I_1 I_2 + \alpha_3 I_3 \tag{16}$$

we get

$$A_1 = \lambda I_1 + 3\alpha_a I_1^2 + \alpha_2 I_2 \qquad A_2 = 2G + 2\alpha_2 I_1 \qquad A_3 = 3\alpha_3 \tag{17}$$

Potential (16) gives the general form of the squared dependence of stresses on strains. If $\alpha_1 = \alpha_2 = \alpha_3 = 0$ then formulas (15) lead to Hooke's law. Let the orders of the elastic modulus α_j in equation (16) not exceed the orders of the moduli λ, G, and let the stresses be referred to λ (or to G). Then the stresses become dimensionless and they may be expanded into a power series of μ.

4 Asymtotic Solution of System (11)

Firstly let $p^\pm_j = 0,\ j = 1, 3$. We integrate equations (11) in z from $-h_2$ to h_2. Then the first of equations (11) we multiply by z and integrate within the same limits. As a result we obtrain three integral equilibrium equations

$$\begin{aligned}
&\langle\sigma^o_{11}\rangle' + \langle\Gamma^1_{11}\sigma^o_{11}\rangle + 2\langle\Gamma^1_{13}\sigma^o_{13}\rangle + \langle\Gamma^1_{22}\sigma^o_{22}\rangle + \langle\Gamma^1_{33}\sigma^o_{33}\rangle = 0\\
&\langle\sigma^o_{13}\rangle' + \langle\Gamma^3_{11}\sigma^o_{11}\rangle + 2\langle\Gamma^3_{13}\sigma^o_{13}\rangle + \langle\Gamma^3_{22}\sigma^o_{22}\rangle + \langle\Gamma^3_{33}\sigma^o_{33}\rangle = 0\\
&\langle z\sigma^o_{11}\rangle' - \langle\sigma^o_{13}\rangle + \langle z\Gamma^1_{11}\sigma^o_{11}\rangle + 2\langle z\Gamma^1_{13}\sigma^o_{13}\rangle + \langle z\Gamma^1_{22}\sigma^o_{22}\rangle + \langle z\Gamma^1_{33}\sigma^o_{33}\rangle = 0
\end{aligned} \tag{18}$$

where

$$\langle Y\rangle \equiv \frac{1}{h}\int_{-h_2}^{h_2} Y dz \tag{19}$$

According to estmations (9) we get

$$A_1 \sim \mu \qquad A_2 \sim 1 \qquad A_3 = O(\mu^{-1}) \tag{20}$$

$$\left\{\overset{*}{\sigma}_{11}, \overset{*}{\sigma}_{22}, \overset{*}{\sigma}_{33}\right\} = O(\mu) \qquad \overset{*}{\sigma}_{13} = O\left(\mu^2\right) \tag{21}$$

We remind that the symbol "~" provides the exact order estimation and the symbol "O" gives the upper order estimation.

A comparison of the terms in equations (11) and (18) enables us to make estimations (21) more precise, namely

$$\left\{\sigma_{11}^{o}, \sigma_{22}^{o}\right\} \sim \mu \quad \text{and} \quad \left\{\sigma_{13}^{o}, \sigma_{33}^{o}\right\} \sim \mu^2 \tag{22}$$

and to obtain the estimations for the integral variables in equations (18).

$$\left\langle \sigma_{22}^{o} \right\rangle \sim \mu \qquad \left\{\left\langle \sigma_{11}^{o} \right\rangle, \left\langle \sigma_{13}^{o} \right\rangle\right\} \sim \mu^2 \qquad \left\langle z\sigma_{11}^{o} \right\rangle \sim \mu^3 \tag{23}$$

The disagreement between the orders of the variables σ_{11}^{o} and $\left\langle \sigma_{11}^{o} \right\rangle$ is caused by the fact that the average value of σ_{11}^{o} is close to zero. We can now give the approximate expressions for the stresses $\overset{*}{\sigma}_{ij}$

$$\begin{aligned}
\overset{*}{\sigma}_{11} &= \mu_0\left\{\lambda(\varepsilon_1 + \varepsilon_2 + z\theta' + w_z) + 2G(\varepsilon_1 + z\theta')\right\} + O\left(\mu^2\right) \\
\overset{*}{\sigma}_{13} &= \mu_0^2 G(u_z + w') + O\left(\mu^3\right) \\
\overset{*}{\sigma}_{22} &= \mu_0\left\{\lambda(\varepsilon_1 + \varepsilon_2 + z\theta' + w_z) + 2G\varepsilon_2\right\} + O\left(\mu^2\right) \\
\overset{*}{\sigma}_{33} &= \mu_0\left\{\lambda(\varepsilon_1 + \varepsilon_2 + z\theta' + w_z) + 2Gw_z\right\}\mu_0^2\left\{\lambda\left((\varepsilon_1 + z\theta')^2/2 + w\theta' - z\theta_0' + \varepsilon_2^2/2 + z\kappa_2^0 + w_z^2/2\right)\right. \\
&\quad \left. + Gw_z^2 + 3\alpha_3 w_z^2 + 2\alpha_2(\varepsilon_1 + \varepsilon_2 + z\theta' + w_z) + 3\alpha_1(\varepsilon_1 + \varepsilon_2 + z\theta' + w_z)^2 + \alpha_2\left((\varepsilon_1 + z\theta')^2 + \varepsilon_2^2 + w_z^2\right)\right\} + O\left(\mu^3\right)
\end{aligned} \tag{24}$$

