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Random Vibrations of Elastic-plastic Structures

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The basic finite element method solutions for the description of bar structure random vibrations are obtained on the basis of small elasto-plastic strains theory and the piecewise approximation of the hysteresis loop. The stochastic dynamics problem solution was carried out for the stationary case. Executed calculations demonstrate the efficacy of the ordered methodology.

1 Introduction

For large machine tool frames a random cyclic input and the occurrence of local plastic strains in consequence of this input are a typical attribute. In this case in order to calculate the low-cycle strength it is necessary to know the mechanism of the cyclic deformation of the frame material. The reasons which make these calculations difficult include the provision of a correct description of the elasto-plastic deformation diagram in the form of some hysteresis loop and the subsequent solution of the nonlinear stochastic dynamics problem. In this work the piecewise approximation of cyclic deformation is used, because it is possible to take into account all peculiarities of the diagram and the possibility of its change for different load cycles. For the solution of random nonlinear vibrations the Markov process theory together with the finite element method (FEM) are used.

2 The Main Hypotheses

Solution of the stochastic dynamics problem of bar structures subject to spatial flexural-longitudinal-torsional vibrations is considered in view of random loading and the advent of local elasto-plastic strains. The discrete model of the structure is based on finite elements having 12 degrees of freedom. Finite element and positive directions of generalized unit displacements vector components $\mathbf{Y}(t)$ are shown in Figure 1. Models of elastic supports consist of a 6 stiffnesses (3 forces and 3 moments) and a mass. For the finite element chosen the stiffness matrix is constructed by using the dynamic stiffness matrices of individual elements at elastic strains. In the case of the beginning of local elasto-plastic strains it is supposed that

- the plastic strains are local, that natural frequencies and modes with elasto-plastic strains are only slightly different from those of the elastic structure;
- the construction material is cyclic stable and the diagram of cyclic deformation has a form as shown in Figure 2;
- only a single mode is excited.

3 Derivation of Elasto-plastic Structures Vibration Equations

For the notation of the structural vibration equations with elasto-plastic strains a conventional FEM procedure is used, in which there are physical relations based on the theory of small elasto-plastic strains (Pisarenko and Mozharovsky, 1981) with nonlinear relations (in form of a hysteresis loop as in Figure 2) between stress and strain intensities. For an explanation of this nonlinear relation, piecewise approximation is used and written as

$$\sigma_x = E\,(y,z)\varepsilon_x + B\,(y,z) \tag{1}$$

where

$$\varepsilon_x = \varepsilon_{x0} + y\kappa_z + z\kappa_y = \varepsilon_A \sin\psi \tag{2}$$

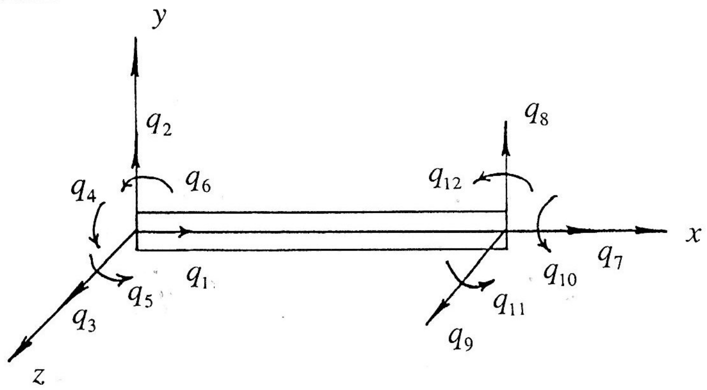


Figure 1. The Finite Element and Positive Directions of Generalized Unit Displacements Vector Components **Y**(t)

