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MHD Squeezing Flow Between Two Parallel Discs

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The unsteady motion of a viscous, incompressible and electrically conducting fluid squeezed between two parallel discs, in which the lower disc is rotating with an arbitrary time-dependent angular velocity, while the upper disc approaches the lower one with a time-dependent velocity, is studied. A numerical solution of the governing partial differential equations is obtained through a fourth-order accurate Hermitian finite-difference scheme. Results for the velocity field, normal pressure forces (load) and the torque which the fluid exerts on the discs are presented for some values of the Reynolds number and magnetic field parameter at various non-dimensional times. It is found that the load on the upper disc increases significantely with the increase of the magnetic field parameter. It also increases with the decrease of the gap between the discs. The torque on the lower disc is shown to increase with the increase in the magnetic field parameter as well as with the angular velocity of this disc.

1 Introduction

The study of the squeezing flow between two planes is of special interest for applications to bearings with liquid-metal lubrication. The interaction of the flowing liquid-metal lubricant with the applied magnetic field can be used to increase the total load which the rotor can support and to reduce the viscous drag on the rotor. Studies on the effects of an applied magnetic field in lubrication were made by Hughes and Elco (1962), Kuzma et al. (1964), Krieger et al. (1967) and Kamiyama (1969). In these investigations the authors have considered the magnetic force term but neglected some or all the inertia terms in the Navier-Stokes equations. Hamza (1988) studied the squeezing flow between two non-rotating discs in the presence of a magnetic field acting perpendicular to the discs by taking into account all the inertia terms. In a subsequent paper Hamza (1989) obtained a similarity solution of the governing equations, where the axial magnetic field is assumed to be of a particular time-dependent form.

The study of similarity solutions for the steady motion of an incompressible viscous fluid between two rotating discs was initiated by Batchelor (1951). Using von Karman similarity transformation the governing equations were reduced to two coupled fourth-order ordinary differential equations. The solution of these equations was expressed in power series of small values of the Reynolds number Re . However, Lance and Rogers (1962) developed a shooting method to solve numerically the ordinary differential equations for different values of Re. Subsequently Holodniok et al. (1977, 1981) have developed a finite-difference scheme along with Newton´s iteration for obtaining solutions at higher values of Re. A detailed review of these studies was made by Zandbergen and Dijkstra (1987).

Ishizawa (1966) has shown that when the angular velocities of the discs are time-dependent, the Navier-Stokes equations describing the flow between two discs can also be reduced to a pair of coupled non-linear ordinary differential equations. Hamza and MacDonald (1984) considered the case where two parallel discs in an unsteady rotation have also a velocity component in a direction perpendicular to their planes and obtained a similarity solution of the governing equations. Their solution requires that at time t, the separation of the discs must be proportional to $(1-\alpha t)^{1/2}$ and the angular velocities of the discs are proportional to $(1-\alpha t)^{-1}$, where α^{-1} denotes a characteristic time.

In the present paper, we consider the unsteady flow of a viscous, incompressible and electrically conducting fluid between two parallel discs of a small gap width, where the lower disc rotates with an arbitrary time-dependent velocity and the upper disc approaches the lower one with a time-dependent velocity. We assume that a uniform magnetic field is applied perpendicular to the planes of the discs. Since the gap width between the discs at any time is small compared to their diameters, the edge effects can be neglected. Numerical solutions of the governing non-linear parabolic partial differential equations are obtained through a higher-order accurate Hermitian finite-difference scheme. Effects of the magnetic field parameter, the squeezing parameter and the Reynolds number on the flow field, on the normal pressure forces exerted on the upper disc (load) and on the torque on the lower disc are determined at different times.

2 Formulation

We consider the flow of a viscous, incompressible and electrically conducting fluid between two parallel discs in the presence of a uniform magnetic field B_0, which is assumed to be applied perpendicular to the planes of the discs. Let (r, θ, z) be cylindrical polar coordinates with the discs occupying the planes $z = 0$ (the lower disc) and $z = d(t^*)$ (the upper disc), respectively, where d is the distance between the discs and t^* is the nondimensional time. It is further assumed that the lower disc rotates with the angular velocity $\omega(t^*) = \Omega\phi(t^*)$ in its own plane and the upper disc approaches the lower disc with a constant velocity W_d. Initially ($t^* = 0$) the discs are at the constant distance H and the lower disc is rotating with a uniform angular velocity Ω. This implies that $d(0) = H$ and $\phi(0) = 1$.

