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"The Given Force Cannot be a Function of Acceleration"

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The modelling of reactive forces, those that are applied by constitutive components and are generated by their deformation typically assumes that various deformation derivatives can be included. The inclusion of the deformation and the first derivative, that is extension and velocity, or strain and strain rate are well established. However, there is a problem when the acceleration or rate of strain rate is included in the modelling of constitutive components. It will be shown that these will cause at the very least a non-uniqueness in motion, and consequently such modelling is adverse to Newtonian mechanics.

1 Introduction

Pars (1965) in his book "A Treatise on Analytical Dynamics", pages 12-13, states that "forces depending on the acceleration are not admissible in Newtonian dynamics". This is correct but the method used by Pars to show this is flawed. In the present paper the statement on admissibility is examined and a more correct method is used to confirm the Pars statement. The following is presented using the notation of Pars and part of his development is reproduced with annotations introduced by the present author.

2 Background

First the development by Pars is outlined here. Consider a particle mass m, moving along a line Ox and consider two forces $m\phi$ and $m\psi$ that can act on this particle; the multiplier m, used by Pars is taken as an expedience and will be retained in the present paper. The functions ϕ and ψ can be functions of the particle position x, the particle velocity $v(= dx/dt)$ and the time t; these functions can also be functions of the motion history but this is not relevant to this article. Since these functions ϕ and ψ are functions of variables x and v that are themselves functions of time although the functional behaviour of x and v is not known a priori then it is proper to state that ϕ and ψ are functionals of x and v. Now also let these functionals be functions of the acceleration $f(= dv/dt)$ and it is this dependence that is the primary concern.

Consider three experiments, the first where the particle is acted on by the force $m\phi\,(f)$, the second by $m\psi\,(f)$ and in the third by the combinations of forces $m(\phi(f) + \psi(f))$. Pars states here that the values of x, v and t are the same in all three experiments but not how to achieve this. Denoting the resulting accelerations of each experiment by f_1, f_2 and f_3 then

$$f_1 = \phi(f_1) \tag{1}$$

$$f_2 = \psi(f_2) \tag{2}$$

and

$$f_3 = \phi(f_3) + \psi(f_3) \tag{3}$$

The first point is that there should be uniqueness in f and thus the force functionals ϕ and ψ should be linear in f. As Pars has done, these functionals are prescribed such that the f can be uniquely determined.

Pars then invokes the fundamental postulate of Newtonian mechanics that when two forces act simultaneously on a particle the effect is the same as that of a single force equal to the vector sum, and equivalently each force gives rise to an acceleration that it would produce if the other were absent, thus

$$f_3 = f_1 + f_2 \tag{4}$$

and combining this with the equation (3) above for f_3 gives

$$f_1 + f_2 = \phi(f_1 + f_2) + \psi(f_1 + f_2) \tag{5}$$

and Pars then continues to show correctly that this latter equation is inconsistent with equations (1) and (2); here this will not be pursued further but the introduction of equation (4) will be queried.