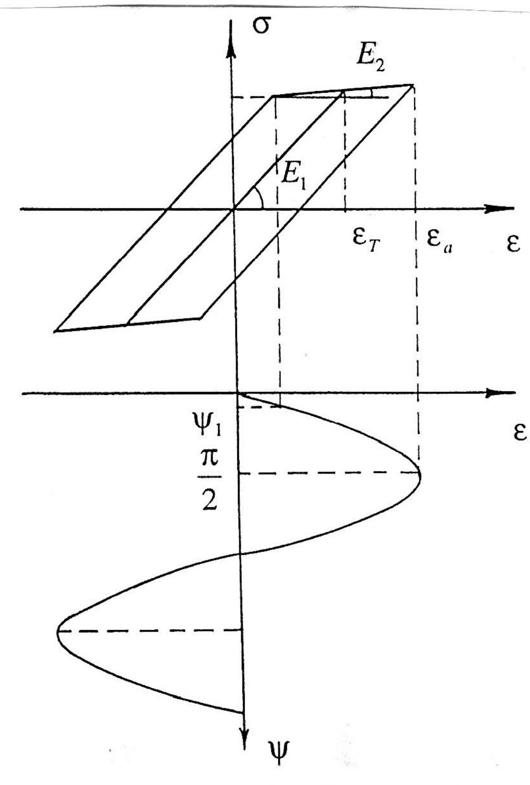


Figure 2. Diagram of Cyclic Deformation

$$E = \begin{cases} E_1 \\ E_2 \\ E_1 \end{cases} \qquad E = \begin{cases} (E_1 - E_2)(\varepsilon_A - \varepsilon_T) & 0 < \psi < \psi_1 \\ \sigma_T - E_2 \varepsilon_T & \psi_1 < \psi < \pi / 2 \\ (E_1 - E_2)(\varepsilon_A - \varepsilon_T) & \end{cases} \tag{3}$$

In equation (3), ψ_1 represents the value of phase ψ according to point 1 of the hysteresis loop (Figure 2). From equations (1) to (3) the components of generalized unit force and strain vectors are defined as

$$\begin{aligned} N_x &= \int_F \sigma_x \, dF = \int_F E \left(\varepsilon_{x0} + y\kappa_z + z\kappa_y \right) dF + \int_F B \, dF \\ &= E_1 \left[\varepsilon_{x0} F^* + \kappa_z S_z^* + \kappa_y S_z^* \right] + N_x^* \end{aligned} \tag{4}$$

where

$$F^* = \int_F \frac{E}{E_1}\, dF \qquad S_z^* = \int_F \frac{E}{E_1}\, ydF \qquad S_y^* = \int_F \frac{E}{E_1}\, zdF \qquad N_x^* = \int_F B\; dF \tag{5}$$

Expressions for bending moments and for the torque are obtained similarly.

$$\begin{Bmatrix} M_y \\ M_z \end{Bmatrix} = \int_F \sigma_x \begin{Bmatrix} y \\ z \end{Bmatrix} dF = \int_F E\left(\varepsilon_{x0} + y\kappa_z + z\kappa_y\right)\begin{Bmatrix} y \\ z \end{Bmatrix} dF + \int_F B \begin{Bmatrix} y \\ z \end{Bmatrix} dF$$

$$= E_1 \left[\varepsilon_{x0} \begin{Bmatrix} S_y^* \\ S_z^* \end{Bmatrix} + \kappa_z \begin{Bmatrix} I_{yz}^* \\ I_z^* \end{Bmatrix} + \kappa_y \begin{Bmatrix} I_y^* \\ I_{yz}^* \end{Bmatrix} \right] + \begin{Bmatrix} M_y^* \\ M_z^* \end{Bmatrix} \tag{6}$$

where

$$I_{yz}^* = \int_F \frac{E}{E_1}\, yzdF \qquad I_y^* = \int_F \frac{E}{E_1}\, z^2 dF \qquad M_y^* = \int_F BzdF$$

$$I_z^* = \int_F \frac{E}{E_1}\, y^2 dF \qquad M_z^* = \int_F BydF \tag{7}$$

$$M_x = \tau W_p = GW_p \gamma_{yz} = \frac{E_1 I_p}{2(1+\nu)r}\, \gamma_{yz} \tag{8}$$

According to equations (4) to (8) the relationship between generalized forces and strains is defined as

$$\boldsymbol{\sigma} = \mathbf{D}^* \boldsymbol{\varepsilon} + \boldsymbol{\sigma}^* \tag{9}$$

where

$$\mathbf{D}^* = \begin{bmatrix} F^* & 0 & S_y^* & S_z^* \\ 0 & \dfrac{I_p}{2(1+\nu)r} & 0 & 0 \\ S_y^* & 0 & I_y^* & I_{yz}^* \\ S_z^* & 0 & I_{yz}^* & I_z^* \end{bmatrix} \qquad \boldsymbol{\sigma}^* = \left[N_x^*, 0, M_y^*, M_z^* \right] \tag{10}$$