The Navier-Stokes equations for the governing unsteady axisymmetric flow can be written as

$$\frac{\partial \varsigma}{\partial t} + u\frac{\partial \varsigma}{\partial r} + w\frac{\partial \varsigma}{\partial z} - \frac{u\varsigma}{r} - 2\frac{v}{r}\frac{\partial v}{dz} = \nu\left(\nabla^2\varsigma - \frac{\varsigma}{r^2}\right) - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\sigma B_0^2 u\right) \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left(\nabla^2 v - \frac{v}{r^2}\right) - \frac{\sigma B_0^2}{\rho}v \tag{2}$$

where u, v and w are the velocity components in the radial (r), tangential (θ) and axial (z) directions, and ς is the circumferential component of vorticity given by

$$\varsigma = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \tag{3}$$

If the stream function ψ is defined by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z} \qquad w = -\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{4}$$

thus ς becomes

$$\varsigma = \frac{1}{r}\left(\nabla^2\psi - \frac{2}{r}\frac{\partial \psi}{\partial r}\right) \tag{5}$$

where the Laplacian operator ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{6}$$

In writing equations (1) and (2), the effect of the induced magnetic field on the flow is neglected and this is justified for flow at a small magnetic Reynolds number. This is indeed true for flow of liquid metals, e.g., mercury or liquid sodium.

We now introduce the following non-dimensional variables:

$$\eta = z/d\left(t^*\right) \qquad \beta\left(t^*\right) = d\left(t^*\right)/H \qquad \phi\left(t^*\right) = \omega\left(t^*\right)/\Omega$$

$$t^* = \Omega t \qquad f\left(\eta, t^*\right) = \psi/r^2 H\omega\left(t^*\right) \tag{7}$$

$$h\left(\eta, t^*\right) = \varsigma H/r\omega\left(t^*\right) \qquad g\left(\eta, t^*\right) = v/r\omega\left(t^*\right)$$

Using these variables, equations (1) to (3) then transform to

$$Re^{-1}\frac{\partial^2 h}{\partial\eta^2} + 2\phi\beta\left(f\frac{\partial h}{\partial\eta} + g\frac{\partial g}{\partial\eta}\right) + \beta\beta_{,t^*}\eta\frac{\partial h}{\partial\eta} - \beta^2\left(h_{,t^*} + h\phi_{,t^*}\phi^{-1}\right) - M\beta^2 h = 0 \tag{8}$$

$$Re^{-1}\frac{\partial^2 g}{\partial\eta^2} + 2\phi\beta\left(f\frac{\partial g}{\partial\eta} - g\frac{\partial f}{\partial\eta}\right) + \beta\beta_{,t^*}\eta\frac{\partial g}{\partial\eta} - \beta^2\left(g_{,t^*} + g\phi_{,t^*}\phi^{-1}\right) - M\beta^2 g = 0 \tag{9}$$

$$\frac{\partial^2 f}{\partial\eta^2} - \beta^2 h = 0 \tag{10}$$

subject to the boundary conditions (for $t^* > 0$)

$$\begin{array}{llll} \eta = 0 : f = 0 & \dfrac{\partial f}{\partial\eta} = 0 & g = 1 & h = \dfrac{1}{\beta^2}\dfrac{\partial^2 f}{\partial\eta^2} \\ \eta = 1 : f = \varepsilon/\left[2f\left(t^*\right)\right] & \dfrac{\partial f}{\partial\eta} = 0 & g = 0 & h = \dfrac{1}{\beta^2}\dfrac{\partial^2 f}{\partial\eta^2} \end{array} \tag{11}$$

where $g_{,t^*} = dg/dt^*$, $h_{,t^*} = dh/dt^*$, $\beta_{,t^*} = d\beta/dt^*$ *and* $\phi_{,t^*} = d\phi/dt^*$. We now assume that the upper disc approaches the lower one with a velocity $W_d = \Omega H_\varepsilon$ where ε is a positive constant called squeezing parameter. Hence

$$-W_d = w\left(1, t^*\right) = -2\omega\left(t^*\right)Hf\left(1, t^*\right) \tag{12}$$

