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Free Vibrations of a Non-uniformly Heated Viscoelastic Cylindrical Shell

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Low-frequency free vibrations of a viscoelastic cylindrical shell with free edges taking into account a non-uniform temperature field is investigated. Using Tovstik's asymptotic method, the solution of the shell equations are derived in the form of functions, quickly oscillating and damping near "the coldest" generatrix.

1 Introduction

We will study low-frequency free vibrations of a non-uniformly heated viscoelastic cylindrical shell. Temperature stresses are supposed to be absent here, and a relaxation kernel is assumed to be some function of the non-uniform temperature.

The characteristic property of the problem under consideration is the localization of vibration modes in a vicinity of "the weakest generatrix", which is "the coldest one" here. For the first time, the localization of vibration modes of an elastic noncircular cylindrical shell with slanted edges was studied by Tovstik (1983). Later viscoelastic cylindrical shells were investigated by Mikhasev (1992).

2 Problem Setup

Consider a thin circular cylindrical shell of constant length l and thickness h. We introduce an orthogonal coordinate system s, φ, so that the first quadratic form of the middle surface takes the form $R^2\left(ds^2+d\varphi^2\right)$, where R is the radius of the middle surface of the shell, and s and φ are the axial and circumferential coordinates, respectively. The shell material is assumed to be linearly viscoelastic with instantaneous Young's modulus E and Poisson's ratio ν.

Suppose the shell edges are free, and the temperature distribution in the shell is

$$T = T_0 + \mu\ (1-\cos\varphi)$$

where $0<\mu$, T_0 are constants. Then initial temperature stresses are missing (Podstrigach and Schvets, 1978), and for analysis of the lowest part of the spectrum of free vibrations, the following basic equations, written in dimensionless form, can be used:

$$\begin{gathered}
\varepsilon^4\Delta^2\left[W-\int_{-\infty}^{t}\tilde{K}(t-\tau,T(\varphi))W(\tau)d\tau\right]+\frac{\partial^2\Phi}{\partial s^2}+\frac{\partial^2 W}{\partial t^2}=0 \\
\varepsilon^4\Delta^2\Phi-\frac{\partial^2}{\partial s^2}\left[W-\int_{-\infty}^{t}\tilde{K}(t-\tau,T(\varphi))W(\tau)d\tau\right]=0
\end{gathered}\tag{1}$$

$$\Delta=\partial^2/\partial s^2+\partial^2/\partial\varphi^2 \qquad \varepsilon^8=h^2/\left[12R^2(1-\nu^2)\right]$$

$$W=\varepsilon^4R^{-1}W^* \qquad \Phi=\Phi^*/(\varepsilon^4Eh) \qquad t=t'/t_c \qquad t_c=\rho R^2/(\varepsilon^4E)$$

Here W^*, Φ^* are the normal deflection and the stress function, respectively, ρ is the mass density, ε is a natural small parameter, $\tilde{K}(t, T(\varphi))$ is the relaxation kernel of the shell material, and t_c is the characteristic time.

According to the temperature-time analogy (Ferry, 1963), the relaxation kernel may be represented in the form

$$\tilde{K}(t, T(\varphi)) = K(t'(T(\varphi)) \tag{2}$$

where

$$t'(T(\varphi)) = \int_0^t \frac{d\tau}{a_T(T(\varphi))} = \frac{t}{a_T(T(\varphi))} \tag{3}$$

is the reduced time and $a_T(T(\varphi))$ is the coefficient of temperature-time reduction. For many polymers in a large range of temperatures the function $a_T(T(\varphi))$ obeys the William- Landel - Ferry formula (Ferry, 1963).

$$\ln a_T(T(\varphi)) = -\frac{c_1^g(T(\varphi) - T_g)}{c_2^g + T(\varphi) - T_g}$$

where T_g is the reduction temperature, and the coefficients c_1^g and c_2^g are derived experimentally. It is assumed $T_g = T_0$ here.