In equation (10) elements of the matrix $\mathbf{D}^*$ are an area, static moments and section inertia moments in view of plastic deformation. They are functions of the coordinate x, the strain amplitude ε_A and the phase ψ. In the case of an elastic deformation the matrix $\mathbf{D}^*$ is diagonal and its elements are an area and section inertia moments. The vector $\boldsymbol{\sigma}^*$ is a vector of an additional generalized force resulting from the plastic strains in the bar section. After substituting equation (9) into the expression for potential energy of a bar element we have

$$\mathbf{U} = \frac{1}{2}\int_0^L \mathbf{Y}^T \mathbf{R}^T \mathbf{D}^* Py\; dx + \int_0^L \mathbf{Y}^T \mathbf{R}^T \boldsymbol{\sigma}^*\, dx \tag{11}$$

In this relation the matrix $\mathbf{R}$ describes the constraint between the strain vector $\boldsymbol{\varepsilon}$ and the unit displacement vector $\mathbf{Y}$ and it is obtained similarly under an elastic deformation (Gallagher, 1975).

$$\boldsymbol{\varepsilon} = \mathbf{RY} \tag{12}$$

Minimizing potential energy with respect to unit displacements, we obtain expressions for stiffness matrix $\mathbf{K}_e$ and elemental additional unit force vector $\mathbf{X}_e^*$, which are functions of strain amplitude ε_A and vibration phase ψ.

$$\mathbf{K}_e = \int_0^L \mathbf{R}^T \mathbf{D}^* \mathbf{R}\, dx \tag{13}$$

$$\mathbf{X}_e^* = \int_0^L \mathbf{R}^T \boldsymbol{\sigma}^*\, dx \tag{14}$$

The derivations of the mass matrix $\mathbf{M}_e$ and the damping matrix $\mathbf{C}_e$ are done similar to the one for solving the problem of bar element vibration in an elastic arrangement (Postnov und Harhurim, 1974). On the basis of the elemental mass, stiffness and damping matrices and using the conventional procedure (Postnov and Harhurim, 1974), we can obtain the frame stiffness matrix in global coordinates and the equation of the bar structure forced vibrations subject to plastic deformation

$$\mathbf{M}\ddot{\mathbf{Y}}(t) + \mathbf{C}\dot{\mathbf{Y}}(t) + \mathbf{K}(\mathbf{Y},\dot{\mathbf{Y}},t)\mathbf{Y}(t) = \mathbf{X}(t) + \mathbf{X}^*(\mathbf{Y},\dot{\mathbf{Y}},t) \tag{15}$$

In equation (15) the stiffness matrix $\mathbf{K}$ is a nonlinear function of the unit displacement vector and its derivative (or of the strain amplitude ε_A and vibration phase ψ, on account of linear function (12) between ε and $\mathbf{Y}$).

4 The Solution of the Stochastic Dynamics Problem

Transforming the stiffness matrix $\mathbf{K}$ into $\mathbf{K} = \mathbf{K} + \mathbf{K}_L - \mathbf{K}_L$, where $\mathbf{K}_L$ is the elastic stiffness matrix, equation (15) may be written as

$$\mathbf{M}\ddot{\mathbf{Y}}(t) + \mathbf{C}\dot{\mathbf{Y}}(t) + \mathbf{K}_L\mathbf{Y}(t) = \mathbf{X}(t) - \left[\mathbf{K}(\mathbf{Y},\dot{\mathbf{Y}},t) - \mathbf{K}_L\right]\mathbf{Y}(t) + \mathbf{X}^*(t) \tag{16}$$

On account of the adopted hypothesis about the initiation of a single mode the unit displacement vector may be defined as

$$\mathbf{Y}(t) = \boldsymbol{\Phi}\, q(t) \tag{17}$$

where $\boldsymbol{\Phi}$ and $q(t)$ are a natural waveform vector and a generalized coordinate of the structure and are in agreement with the natural frequency Ω. Substitution of equation (17) into equation (16) and multiplication from the left by vector $\boldsymbol{\Phi}$ produce the nonlinear equation for the generalized coordinate $q(t)$.

$$\ddot{q}(t) + 2\varepsilon\Omega\dot{q}(t) + \Omega^2 q(t) = \boldsymbol{\Phi}^T\left[\mathbf{X}(t) - (\mathbf{K} - \mathbf{K}_L)\boldsymbol{\Phi}\, q(t) + \mathbf{X}^*(t)\right] \tag{18}$$

Let us introduce the symbols for the stationary broadband random process $\chi(t)$, which is proportional to a small parameter μ.