Due to estimations $\sigma_{33}^{o} \sim \mu^2$ and $\left\langle \sigma_{11}^{o} \right\rangle \sim \mu^2$ we get in zero approximation

$$\begin{aligned}
&\lambda(\varepsilon_1 + \varepsilon_2 + z\theta' + w_z) + 2Gw_z = 0 \\
&\left\langle \lambda(\varepsilon_1 + \varepsilon_2 + z\theta' + w_z) + 2G(\varepsilon_1 + z\theta') \right\rangle = 0
\end{aligned} \tag{25}$$

and the following relations:

$$\begin{aligned}
\varepsilon_1 &= -\nu\varepsilon_2 + O\left(\mu^2\right) & \overset{*}{\sigma}_{11} &= \frac{E}{1-\nu^2} z\theta' + O\left(\mu^2\right) \\
w_z &= -\nu\varepsilon_2 - \frac{\nu}{1-\nu} z\theta' + O\left(\mu^2\right) & w &= -\nu\varepsilon_2 z - \frac{\nu}{2(1-\nu)^2} z^2\theta' + O\left(\mu^2\right)
\end{aligned} \tag{26}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad G = \frac{E}{2(1+\nu)} \tag{27}$$

Here E is Young's modulus. In the first of equations (11) the first and the second terms are asymptotically the main ones, and in the second equation the second and the third terms are the main ones. These terms give

$$\frac{\partial\left(r_0\overset{*}{\sigma}_{11}\right)}{\partial s_0} + r_0\frac{\partial\overset{*}{\sigma}_{13}}{\partial z} = O(\mu) \quad \text{and} \quad \frac{\partial\overset{*}{\sigma}_{33}}{\partial z} - \theta'\overset{*}{\sigma}_{11} = O(\mu) \tag{28}$$

By integrating equations (28) with respect to z according to expression (26) for $\overset{*}{\sigma}_{11}$, and according to the boundary conditions $\overset{*}{\sigma}_{13} = \overset{*}{\sigma}_{33} = 0$ at $z = \pm h_2$, we get

$$\overset{*}{\sigma}_{13} = \frac{E\left(h_2^2 - z^2\right)}{2\left(1-\nu^2\right)r_0}\left(r_0\theta'\right)' + O\left(\mu^3\right) \quad \text{and} \quad \overset{*}{\sigma}_{33} = -\frac{E\left(h_2^2 - z^2\right)r_0}{2\left(1-\nu^2\right)}\theta' + O\left(\mu^3\right) \tag{29}$$

By using expression (24) for σ_{33}^*, we find an expression for w_z which is more exact than that of equation (26), namely

$$w_z = -\mu_0\left(\nu\varepsilon_2 + \frac{\nu}{1-\nu}z\theta'\right) - \frac{\mu_0^2}{\lambda+2G}\left\{\lambda(\varepsilon_1+\nu\varepsilon_2) + b_1\varepsilon_2^2 + z\left[\lambda\left(-\theta_0'+\kappa_2^0\right) + b_2\varepsilon_2\theta'\right] + b_3 z^2\theta'^2 - \sigma_{33}^*\right\} + O\left(\mu^3\right) \tag{30}$$

where

$$\begin{aligned} b_1 &= \frac{E\nu}{2(1-2\nu)} + \alpha_1 3(1-2\nu)^2 + \alpha_2\left(1-2\nu+6\nu^2\right) + 3\nu^2\alpha_3 \\ b_2 &= -\frac{E\nu^2}{(1+\nu)(1-2\nu)} + \alpha_1\frac{6(1-2\nu)^2}{1-\nu} - \alpha_2\frac{6\nu(1-2\nu)}{1-\nu} + \alpha_3\frac{6\nu^2}{1-\nu} \\ b_3 &= \frac{E\nu}{2(1+\nu)(1-2\nu)} + \alpha_1\frac{3(1-2\nu)^2}{(1-\nu)} + \alpha_2\frac{1-4\nu+6\nu^2}{(1-\nu)^2} + \alpha_3\frac{3\nu^2}{(1-\nu)^2} \end{aligned} \tag{31}$$

Then from equations (24) we obtain

$$\begin{aligned} \frac{\sigma_{11}^*}{E} &= \mu_0\frac{z\theta'}{1-\nu^2} + \mu_0^2\frac{1}{1-\nu^2}\left(\varepsilon_1+\nu\varepsilon_2+\nu z\kappa_2^0 - z\theta_0'\right) + \mu_0^2 E\left\{b_{11}\varepsilon_2^2 + b_{12}\varepsilon_2 z\theta' + b_{13}(z\theta')^2 \right. \\ &\quad \left. + b_{14}\theta'^2\left(h_2^2-z^2\right)\right\} + O\left(\mu_0^3\right) \\ \frac{\sigma_{22}^*}{E} &= \mu_0\left(\varepsilon_2+\frac{\nu z\theta'}{1-\nu^2}\right) + \mu_0^2\frac{\nu}{1-\nu^2}\left(\varepsilon_1+\nu\varepsilon_2+\nu z\kappa_2^0 - \nu z\theta_0'\right) + \mu_0^2 E\left\{b_{21}\varepsilon_2^2 + b_{22}\varepsilon_2 z\theta' + b_{23}(z\theta')^2 \right. \\ &\quad \left. + b_{24}\theta'^2\left(h_2^2-z^2\right)\right\} + O\left(\mu_0^3\right) \end{aligned} \tag{32}$$

where b_{ij} are the linear functions of the dimensionless variables $\alpha_j^0 = \alpha_j / E,\ j=1,2,3$ with the coefficients depending on ν.