3 Method of Solution

We shall investigate the lowest part of the spectrum of vibrations, corresponding to the main stress-strain state of the shell (Tovstik, 1983). Then, for the free edges of the shell, the main boundary conditions will take the form

$$\frac{\partial^2 W}{\partial s^2} = \frac{\partial^3 W}{\partial s^3} = 0 \qquad \text{for } s = 0, l \tag{4}$$

The approximate solution of the boundary value problem (1), (4) can be expressed as (Tovstik, 1983; Mikhasev, 1992)

$$W = W_\Sigma(s,\xi)\exp\left\{i\left[\Omega t + \varepsilon^{-1/2} p\xi + \frac{1}{2} b\xi^2\right]\right\}$$

$$W_\Sigma = \sum_{n=0}^{\infty} \varepsilon^{1/2} w_n(s,\xi) \qquad \xi = \varepsilon^{-1/2}(\varphi - \varphi_0) \qquad \operatorname{Im} b \rhd 0 \qquad \operatorname{Im} \Omega \rhd 0 \tag{5}$$

where $\varphi = \varphi_0$ is the weakest generatrix, $w_n(s,\xi)$ are polynomials in ξ. The function **Φ** is found in the same form. The last inequalities guarantee attenuation of the wave amplitudes far from the line $\varphi = \varphi_0$ and damping of vibrations in the course of time.

The required complex frequency Ω and the temperature $T(\varphi)$ can be expanded into the series

$$\Omega = \Omega_0 + \varepsilon\Omega_1 + \varepsilon^2\Omega_2 + \ldots$$
$$T(\varphi) = T_0 + \mu(1 - \cos\varphi_0 + \varepsilon^{1/2}\xi\sin\varphi_0 + \frac{1}{2}\varepsilon\xi^2\cos\varphi_0 + \ldots) \tag{6}$$

The substitution of equations (5) and (6) into equations (1) and (4) produces a sequence of boundary value problems

$$\sum_{j=0}^{n} L_j w_{n-j} = 0 \qquad \text{n} = 0,\ 1,\ 2... \tag{7}$$

$$\frac{\partial^2 w_n}{\partial s^2} = \frac{\partial^3 w_n}{\partial s^3} = 0 \qquad \text{for } s = 0,\ l \tag{8}$$

where

$$L_0 = \frac{(1-C_0)}{p^4}\frac{\partial^4}{\partial s^4} + [p^4(1-C_0) - \Omega_0{}^2] \qquad L_1 = b\frac{\partial L_0}{\partial p}\xi + \frac{\partial L_0}{\partial \varphi_0}\xi - i\frac{\partial L_0}{\partial p}\frac{\partial}{\partial \xi}$$

$$L_2 = \frac{1}{2}\left(b^2\frac{\partial^2 L_0}{\partial p^2} + 2b\frac{\partial^2 L_0}{\partial p \partial \varphi_0} + \frac{\partial^2 L_0}{\partial \varphi_0{}^2}\right)\xi^2 - i\left(b\frac{\partial^2 L_0}{\partial p^2} + \frac{\partial^2 L_0}{\partial p \partial \varphi_0}\right) - \xi\frac{\partial}{\partial \xi} - \frac{1}{2}\frac{\partial^2 L_0}{\partial p^2}\left(iby + \frac{\partial^2}{\partial \xi^2}\right) -$$

$$-\frac{i}{2}\frac{\partial^2 L_0}{\partial p \partial \varphi_0} + \Omega_1\frac{\partial L_0}{\partial \Omega_0}$$

$$C_0 = C_0(\varphi_0;\Omega_0) = \int_0^{+\infty} K\left(\frac{t}{a_T(T(\varphi_0))}\right) e^{-i\Omega_0 t}\,dt$$