$$\chi(t) = \boldsymbol{\Phi}^T\mathbf{X}(t) \tag{19}$$

and for the function $h(q,\dot{q},t)$, having μ^2 of infinitesimal order,

$$h(q,\dot{q},t) = 2\varepsilon\Omega\dot{q} + \boldsymbol{\Phi}(t)\left\{\left[\mathbf{K}(q,\dot{q},t) - \mathbf{K}_L\right]\boldsymbol{\Phi} q - \mathbf{X}^*\right\} \tag{20}$$

Then equation (18) becomes

$$\ddot{q} + \Omega^2 q = -h(q,\dot{q},t) + \chi(t) \tag{21}$$

For the initiating coordinate $q(t)$ the last of equations (21) is the nonlinear stochastic differential equation with the broadband random input $\chi(t)$, whose correlation function $K_\chi(\tau)$ and the spectral density $S_\chi(\omega)$ are obtained by certain probability performances of the unit forces vector $\mathbf{X}(t)$ according to relation (19). The solution of the stochastic dynamics problem for equation (21) is carried out on the basis of Krylov-Bogolubov's averaging principle in combination with the methods of the Markov process theory (Dimentberg, 1980). For this solution it is nesessary to move to the new „slow“ variables - the amplitude $A(t)$ and the phase $\varphi(t)$ - according to relations

$$q(t) = A(t)\sin(\Omega t + \varphi(t)) = A(t)\sin\psi(t)$$
$$\dot q(t) = \Omega A(t)\cos\psi(t) \qquad \psi(t) = \Omega t + \varphi(t) \tag{22}$$

Using relations (22) and (21), we can obtain two first order equations for the „slow“ variables A and φ (Dimentberg, 1980).

$$\dot A(t) = \Omega^{-1}\cos\psi\left[-h(A,\varphi,t)+\chi(t)\right]$$
$$\dot\varphi(t) = -(\Omega A)^{-1}\sin\left[-h(A,\varphi,t)+\chi(t)\right] \tag{23}$$

Let

$$g_1(\varphi,t) = \Omega^{-1}\cos\psi \qquad V_1(A,\varphi,t) = -g_1(\varphi,t)h(A,\varphi,t)$$
$$g_1(\varphi,t) = -(\Omega A)^{-1} \qquad V_2(A,\varphi,t) = -g_2(\varphi,t)h(A,\varphi,t) \tag{24}$$

then system (23) may be written as

$$\dot A(t) = V_1(A,\varphi,t) + g_1(\varphi,t)\chi(t)$$
$$\dot\varphi(t) = V_2(A,\varphi,t) + g_2(\varphi,t)\chi(t) \tag{25}$$

If the small parameter μ tends to zero, $A(t)$ and $\varphi(t)$ may be considered as a two-dimentional diffusional Markov process with the coefficients of drift κ_i and diffusion κ_{ii} (Dimentberg, 1980).

$$\kappa_i = \frac{1}{T}\left[\int_0^T V_i(A,\varphi,t)dt + \int_0^T dt\int_0^\infty \frac{\partial g_i(A,\varphi,t)}{\partial u_i} g_j(A,\varphi,t+s)\,K_\chi(s)\,ds\right]$$
$$\kappa_{ii} = \frac{1}{T}\left[\int_0^T dt\int_0^\infty g_i(A,\varphi,t)g_i(A,\varphi,t)K_\chi(s-t)\,ds\right] \qquad (i=1,2) \tag{26}$$
$$\kappa_{12} = \kappa_{21} = 0 \qquad u_1 = \varphi_1 \qquad \varphi_2 = A \qquad T = 2\pi/\Omega$$

Making the computations with equations (24) and (26), we can write

$$\kappa_1(A) = -\frac{1}{2\pi\Omega}\int_0^{2\pi} h(A,\psi)\cos\psi\,d\psi + \frac{S_0}{2\Omega^2 A}$$
$$\kappa_2(A) = \frac{1}{2\pi\Omega A}\int_0^{2\pi} h(A,\psi)\sin\psi\,d\psi \qquad \kappa_{11} = \frac{S_0}{2\Omega^2} \qquad \kappa_{22} = \frac{S_0}{2\Omega^2 A^2} \tag{27}$$

where $S_0 = 2\pi S_\chi(\Omega)$ is the intensity of transforming white noise. Using the coefficients κ_i, κ_{ii} it is possible to write the Fokker-Planck-Kolmogorov (FPK) equation for the transition probability $f(A,\varphi,t)$ using a two-dimensional Markov process $[A(t),\varphi(t)]$