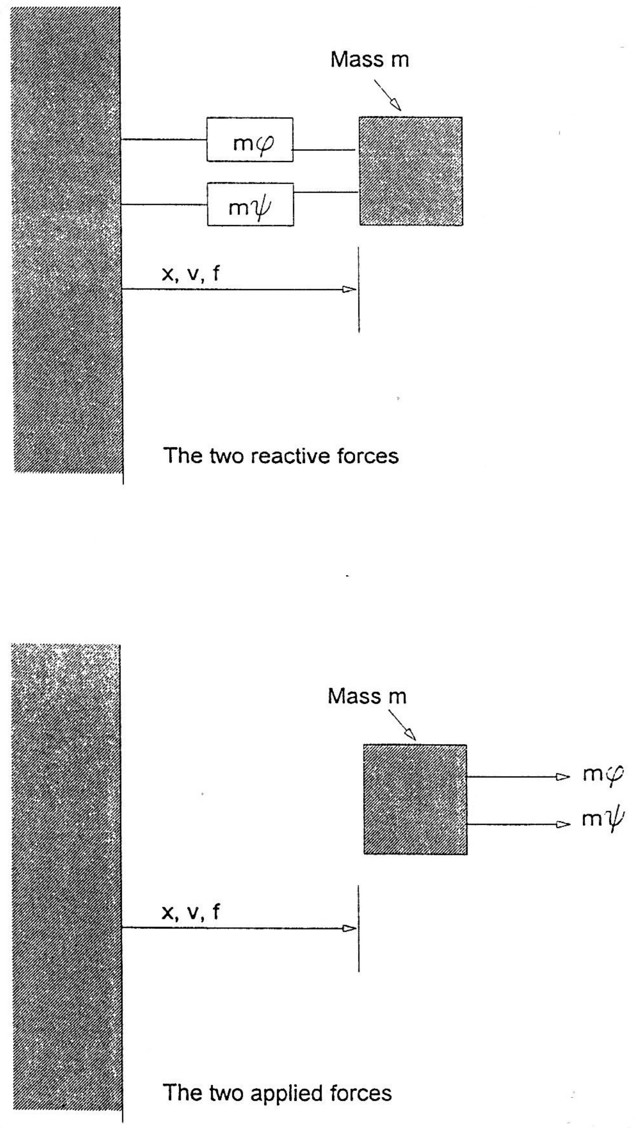

Figure 1. A Mass Subject to the Action of Two Forces

3 The Contradiction and the Resolution

A system which incorporates the two functionals ϕ and ψ is shown in Figure 1. It is important to distinguish between the two types of forces shown; the forces considered by Pars are applied forces and act independently of each other and hence equation (4) can be used. However if the forces are reactive as they must be if they are dependent on the motion, the accelerations that they cause are not independent of the action of the other force. The combined action of the two reactive forces is modelled by equation (3).

Returning to the Pars statement, "forces depending on the acceleration are not admissible in Newtonian dynamics", it is necessary to define a *Newtonian space*, as a field of behaviour of systems that are governed by or adhere to the three Newton laws; this is the equivalent to Pars' *Newtonian dynamics*. There are systems that are not in the Newtonian space and those systems that use force functionals that contain accelerations and higher displacement derivatives, i.e. rate of change of acceleration $(= df / dt)$, will be seen to be in a non-Newtonian space.

Consider a mass, moving in x-direction and acted on by a force component, Figure 2, where the behaviour of the force is governed by a constitutive law written functionally as $\psi\ (x, v, f)$. This force functional is a function of the mass position x, its velocity v and its acceleration f. Obviously it could also be a function of other differential and integral operations on x, but it will suffice here to consider only those three.

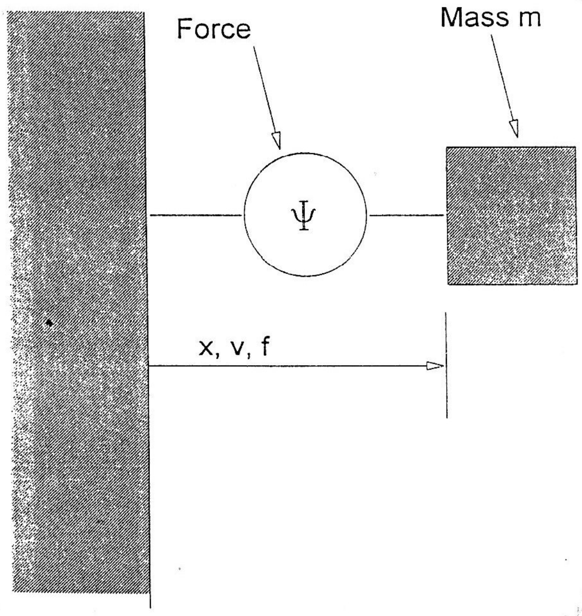

Figure 2. A Single Mass and a Single Force

Applying Newton's second law gives

$$mf = \psi(f, v, x) \tag{6a}$$

where

$$v = \frac{dx}{dt} \tag{6b}$$

and

$$f = \frac{d^2x}{dt^2} \tag{6c}$$

The velocity and acceleration are only related to the mass position by compatibility requirements (6b and c), they are quite independent of each other in equation (6a). Thus the force element can be examined on its own away from the mass and could be visualised as a specimen undergoing tests in a testing machine. (A testing machine is defined as a machine that can impose on a test specimen a specified kinematic deformation and will measure the force exerted on the machine by the test specimen. The specified kinematic deformation will be determined from the requirements of the test and can include all history and rate deformation paths.)