$$\begin{aligned} b_{11} &= \frac{\nu}{2(1-\nu)} + \alpha_1^0\frac{3(1-2\nu)^3}{(1-\nu)} + \alpha_2^0\frac{\left(1-2\nu+6\nu^2\right)(1-2\nu)}{1-\nu} + \alpha_3^0\frac{3\nu^2(1-2\nu)}{1-\nu} \\ b_{12} &= -\frac{2\nu}{1-\nu^2} + \alpha_1^0\frac{6(1-2\nu)^3}{(1-\nu)^2} + \alpha_2^0\frac{2\left(1-4\nu+6\nu^2\right)(1-2\nu)}{(1-\nu)^2} - \alpha_3^0\frac{6\nu^2\left(1-2\nu+2\nu^2\right)}{(1-\nu)^2} \\ b_{13} &= -\frac{1-2\nu}{2(1-\nu)\left(1-\nu^2\right)} + \alpha_1^0\frac{3(1-2\nu)^3}{1-\nu} + \alpha_2^0\frac{\left(3-2\nu+4\nu^2\right)(1-2\nu)}{1-\nu} + \alpha_3^0\frac{3\left(1-\nu+\nu^2\right)}{(1-\nu)^3} \\ b_{21} &= -\frac{1-\nu+\nu^2}{2(1-\nu)} + \alpha_1^0\frac{3(1-2\nu)^3}{(1-\nu)^2} + \alpha_2^0\frac{2\left(1-2\nu+4\nu^2\right)(1-2\nu)}{(1-\nu)^2} + \alpha_3^0\frac{3\left(1-\nu-\nu^3\right)}{1-\nu} \\ b_{22} &= -\frac{2\nu^2}{1-\nu^2} + \alpha_1^0\frac{6(1-2\nu)^3}{(1-\nu)^2} + \alpha_2^0\frac{2\left(1-2\nu+4\nu^2\right)(1-2\nu)}{(1-\nu)^2} - \alpha_3^0\frac{6\nu^3}{(1-\nu)^2} \\ b_{23} &= -\frac{\nu(1-2\nu)}{2(1-\nu)\left(1-\nu^2\right)} + \alpha_1^0\frac{3(1-2\nu)^3}{(1-\nu)3} + \alpha_2^0\frac{\left(1-2\nu+4\nu^2\right)(1-2\nu)}{(1-\nu)^3} - \alpha_3^0\frac{3\nu^3}{(1-\nu)^3} \\ b_{14} &= b_{24} = -\frac{\nu}{2(1-\nu)\left(1-\nu^2\right)} \qquad \alpha_j^0 = \frac{\alpha_j}{E} \qquad j = 1,2,3 \end{aligned} \tag{33}$$

5 Two-dimensional Equilibrium Equations

We consider the curvilinear parallelepiped which before the deformation occupies the domain $s_0,\ s_0+ds_0;\ \varphi,\ \varphi+d\varphi;\ -h_2 \le z \le h_2$. In this case the equilibrium equations in projections on the tangent and on the normal to the neutral surface after deformation have the form

$$
\begin{aligned}
&\left(r_0 T_1\right)' - T_2\cos\theta + r_0\theta' Q_1 + r_0 p_1 = 0\\
&\left(r_0 Q_1\right)' - T_2\sin\theta - r_0\theta' T_1 + r_0 p_3 = 0\\
&\left(r_0 M_1\right)' - M_2\cos\theta - r_0(1+\varepsilon_1)Q_1 = 0
\end{aligned}
\tag{34}
$$

where the projections of stress resultants and stress couples $T_j,\ Q_1,\ M_j$ are referred to the unit length of the neutral surface before deformation and are defined by formulas

$$
\begin{aligned}
&(T_1\mathbf{i}_1+Q_1\mathbf{i}_3)r_0 d\varphi = \left\langle \frac{S_1^*}{S_1}\sigma_1\right\rangle h r_0 d\varphi \qquad && M_1\mathbf{i}_2 r_0 d\varphi = \left\langle \frac{S_1^*}{S_1}\mathbf{R}^1\times\sigma_1\right\rangle h r_0 d\varphi\\
&T_2\mathbf{i}_2 ds_0 = \left\langle \frac{S_2^*}{S_2}\sigma_2\right\rangle h ds_0 && M_2\mathbf{i}_1 ds_0 = -\left\langle \frac{S_2^*}{S_2}\mathbf{R}^1\times\sigma_2\right\rangle h ds_0
\end{aligned}
\tag{35}
$$