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial A}[\kappa_1(A)f] - \frac{\partial}{\partial\varphi}[\kappa_2(A)f] + \frac{\kappa_{11}}{2}\frac{\partial^2 f}{\partial A^2} + \frac{1}{2}\frac{\partial^2}{\partial\varphi^2}[\kappa_{22}(A)f] \tag{28}$$

In equation (28) the coefficients κ_i and κ_{ii} are functions of the amplitude A only, because the integration of equation (28) for variable p going from 0 to 2π yields the FPK equation for the stationary one-dimensional transition density of amplitude $A(t)$.

$$\frac{d}{dA}[\kappa_1(A)f(A)] = \frac{\kappa_{11}}{2}\frac{\partial^2 f}{\partial A^2} \tag{29}$$

The plastic strains in the frame begin after the amplitude $A(t)$ runs into the value A^* obtained from the condition of attainment of stress intensity of yield limit. During the change in $A(t)$ going from 0 to A^* elastic strains remain in a structure. The coefficient $\kappa_i(A)$ for different amplitude values may be written as

$$\kappa_1(A) = \begin{cases} -\dfrac{1}{2\pi\Omega}\displaystyle\int_0^{2\pi} 2\varepsilon A\Omega^2\cos^2\psi\ d\psi = \varepsilon A\Omega & 0 \le A(t) < A^* \\ -\dfrac{1}{2\pi\Omega}\displaystyle\int_0^{2\pi} h(A,\psi)\cos\psi\ d\psi + \dfrac{S_0}{4\Omega^2 A} & A(t) \ge A^* \end{cases} \tag{30}$$

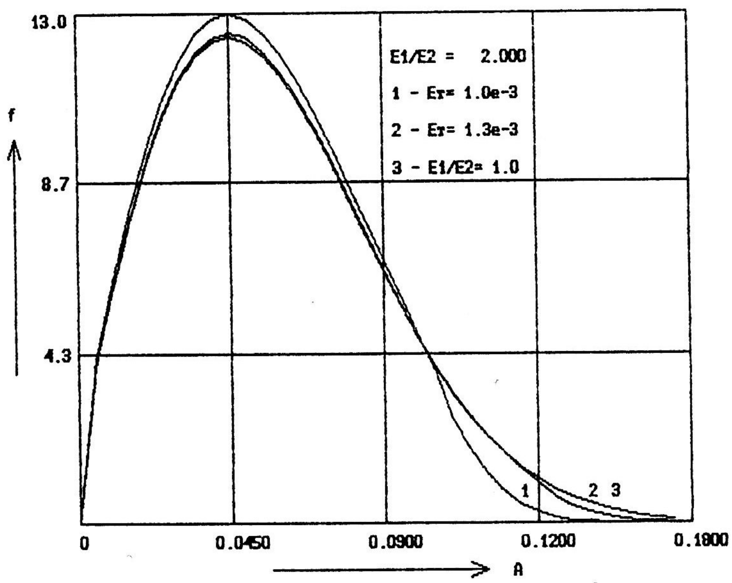
In view of expression (30) and equation (20) the solution of equation (29) may be written as

$$f(A) = \begin{cases} C_1 A\exp\left[-\dfrac{2\Omega^2\varepsilon}{S_0}A^2\right] & 0 \le A(t) < A^* \\ C_2 A\exp\left[-\dfrac{2\Omega^2\varepsilon}{S_0}A^2 + \dfrac{1}{\pi}\displaystyle\int_{A^*}^{A}\int_0^{\pi}\mathbf{\Phi}^T\left[\mathbf{K}(A,\psi)\mathbf{\Phi}A\sin\psi - \mathbf{X}^*(A,\psi)\right]\cos\psi\ d\psi\ dA\right. & A(t) \ge A^* \end{cases} \tag{31}$$

In equation (31) the constants C_1 and C_2 are obtained from conjugate conditions in the point $A = A^*$ and from a probability density normalizing condition.

5 The Calculations

On the basis of the worked out methodology a random vibration calculation program was created in view of the elasto-plastic properties of a material. Test calculations were carried out for a beam with hinges at the ends under a uniformly distributed load. The length of the beam is 0.3 m, the cross section is 0.02 m², Young´s modulus $E_1 = 2.01{*}10^{11}$ Pa, the density $\rho = 7.8{*}10^3$ kg/m^3. Calculations show that 8 elements are sufficient for the attainment of the required accuracy. Distributions *P*(*A*) were taken for different values of the yield limit ε_T and different variants of ratio E_1/E_2. Several *P*(*A*) plots are shown in Figure 3. The ratio $E_1/E_2 = 1$ according to the elastic problem solution with Rayleigh´s density. The probability density of the generalized coordinate amplitude of the turbine plant K-550-6.5 oilpipe (Figure 4) is shown in Figure 5. This plot is for the first natural frequency. The natural mode for this frequency is shown on Figure 4. Pipe length was 3.61 m, pipe diameter 0.219 m, wall thickness 0.009 m, material density $\rho = 7.8{*}10^3\,\mathrm{kg/m^3}$. The region of maximum plastic strains is marked by a dotted line.