It is observed that acceleration appears on both sides of equation (6a); as Pars has stated there must be uniqueness in the acceleration term. To achieve this it is necessary to impose linearity of f in the force functional ψ; any nonlinearity could result in a state in f-v-x space, Figure 3, where a specified point x, v gives a nonunique value for f resulting from the application of equation (6a). To ensure linearity the following form for ψ will be introduced.:

$$\psi(f, v, x) = \Omega(v, x) - \Lambda(v, x)f \tag{7}$$

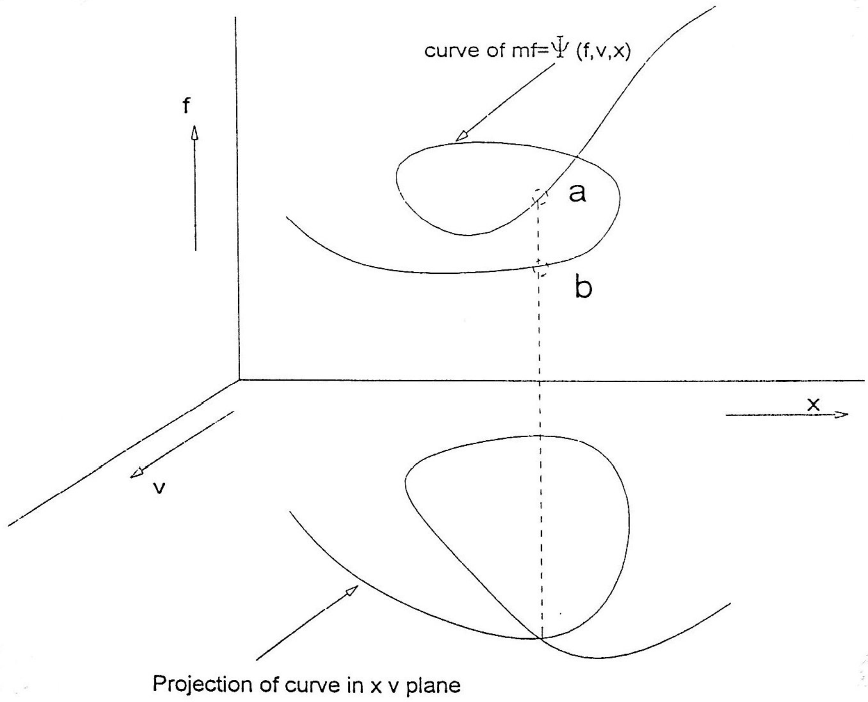

Figure 3. Solution to Equation (6a)

where Ω and Λ are two new functionals; the negative sign before the second term is a convenience. Substituting equation (7) in equation (6) results in the following:

$$mf = \Omega(v, x) - \Lambda(v, x)f \tag{8}$$

The term Λ represents the added (or equivalent) mass of the system and this will be discussed later. Equation (8) can be rewritten as

$$(m + \Lambda)f = \Omega \tag{9}$$

$\Lambda(v, x)$ is a functional that can vary as the system configuration changes; again it could be "experimentally" determined together with $\Omega(v, x)$ by recourse to the testing machine. First, if Λ is at any time negative then by designing the system with the appropriate mass m such that at that point in the v-x space, $m = |\Lambda|$, the system would respond with an infinite acceleration. Second if $(m + \Lambda) \leq 0$ the resulting acceleration would be in the opposite sense (vectorially opposite) to the force. For positive definiteness of inertia $m \geq |\Lambda|$. The value of m is assumed to take any positive value.