Here $\sigma_i = \sigma_{ij}\mathbf{k}_j$ are the actual stresses after deformation, and $\mathbf{R}^1$ is given by formula (5). After projecting relations (35) on the unit vectors $\mathbf{i}_j$, substituting expressions (29) and (32) and integrating with respect to z, we obtain the nonlinear two-dimensional elasticity relations

$$
\begin{aligned}
T_1 &= \mu_0^2\left\{K(\varepsilon_1+\nu\varepsilon_2) + Eb_{11}h\varepsilon_2^2 + Da_1\theta'^2\right\} + O\left(h\mu^3\right)\\
T_2 &= \mu_0 Eh\varepsilon_2 + \mu_0^2\left\{K\nu(\varepsilon_1+\nu\varepsilon_2) + Eh(1+b_{21})\varepsilon_2^2 + Da_2\theta'^2\right\} + O\left(h\mu^3\right)\\
M_1 &= \mu_0^3 D\theta' + \mu_0^4\left\{D\left(-\theta_0'+\nu\kappa_2^0\right) + Da_3\varepsilon_2\theta'\right\} + O\left(h\mu^5\right)\\
M_2 &= \mu_0^3 D\nu\theta' + \mu_0^4\left\{D\left(\kappa_2^0-\nu\theta_0'\right) + Da_4\varepsilon_2\theta'\right\} + O\left(h\mu^5\right)
\end{aligned}
\tag{36}
$$

where

$$
\begin{aligned}
&K = \frac{Eh}{1-\nu^2} \qquad D = \frac{Eh^3}{12\left(1-\nu^2\right)} \qquad a_1 = 1 + (b_{13}+2b_{14})\left(1-\nu^2\right)\\
&a_2 = (b_{23}+2b_{24})\left(1-\nu^2\right) \qquad a_3 = b_{12}\left(1-\nu^2\right) - 2\nu \qquad a_4 = b_{22}\left(1-\nu^2\right) - \frac{\nu(\nu+1)}{2}
\end{aligned}
\tag{37}
$$

We exclude the term $K(\varepsilon_1+\nu\varepsilon_2)$ from the first two relations (36) and rewrite these relations with an accuracy sufficient for the calculations following.

$$
\begin{aligned}
M_1 &= D\left(\theta' - \theta_0' + \nu\kappa_2^0\right) + 2Dc_2\varepsilon_2\theta' + O\left(Eh^2\mu^3\right)\\
T_2 &= \nu T_1 + Eh\left(\varepsilon_2 + c_1\varepsilon_2^2\right) + Dc_2\theta'^2 + O\left(Eh\mu^3\right)\\
\varepsilon_1 &= -\nu\varepsilon_2 + O\left(\mu^2\right) \qquad M_2 = \nu M_1 + O\left(Eh^2\mu^2\right)
\end{aligned}
\tag{38}
$$

The dependence on the nonlinear elastic properties of the material is shown only through the terms with coefficients c_1 and β_2, which are equal to

$$
\begin{aligned}
c_1 &= \frac{3}{2} + 3\alpha_1^0(1-2\nu)^3 + 3\alpha_2^0(1-2\nu)\left(1+2\nu^2\right) + 3\alpha_3^0\nu\left(1-2\nu^3\right)\\
c_2 &= -2\nu + \frac{1+\nu}{1-\nu}\left[3\alpha_1^0(1-2\nu)^3 + \alpha_2^0(1-2\nu)\left(1-4\nu+6\nu^2\right) - 3\alpha_3^0\nu\left(1-2\nu+2\nu^2\right)\right]
\end{aligned}
\tag{39}
$$

System (34) together with the geometric relation

$$(r_0\varepsilon_2)' = (1+\varepsilon_1)\cos\theta - \cos\theta_0 \tag{40}$$

which follows from formulas (3), and with relations (38) forms the closed system for 8 unknown functions T_j, M_j, ε_j, Q_1, θ. This system is of the 5th order and the main unknown functions are T_1, M_1, ε_2, Q_1, θ, at the same time the functions T_2, M_2, ε_1, may be expressed through the main functions by formulas (38).

The formal error of the resulting system (34), (38), (40) is of the order μ^2 or the order of the relative shell thickness. Here we only discuss the error of the construction of the internal stress state. The problem of its interaction with the boundary layer (see Goldenveizer, 1994) is not studied. The elastic shell potential energy is equal to

$$\Pi = \iiint \Phi H_1 H_2 H_3 \, dq_1 \, dq_2 \, dq_3 \tag{41}$$

Substituting expression (10) for deformations into formula (41) and integrating with respect to z we get

$$\Pi = \pi\int_{s_1}^{s_2} r_0 \left[Eh\left(\varepsilon_2^2 + \frac{2c_1}{3}\varepsilon_2^3\right) + D\left((\theta'-\theta_0')^2 + 2\nu(\theta'-\theta_0')^2\kappa_2^0 + \left(\kappa_2^0\right)^2 + 2c_2\varepsilon_2(\theta')^2\right) + O\left(\frac{Eh^3}{R^2}\right)\right] ds_0 \tag{42}$$

where the constants c_1 and c_2 are the same as in formulas (38).