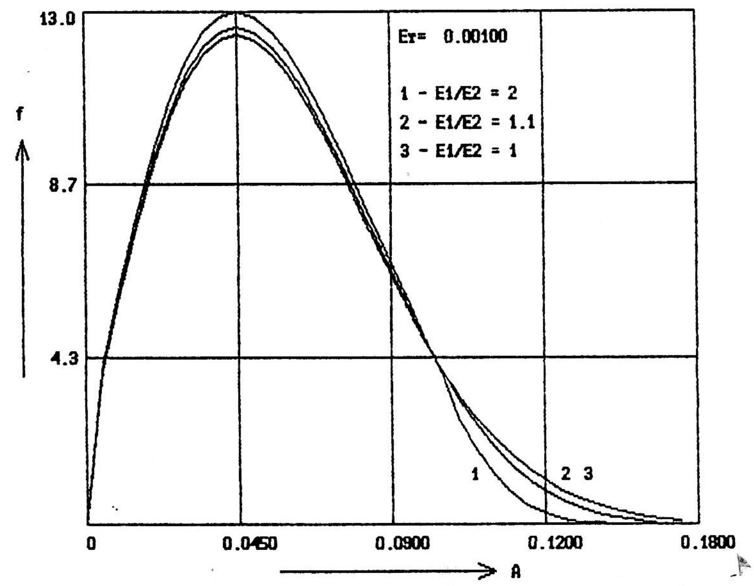


Figure 3. The Probability Density of Beam Generalized Coordinate Amplitude for Different Parameters of Hysteresis Loop

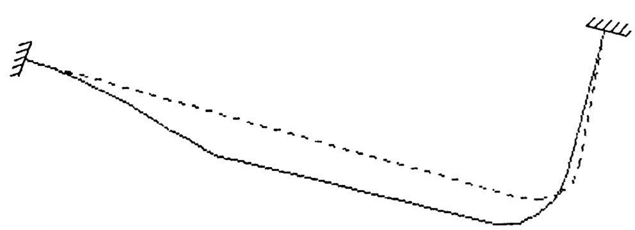


Figure 4. The Model and Natural Mode of the Turbine Plant Oil Pipe

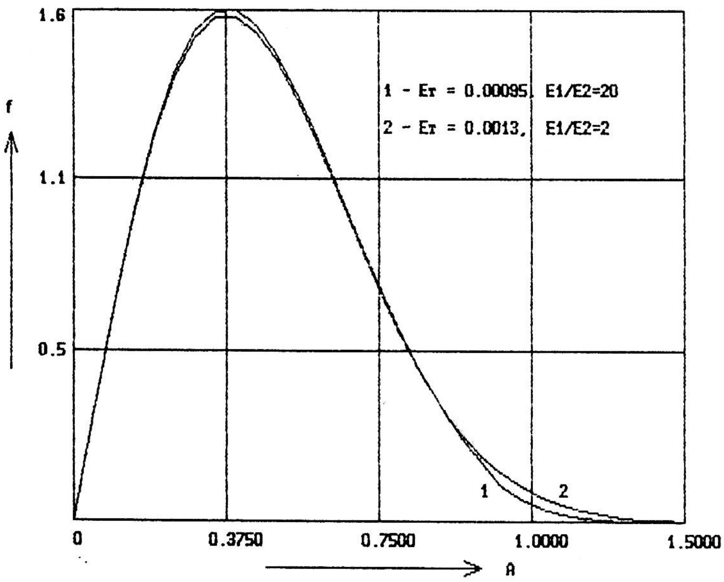


Figure 5. The Probability Density of Pipe Generalized Coordinate Amplitude for Different Parameters of Hysteresis Loop

6 Conclusions

The stochastic dynamics problem solution methodology considered allows the study of effects caused by local plastic strain, and calculations with different hysteresis loops. The analysis has shown that for the range of hysteresis loop parameters investigated, the variation the probability density of the generalized coordinate amplitude is appreciably different from Rayleigh's density and there is a considerable decrease of strain peak amplitude.

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