4 Zeroth- , First- and Second-order Approximations

The solution of problem (7), (8) for $n = 0$ is easily seen to be

$$w_0 = P_0(\xi)\, y_m(s) \tag{9}$$

under the condition

$$H(p,\varphi_0;\Omega_0) \equiv \lambda_m^4(1-C_0) - \Omega_0^2 p^4 + (1-C_0)p^8 = 0 \tag{10}$$

where P_0 is a polynomial in ξ,

$$y_m(s) = \left[S_K(\lambda_m s) - \frac{T_K(\lambda_m l)}{U_K(\lambda_m l)} T_K(\lambda_m s)\right]$$

$$S_K(x) = \frac{1}{2}(\cosh x + \cos x)$$

$$T_K(x) = \frac{1}{2}(\sinh x + \sin x)$$

$$U_K(x) = \frac{1}{2}(\cosh x - \cos x)$$

and λ_m is a root of the transcendental equation

$$\cosh(\lambda l)\cos(\lambda l) - 1 = 0$$

Separating in equation (10) the real and imaginary parts , we obtain

$$\frac{\omega_0{}^2-\alpha_0{}^2}{1-B_0}=\frac{2\alpha_0\omega_0}{A_0} \tag{11}$$

$$\frac{\omega_0{}^2-\alpha_0{}^2}{1-B_0}=\frac{\lambda_m^4}{p^4}+p^4 \tag{12}$$

where $\omega_0=\operatorname{Re}\Omega_0$, $\alpha_0=\operatorname{Im}\Omega_0$, $B_0=\operatorname{Re}C_0$, $A_0=-\operatorname{Im}C_0$. It follows from equations (11) and (12) that $\alpha_0=g(p,\varphi_0)$, $\omega_0=f(p,\varphi_0)$. The minimization of the last function over p and φ_0 yields

$$\omega_0^o=\min f(p,\varphi_0)=f\left(p^o,\varphi_0^o\right) \tag{13}$$

$$\alpha_0^o=g\left(p^o,\varphi_0^o\right)$$

where

$$p^o=\lambda_m^{1/2} \qquad \varphi_0^o=0 \tag{14}$$

Thus, the weakest line $\varphi_0=0$ is the coldest one here.

It is of interest to note that

$$\omega_e^0=\sqrt{\left(p^o\right)^4+\frac{\lambda_m^4}{\left(p^o\right)^4}}=\sqrt{2}\,\lambda_m$$

is the lowest frequency of the elastic shell vibrations corresponding to the eigenvalue λ_m .

Taking into account equations (13) and (14), equation (7) is reduced to the identity $L_1 w_0\equiv 0$. Then the solution of problem (7), (8) for n = 1 can be represented in the form

$$w_1=P_1(\xi)y_m(s) \tag{15}$$

where $P_1(\xi)$ is a polynomial in ξ again.

The compatibility condition of boundary value problem (7) and (8) for n = 2 gives the equation

$$\frac{1}{2}\left(\frac{\partial^2H^o}{\partial p^2}b^2+\frac{\partial^2H^o}{\partial\varphi_0^2}\right)\xi^2P_0-ib\frac{\partial^2H^o}{\partial p^2}\left(\frac{1}{2}P_0+\xi\frac{\partial P_0}{\partial\xi}\right)-\frac{1}{2}\frac{\partial^2H^o}{\partial p^2}\frac{\partial^2P_0}{\partial^2\xi}-$$
$$-2\Omega_1\lambda_m^2\left(\Omega_0^o+\lambda_m^2\frac{\partial C_0^o}{\partial\Omega_0}\right)P_0=0 \tag{16}$$

The mark o in equation (16) and below means that all functions are calculated at $p=\lambda_m^{1/2}$, $\varphi_0=0$, $\omega_0=\omega_0^o$, $\alpha_0=\alpha_0^o$. Equation (16) has the solution $P^o(\xi)=a_N\xi^N+\ldots+a_1\xi+a_0$ if

$$b^2=-\frac{\partial^2H^o/\partial\varphi_0^2}{\partial^2H^o/\partial p^2} \tag{17}$$