6 Shell Loaded by an Axial Force

Let the axial force P be applied to the shell edges. We study the deformed shell state which is close to its mirror image which is obtained by the reflection from the plane Q which is perpendicular to the shell axis and which contains the given parallel s_*. We assume that this parallel is sufficiently far from the shell edges. Let at $s_0^1 \le s_0 < s_0^*$ the shell form be close to the initial form, and at $s_0^* < s_0 \le s_0^2$ the shell segment is specular reflected. Both at $s_0^1 \le s_0 < s_0^*$ and at $s_0^* < s_0 \le s_0^2$ the shell stress state is close to the membrane state. Near the parallel $s_0 = s_0^*$ there is the internal edge effect which is described by equations (34), (38) and (40). We seek the axial force P which holds the shell in the given position. (Stated more exactly the edge fixing in the axial direction holds the shell and the force P appears). We also assume that the angle $\gamma = \theta_0\left(s_0^*\right)$. The stress state which satisfies estimations (9) is realized near the parallel $s_0 = s_*$. We introduce dimensionless variables by means of the following formulas:

$$\begin{gathered} \{s_0, r_0\} = R\{s_0^o, r_0^o\} \qquad \{\varepsilon_1, \varepsilon_2\} = \mu\{\varepsilon_1^o, \varepsilon_2^o\} \qquad \{p_1, p_3\} = Eh\mu^2 R^{-1}\{p_1^o, p_3^o\} \\ \{T_1, Q_1\} = Eh\mu^2\{T_1^o, Q_1^o\} \qquad T_2 = Eh\mu T_2^o \qquad \{M_1, M_2\} = EhR\mu^3\{M_1^o, M_2^o\} \end{gathered} \tag{43}$$

where the dimensionless variables are denoted by the symbol ^{o}. Substitutions (43) are introduced in such a manner that near the parallel $s_0 = s_*$ the values marked with the symbol ^{o} are of the order 1. We introduce the projections V and U of the stress resultants on the axial and the normal directions correspondingly.

$$T_1^o = U\cos\theta + V\sin\theta \quad \text{and} \quad Q_1^o = U\sin\theta - V\cos\theta \tag{44}$$

Later on we shall omit the symbol ^{o}. Then the system of equations for variables

$$V,\ U,\ \varepsilon_2,\ M_1,\ \theta \tag{45}$$

assumes the form

$$
\begin{aligned}
&(r_0V)' = 0 \\
&\mu(r_0U)' = \varepsilon_2 + \mu\nu(U\cos\theta + V\sin\theta) + \mu c_1\varepsilon_2^2 + \mu^3 c_2\theta'^2 \\
&\mu(r_0\varepsilon_2)' = (1-\mu\nu\varepsilon_2)\cos\theta - \cos\theta_0 \\
&\mu(r_0M_1)' = r_0(1-\mu\nu\varepsilon_2)(U\sin\theta - V\cos\theta) + \mu\nu M_1\cos\theta \\
&\mu\theta' = M_1 + \mu(\theta_0' - \nu\kappa_2^0) - 2\mu^2 c_2\varepsilon_2\theta'
\end{aligned}
\tag{46}
$$

The first of equations (46) gives

$$
V = \frac{C}{r_0} \qquad P = 2\pi EhR\mu^2 C = \frac{2\pi Eh^2}{\sqrt{12(1-\nu^2)}}C \tag{47}
$$

where P is the unknown axial force. Far from shell edges and from the parallel $s_0 = s_*$ the solution of system (46) is given as

$$
\begin{aligned}
&\theta = \theta_0 + O(C\mu^2) \qquad M_1 = O(C\mu^3) \qquad U = \frac{C\cos\theta_0}{r_0\sin\theta_0} \qquad \varepsilon_2 = O(C\mu) \qquad \text{at } s_0 = s_* \\
&\theta = -\theta_0 + O(C\mu^2) \qquad M_1 = -2\mu m_1 + O(C\mu^3) \qquad m_1 = \frac{1}{R_{10}} + \frac{\nu}{R_{20}} \\
&U = -\frac{C\cos\theta_0}{r_0\sin\theta_0} \qquad \varepsilon_2 = O(C\mu) \qquad \text{at } s_0 > s_*
\end{aligned}
\tag{48}
$$

7 Asymptotic Solution of System (46)

Near the parallel $s_0 = s_*$ there exists an internal edge effect and the main deformations are concentrated here. We write equations (46) in vector form.

$$
\mu\mathbf{x}' = \mathbf{F}(\mathbf{x}, s_0, C, \mu) \qquad \mathbf{x} = \{U, \varepsilon_2, M_1, \theta\} \tag{49}
$$

introduce a rescaling

$$
\xi = \frac{s_0 - s_2}{\mu} \tag{50}
$$

and seek a solution in the form

$$
\mathbf{x} = \mathbf{x}^0(\xi) + \mu\mathbf{x}^1(\xi) + O(\mu^2) \qquad C = C^0 + \mu C^1 + \mu^2 C^2 + O(\mu^3) \tag{51}
$$