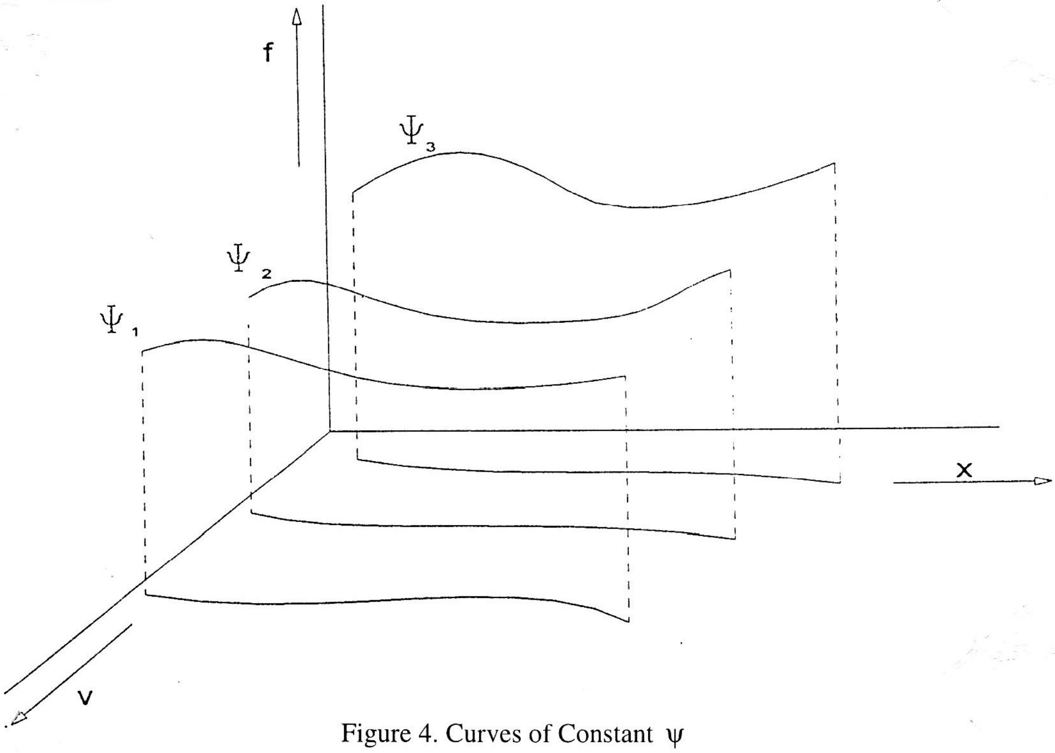

Figure 4. Curves of Constant ψ

This has suggested some problems with the acceptance of $\psi(f, v, x)$; however there is a more significant problem and that arises again from taking the component ψ and examining its behaviour in a testing machine. The machine is used to record the force $\psi(f, v, x)$ exerted by the component for a given profile of component stretch x, stretch rate v, and rate of change of stretch rate f. Figure 4 illustrates qualitatively, a typical behaviour for such a test where traces of constant ψ are shown.

Now consider any point in f-v-x space; this gives the reactive force supplied against the testing machine. Consider that state as one through which the system of Figure 2 passes according to equations (6). The force element has shown for a specific acceleration f, velocity v, and position x a resultant reactive force ψ that is applied to the mass. The acceleration of the mass is thus ψ / m and this is in general not equal to f. This has been shown for the chosen state in the f-v-x configuration space, but it could equally apply to any state in that space.

There is thus an error in the events described, either the force element cannot be tested in the manner described: but this is the conventional, necessary and only method for identifying the behaviour and characteristics of force elements used within a dynamic system, or the modelling and analysis of systems using such a force cannot be resolved using Newton's second law, i.e. the system is in a *non-Newtonian space.*

The added or equivalent mass referred to previously is an inherent inertia within the force element; if such a force element was tested in testing machine, the effect of acceleration would be measured. Its effect would be to resist change in velocity and thus Λ would be positive semidefinite. Within equation (9) the added mass must be kinematically tied to the particle mass m in the system. This is the only type of acceleration entry into the force constitutive that is consistent with Newtonian space. Non-inertial acceleration and higher time differentials $\left(d^n x / dt^n\right)$ whilst not necessarily incorrect, cannot be used within Newtonian space. This exclusion from Newtonian space has implication in many applications; within the field of viscoelasticity it must exclude from the material constitutive relations the dependence of stress on strain acceleration. Other applications include solid-fluid coupled systems where the characteristic of the fluid domain is represented by an equivalent force component and this is assembled with the solid force components to give an equivalent system; it is useful to give a mass quantity to the fluid and this is the quantity Λ referred to in equation (8); this would be usually bounded from below, Λ being greater than 0. It is conceivable that Λ could be less than 0 for systems where there is a reduction or removal of the coupled fluid mass in, for example, a fluid driven vibratory system, but this is beyond the scope of this note.

4 Conclusion

The implications of including the acceleration and higher derivatives in the constitutive form for applied forces, i. e. those generated by non-inertial components is discussed. Their inclusion leads to nonuniqueness in the dynamic motion of the system and thus is in opposition to Newtonian dynamics. The implications of this extend into the theory of viscoelasticity and must restrict the form of constitutive equations if Newton's second law is to be applied.

Literature

1. Pars, L. A.: Treatise on Analytical Dynamics, Heinemann, (1965), pp. 12-13.

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