Then

$$\Omega_1 = \frac{i(N+1/2) b \partial^2 H^o / \partial p^2}{2\lambda_m^2 \left(\Omega_0^o + \lambda_m^2 \partial C_0^o / \partial \Omega_0\right)} \tag{18}$$

5 Example

Numerical computations for a shell made from a polymer material , with $c_1^g = 18{,}1$, $c_2^g = 45$, $T_g = 276$ K (Ferry, 1963), and the kernel (Rzhanitsyn, 1968)

$$K(t) = \frac{0{,}4\, e^{-t}}{\Gamma(0{,}1)\, t^{0,9}} \tag{19}$$

were performed. Here $\Gamma(x)$ is the gamma function. The calculations were conducted for $N = 0,\ m = 1$.

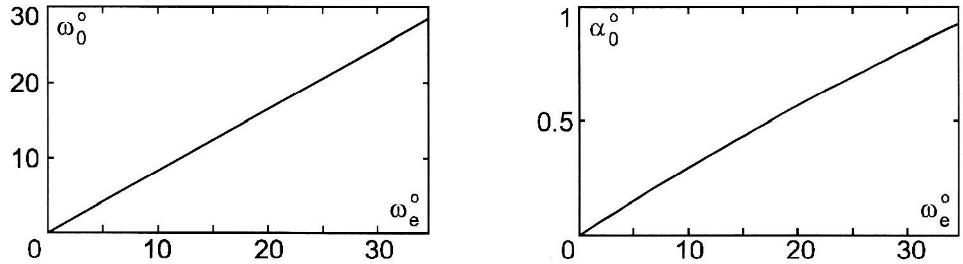

Figure 1. Fundamental frequency ω_0^o and damping decrement α_0^o vs. ω_e^o

Figure 1 shows that the relationships between the parameters ω_0^o, α_0^o and the frequency ω_e^o of the elastic vibrations are almost linear.

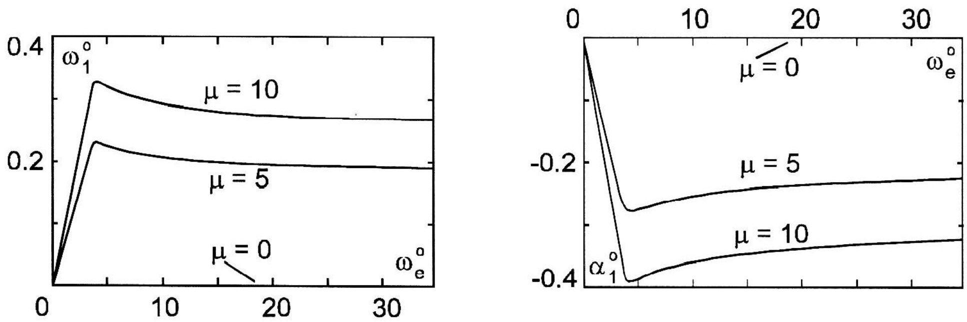

Figure 2. Parameters ω_1^o and α_1^o vs. ω_e^o

Figure 2 represents the graphs of the functions $\omega_1^o = \omega_1^o\left(\omega_e^o\right)$, $\alpha_1^o = \alpha_1^o\left(\omega_e^o\right)$ for various μ. Here $\omega_1^o = \mathrm{Re}\,\Omega_1$, $\alpha_1^o = \mathrm{Im}\,\Omega_1$. It may be seen that the corrections $\varepsilon\omega_1^o$, $\varepsilon\alpha_1^o$ (see formula (6)) for the frequency ω_0^o and the damping decrement α_1^o , respectively, depend on parameter μ, which specifies the power of nonhomogeneity of the temperature field. In the case of Rzhanitsin's kernel (19), the influence of parameter μ is more pronounced at $\omega_e^o \approx 4$.

6 Conclusion

The influence of a nonuniform temperature field on low-frequency free vibrations of a viscoelastic cylindrical shell with free edges has been demonstrated.

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