The precision system (46) enables us to find the written out terms in series (51). We use the following condition as the boundary one: solution (51) as $\xi \to \pm\infty$ corresponds to solution (48) at $s_0 > s_*$. Taking into account that

$$
\begin{aligned}
&r_0(s_0) = \rho + \mu\xi\cos\gamma + O(\mu^2) \qquad \theta_0(s_0) = \gamma + \mu\xi k_1 + O(\mu^3) \\
&\gamma = \theta_0(s_*) \qquad \rho = r_0(s_*) \qquad k_1 = \gamma'
\end{aligned}
\tag{52}
$$

we obtain the following nonlinear system in zero approximation:

$$
\begin{aligned}
\rho\dot{U}_0 &= \varepsilon_2^0 \\
\rho\dot{\varepsilon}_2^0 &= \cos\theta^0 - \cos\lambda \\
\dot{M}_1^0 &= U^0\sin\theta^0 - C^0\rho^{-1}\cos\theta^0 \\
\dot{\theta}^0 &= M_1^0 \qquad\qquad (\dot{\ }) \equiv d(/d\xi)
\end{aligned}
\tag{53}
$$

It is known (see Kriegsmann and Lange, 1980; Evkin and Korovaitsev, 1992), that this system has a solution satisfying the boundary conditions

$$
U^0 = \mp\frac{C^0\cos\gamma}{\rho\sin\gamma} \qquad \varepsilon_2^0 = M_1^0 = 0 \qquad \theta^0 = \mp\gamma \qquad \text{as } \xi \to \pm\infty \tag{54}
$$

only if $C^0 = 0$. This solution satisfies the relations

$$
U^0(-\xi) = U^0(\xi) \qquad \varepsilon_2^0(-\xi) = -\varepsilon_2^0(\xi) \qquad M_1^0(-\xi) = M_1^0(\xi) \qquad \theta^0(-\xi) = -\theta^0(\xi) \tag{55}
$$

In the first approximation we get the system

$$
\begin{aligned}
\rho\dot{U}^1 &= \varepsilon_2^1 + g_1 \\
\rho\dot{\varepsilon}_2^1 &= -\theta^1\sin\theta^0 + g_2 \\
\dot{M}_1^1 &= U^1\sin\theta^0 + \left(U^0\cos\theta^0 + C^0\sin\theta^0\right)\theta^1 + g_3 \\
\dot{\theta}^1 &= M_1^1 + g_4
\end{aligned}
\tag{56}
$$

where

$$
\begin{aligned}
g_1 &= -\xi\dot{U}^0\cos\gamma + U^0\left(\nu\cos\theta^0 - \cos\gamma\right) + c_1\varepsilon_2^{0^2} + c_2 M_1^{0^2} \\
g_2 &= -\xi\dot{\varepsilon}_2^0\cos\gamma - \varepsilon_2^0\left(\nu\cos\theta^0 + \cos\gamma\right) + k_1\xi\sin\gamma \\
\rho^{-1}g_3 &= -\nu\varepsilon_2^0 U^0\sin\theta^0 - C^1\cos\theta^0 + M_1^0\left(\nu\cos\theta^0 - \cos\gamma\right)
\end{aligned}
\tag{57}
$$

and the following boundary conditions from formulas (48):

$$
\begin{aligned}
&U^1(-\xi) = -U^1(\xi) \qquad \varepsilon_2^0(-\xi) = \varepsilon_2^0(\xi) \qquad M_1^0(-\xi) = -M_1^0(\xi) - 2m_1(s_*) \\
&\theta^0(-\xi) = \theta^0(\xi) \qquad \text{at} \qquad \xi > 0
\end{aligned}
\tag{58}
$$

The value C_2 may be found from the existence condition for the solution in the second approximation. But it is more convenient to use the energy relations which are obtained below.

8 The Axial Force Calculation

The work done by the axial force is equal to $A = Pz$, where z is the axial displacement of the edge s_2 with respect to the edge s_1.

$$
z = \int_{s_1}^{s_2}\left[(1+\varepsilon_1)\sin\theta - \sin\theta_0\right]ds_0 \tag{60}
$$

In the case of the internal edge effect near the point $s_0 = s_*$ we get

$$
z = -2\int_{s_*}^{s_2}\sin\theta_0 ds_0 + O\left(\mu^2\right) \tag{61}
$$

The term with the factor μ is equal to zero since the function $\theta^0(\xi)$ is odd. We get the equilibrium equation from the equality $\delta\Pi = \delta A$.

$$
\frac{d\Pi}{ds_*} = P\frac{dz}{ds_*} \tag{62}
$$

wherefrom we find the force P. If we use the dimensionless variables (43) in formula (42) we obtain

$$\Pi = 2\pi EhR^2\mu^3\left(\Pi^0 + \mu\Pi^1 + O\left(\mu^2\right)\right) \tag{63}$$

where

$$\Pi^0 = \frac{\rho}{2}\int_{-\infty}^{\infty}\left(\left(\varepsilon_2^0\right)^2 + \left(\dot\theta^0\right)^2\right)d\xi$$
$$\Pi^1 = \rho\int_{-\infty}^{\infty}\left[\frac{1}{2}\left(\left(\varepsilon_2^0\right)^2 + \left(\dot\theta^0\right)^2\right)\xi\cos\gamma + \varepsilon_2^0\varepsilon_2^1 + \dot\theta^0\dot\theta^1 + \dot\theta^0\left(\nu\kappa_2^0 - \theta_0'\right) + \frac{c_1}{3}\left(\varepsilon_2^0\right)^3 + c_2\varepsilon_2^0\left(\dot\theta^0\right)^2\right]d\xi \tag{64}$$