This explains the boundary condition on f at $\eta = 1$ given by equation (11). In equations (8) and (9) $Re = H^2\Omega/\nu$ is the Reynolds number and $M = B_0^2\sigma/\rho\Omega$ is the magnetic interaction parameter which measures the strength of the electromagnetic body force relative to the Coriolis force. Here ν is the kinematic viscosity, σ is the electrical conductivity and ρ is the density of the fluid.

Since the gap $d(t^*)$ between the two discs decreases with time, we shall assume here that

$$\beta\left(t^*\right) = 1 - \varepsilon t^* \tag{13}$$

The initial conditions of equations (8) to (10) (i.e., at $t^* = 0$) are governed by the solution of the corresponding steady-state similarity equations for the flow due to a uniform rotation of the lower disc. These equations can be obtained from equations (8) to (10) by setting

$$\beta\left(t^*\right) = \phi\left(t^*\right) = 1 \qquad \text{and} \qquad \frac{\partial}{\partial t^*} = 0 \tag{14}$$

From the axial momentum equation it can be shown that $\dfrac{\partial^2 p}{\partial\eta\partial r} = 0$, i.e., the radial pressure gradient $\dfrac{\partial p}{\partial r}$ is independent of η, and this is given by

$$\frac{1}{r}\frac{\partial p}{\partial r} = \rho\Omega^2\phi\left[\frac{Re^{-1}}{\beta^3}\frac{\partial^3 f}{\partial\eta^3}\left(0, t^*\right) + \phi g^2\left(0, t^*\right)\right] \tag{15}$$

If the discs are assumed to be of finite radius a and of negligible thicknesses, then the load or the normal pressure forces exerted on the upper disc is given by

$$L = 2\pi\int_0^a r\left[P\left(r, 1, t^*\right) - \overline{P}\left(r, 1, t^*\right)\right]dr \tag{16}$$

where $P(r, l, t^*) = p(r, 1, t^*) - p(a, 1, t^*)$ and $\overline{P}\ (r, 1, t^*)$ are defined in a similar way and correspond to the conditions on the outher side of the upper disc. But, it is assumed that $\frac{\partial \overline{P}}{\partial r}\ (r, 1, t^*) = 0$, so that from equations (15) and (16), we obtain

$$L = -\frac{\pi\rho\Omega^2 a^4}{4}\left[\frac{Re^{-1}}{\beta^3}\frac{\partial^3 f}{\partial \eta^3}\left(0, t^*\right) + g^2\left(0, t^*\right)\phi\right]\phi \tag{17}$$

If we now define the non-dimensional load as $L^* = 4L/(\pi\rho\Omega^2 a^4)$ then, we have

$$L^* = -\left[\frac{Re^{-1}}{\beta^3}\frac{\partial^3 f}{\partial \eta^3}\left(0, t^*\right) + g^2\left(0, t^*\right)\phi\right]\phi \tag{18}$$

On the other hand, the torques which the fluid exerts on the discs are also of interest in rotating flow problems. Thus, the torque on the lower disc is defined by

$$T = 2\pi\mu\int_0^a r^2\left(\frac{\partial v}{\partial z}\right)_{z=0} dr \tag{19}$$

Using equation (7), T can be expressed in non-dimensional form as

$$T^* = \phi\left(t^*\right)g_\eta\left(0, t^*\right)/\beta\left(t^*\right) \tag{20}$$

where $T^* = \left(2T/\pi\mu\Omega a^4\right)$.

It is worth mentioning that equations (8) to (10) for the steady-state ($t^* = 0$) non-magnetic ($M = 0$) case reduce to ordinary differential equations and they correspond to the set of equations considered by Rogers and Lance (1962). These equations also correspond to those derived by Hamza (1988) in the absence of rotation of the lower disc $(\Omega = 0)$.