According to relations (55) and (59) most of the terms in the expression for Π^1 are odd and they vanish after integration. As a result we get

$$\Pi^1 = 2\rho\gamma m_1(s_*) = 2\gamma(\rho\gamma' + \nu\sin\gamma) \tag{65}$$

When calculating the derivative $d\Pi^0/ds_*$ it is necessary to have in view that ε_2^0 and θ^0 are functions in $\rho(x_*)$ and in $\gamma(x_*)$. To simplify we exclude the parameter ρ from system (53) by the substitution

$$\xi = \sqrt{\rho}\,\xi^* \qquad U^0 = \frac{U^0}{\rho} \qquad \varepsilon_2^0 = \frac{\varepsilon_2^{0*}}{\sqrt{\rho}} \qquad M_1^0 = \frac{M_1^{0*}}{\sqrt{\rho}} \tag{66}$$

Then

$$\Pi^0 = 2\sqrt{\rho}\int_{-\infty}^{0}\left(\left(\varepsilon_2^{0*}\right)^2 + \left(M_1^{0*}\right)^2\right)d\xi^* \tag{67}$$

and

$$\frac{d\Pi}{ds_*} = 2\frac{a_1}{\sqrt{\rho}}\sin\gamma + 2\sqrt{\rho}\,a_2\gamma'\sin\gamma \qquad\qquad \frac{dz}{ds_*} = 2\sin\gamma$$

where

$$a_1 = \frac{1}{4}\frac{\cos\gamma}{\sin\gamma}\int_{-\infty}^{0}\left(\varepsilon_2^{0*}\right)^2 + \left(M_1^{0*}\right)^2 d\xi^* \qquad \text{and} \qquad a_2 = -\int_{-\infty}^{0}U^{0*}d\xi^* \tag{68}$$

The final expression for the axial force P has the form

$$P(s_*) = \frac{2\pi Eh^2\mu}{\sqrt{12\left(1-\nu^2\right)}}\left(P^1 + \mu P^2 + O\left(\mu^2\right)\right) \tag{69}$$

$$P^1 = \frac{1}{\sqrt{\rho}}\left(a_1 + \rho\gamma' a_2\right) \qquad P^2 = \begin{cases} \dfrac{(\rho\gamma m_1)'}{\sin\gamma} & \text{at } \gamma < \dfrac{\pi}{2} \\[2ex] \dfrac{(\rho(\pi-\gamma)m_1)'}{\sin\gamma} & \text{at } \gamma > \dfrac{\pi}{2} \end{cases} \qquad \text{with } (\ )' \equiv \frac{d}{ds_*}$$

where the coefficients a_1 and a_2 depend only on the angle γ. Their values are found in the paper by Koroteeva, Tovstik and Shuvalkin (1995) by the numerical solution of system (53). For some values $\gamma < \frac{\pi}{2}$ the coefficients are given in Table 1. Only the values of P^1 and P^2 depend on the point s_*.

γ	a_1	a_2	γ	a_1	a_2
5^0	0.00714	0.41076	45^0	0.14806	1.30559
10^0	0.02003	0.58212	50^0	0.16071	1.39607
15^0	0.03627	0.71546	55^0	0.16906	1.48803
20^0	0.05469	0.83023	60^0	0.17201	1.58250
25^0	0.07436	0.93417	65^0	0.16836	1.68055
30^0	0.09440	1.03142	70^0	0.15681	1.78332
35^0	0.11397	1.12458	75^0	0.13592	1.89208
40^0	0.13216	1.21549	80^0	0.10409	2.00831
45^0	0.14806	1.30559	85^0	0.05948	2.13378

Table 1. Coefficients a_1 and a_2

The case $\gamma = \frac{\pi}{2}$ may be reduced to the case considered before, after the substitution $\gamma^o = \pi - \gamma$. As a result we find that

$$P^1(\pi-\gamma) = -P^1(\gamma) \qquad \text{and} \qquad P^2(\pi-\gamma) = P^2(\gamma) \tag{70}$$

As examples we study a conical and a spherical shell.

9 Conical Shell

For a conical shell

$$\theta_0 = \gamma = \text{constant} \qquad r_0 = s_0 \cos\gamma \qquad P^2 = 0 \tag{71}$$

and formula (69) gives

$$P(s_*) = 2\pi E h^2 \sqrt{\frac{h}{12(1-\nu^2)s_* \cos\gamma}} \left(a_1(\gamma) + O\left(\frac{h}{s_*}\right)\right) \tag{72}$$

where s_* is the distance between a cone point and the cone top and the value of a_1 is given in Table 1.

We note that if the wide edge is reflected (see Figure 3a), then $P > 0$ and vice versa.

$$\begin{aligned} P > 0 \quad \text{for} \quad \gamma < \frac{\pi}{2} \quad \text{(see Figure 3a)} \\ P < 0 \quad \text{for} \quad \gamma > \frac{\pi}{2} \quad \text{(see Figure 3b)} \end{aligned} \tag{73}$$

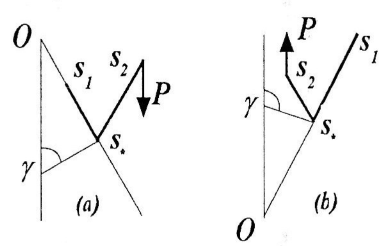

Figure 3. Conical Shell Deformations

In Figure 3 the direction of the force *P* which holds the shell in the reflected state is shown. The absolute value of the force *P* increases simultaneously with the decrease of s_* and with an increase of the angle γ.

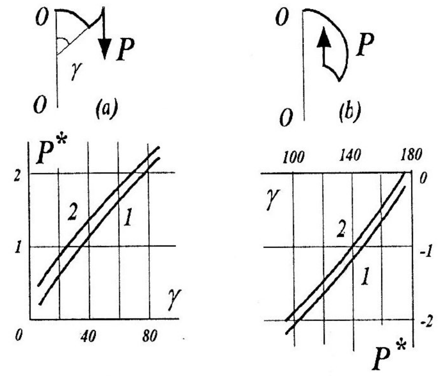

Figure 4. Spherical Shell Deformations and the Corresponding Axial Forces

10 Spherical Shell

For a spherical shell of radius R

$$s_0 = \theta_0 \qquad s_* = \gamma \qquad \rho = \sin\gamma \tag{74}$$

$$P^1 = \frac{1}{\sqrt{\sin\gamma}}(a_1 + a_2 \sin\gamma) \quad \text{and} \quad P^2 = (1+\nu)\left(1+\frac{\gamma\cos\gamma}{\sin\gamma}\right) \quad \text{at} \quad \gamma < \frac{\pi}{2} \tag{75}$$

Let $\nu = 0.3$, $R/h = 100$. The graphics of the functions $P^* = P^1$ (curves 1) and $P^* = P^1 + \mu P^2$ (curves 2) versus γ are shown in Figure 4 for $\gamma < \frac{\pi}{2}$ (Figure 4a) and for $\gamma > \frac{\pi}{2}$ (Figure 4b). The correction term of the order μ appreciably refines the value of the force P especially for those values of γ which are close to 0 or to 180°.

In a paper by Evkin and Korovaitsev (1992) the post-buckling deformations of a spherical shell under an external normal pressure p are studied. It was found that the pressure p is minimal for $\gamma \cong 110^0$. In our problem the axial force P changes monotonically as the angle γ increases.

10 Discussion

Approximate elasticity relations (36) and expression (42) for the potential energy of a shell of revolution made of nonlinear elastic material are obtained for a 5-constant material (16) under the assumption that the deformations are relatively small (have the order μ) and the coefficients α_j of the third order have the order of Young's modulus E. For the deformations of the same order μ the same relations (36) and (42) are also valid for an arbitrary nonlinear elastic material assuming that the coefficients of the fourth and subsequent orders do not exceed essentially the value of E.

Formulas (36) are obtained assuming that a lateral surface loading is absent. It is simple to verify that relations (36) are valid also for the lateral loading $p_j^{\pm} = O(\mu^3)$ with the same error of the order of μ^2. We note that shell bifurcation in the linear approximation occurs for sufficiently smaller loading.

Expression (69) for the axial force under post-buckling shell deformations contains parameters which describe the initial shell neutral surface. The terms of the second order in the nonlinear elasticity relations (the terms with the factors c_1 and c_2 in formula (38)) are not included in this expression since some functions in formula (64) are odd. But these terms are important when we seek an upper limit load.

The bifurcation into a non-symmetric mode may precede the post-buckling axisymmetric deformations. In particular bifurcation occurs for shells with negative Gaussian curvature since for such shells bifurcation takes place for $P = O(Eh^2\mu^{2/3})$ (see Tovstik, 1995). The solution of system (46) may be used in the study of the bifurcation problem.

The construction of a linear two-dimensional thin shell theory based on the three-dimensional theory of elasticity has been investigated in many papers (see Goldenveizer, Kaplunov, and Nolde, 1993; Goldenveizer, 1994).

The direct application of three-dimensional nonlinear theory for large shell deflections is discussed by Sun, Yeh, and Rimrott (1995).

For the large deformations (of the order of unity) the accurate derivation of the shell elasticity relations from the three-dimensional equations of the theory of elasticity is apparently absent. There are the elasticity relations (see Chernykh, 1986) obtained by using hypotheses similar to the Kirchhoff-Love ones. For small (of the order of μ) deformations the comparison with formulas (36) gives a difference in the nonlinear terms.

The system of nonlinear equations similar to (34) which describes the axisymmetric deformations of shells of revolution are given in many papers (see Reissner, 1950; Akselrad, 1976; Valishvili, 1976; etc.). Some terms in these equations differ from another and from equations (34) by the multipliers $1+\varepsilon_1$ or $1+\varepsilon_2$ close to unity and also by the elasticity relations.

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