3 Numerical Method

The non-linear boundary value problem governed by equations (8) to (10) is solved by first differentiating in the t^* – direction and averaging the other terms. The derivatives in η are discretized by using the compact Hermitian formula. The fourth-order accurate method considers as unknowns at each discretized point η_i not only the value of the function f_i itself but also of its first and second derivatives f_i' *and* f_i'', where primes denote differentiation with respect to η_i. The system is closed by considering the following relationships between the function f_i and its derivatives in three successive discretization points.

$$\begin{aligned} f_{i-1}' + 4f_i' + f_{i+1}' &= \frac{3}{k}\left(f_{i+1} - f_{i-1}\right) + 0\left(k^4\right) \\ f_{i-1}'' + 10f_i'' + f_{i+1}'' &= \frac{12}{k^2}\left(f_{i+1} - 2f_i + f_{i-1}\right) + 0\left(k^4\right) \end{aligned} \tag{21}$$

where k is the spatial step of discretization. The second-order derivative f_i'' can be expressed as

$$f_i'' = -\frac{1}{2k}\left(f_{i+1}' - f_{i-1}'\right) + \frac{2}{k^2}\left(f_{i+1} - 2f_i + f_{i-1}\right) \tag{22}$$

which is fourth-order accurate. However, the second-order derivatives can be eliminated in order to reduce the number of unknowns. This method was described in great detail by Adam (1977), and Peyret and Taylor (1983). Loc (1985) employed it successfully for solving the unsteady Navier-Stokes equations.

We shall use here Newton´s linearisation technique to cope with non-linearity. The resulting system of block tri-diagonal equations is solved by a block elimination method (Varga, 1962). The steady-state ordinary differential equations are solved in a similar manner. The boundary conditions on the function h are approximated through a second-order accurate upwind scheme. The iteration starts with assuming a guess solution for f, g and h, satisfying the boundary conditions (11).

4 Results and Discussion

In order to assess the accuracy of the present method, we have applied it to the problem of Lance and Rogers (1962) for the steady flow between two rotating discs without an applied magnetic field ($M = 0$). Results for the radial (f'), tangential (g) and axial (f) velocity components are shown in Figure 1 for $Re = 25$ and 81, and $S = 0$ and 0.5, where S is the ratio of the angular velocities of the two discs and primes denote differentation with respect to η. It is seen from this figure that the present results are in excellent agreement with those of Lance and Rogers (1962). The variation of the non-dimensional load L^* with time was also compared with the perturbation solution obtained by Hamza (1988) and it was found that the maximum percentage difference is about 4%, which is again very good.

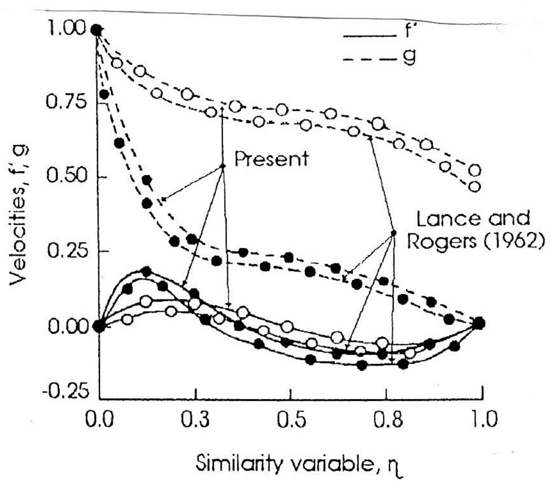

Figure 1. Radial and Tangential Velocity Profiles for $\bullet\, S = 0$, $Re = 81$ and $\circ\, S = 0.5$, $Re = 25$

The effect of the squeezing parameter ε on the velocity profiles $(f, \partial f / \partial\eta, g)$ is described in Figure 2 at $t^* = 1$ for $M = 4$ and $Re = 25$ when the lower disc rotates with a constant angular velocity ($\phi(t^*) = 1$) and $\beta(t^*)$ is given by equation (13). It is seen from this figure that the radial velocity $\partial f / \partial\eta$ increases due to the increase in the squeezing parameter ε. Hence, the radial pressure gradient, which is independent of z, is negative throughout the flow. The negative radial pressure gradient which arises due to squeezing and rotation of the lower disc produces a normal force (or load) on the upper disc and this force is so directed as to push the discs away from each other. Further, we see that the axial (f) and tangential (g) components of velocity also increase with the increase in ε.

Figures 3a and 3b display the radial and tangential velocity profiles $(\partial f / \partial\eta, g)$ for $Re = 25$ and different values ot the magnetic parameter M at $t^* = 1$ with $\phi(t^*) = 1 - 0.2t^{*2}$ (decelerated rotation of the lower disc) and $\beta(t^*) = 1 - 0.25t^*$. It is seen that the radial velocity is outward near each disc. We also see that the magnetic field produces a slight increment in the radial outflow near the discs and the radial velocity profile in the region of the midplane $(\eta = 1/2)$ becomes almost flat. Further, we notice that the radial outflow in the mid-plane decreases with the increase in the magnetic parameter M. Near both discs the tangential velocity decreases with increase in the magnetic field.

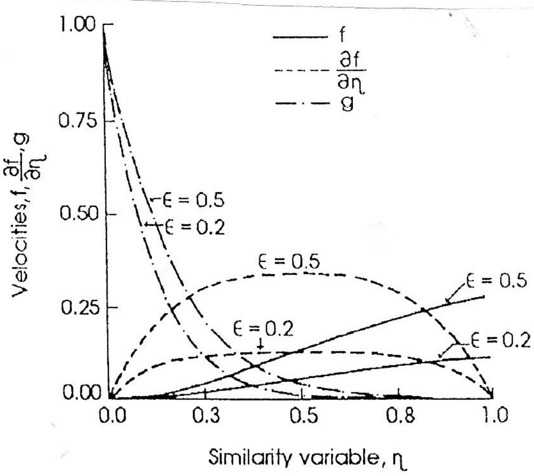

Figure 2. Axial, Radial and Tangential Velocity Profiles at $t^* = 1$ for $\phi(t^*)$, $M = 4$ and $Re = 25$

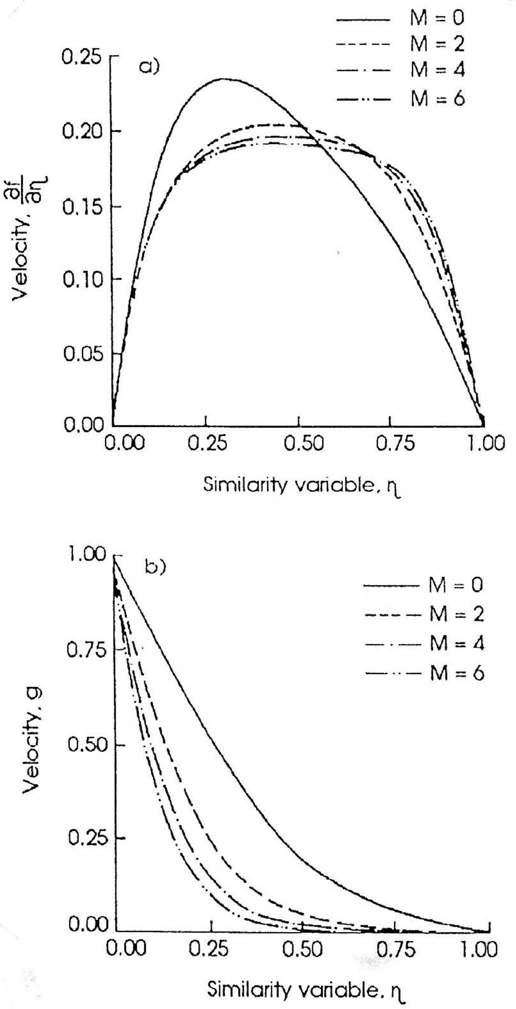

Figure 3. Radial and Tangential Velocity Profiles at $t^* = 1$ for $Re = 25$

Figure 4 shows the variation of the non-dimensional load L^* with time t^* for $M = 0$ (the magnetic field is absent) and $Re = 25$. Graphs are drawn for various values of the squeezing parameter ε with $\phi(t^*) = 1 - 0.3t^{*2}$ and $\beta(t^*)$ given by equation (13) including the flow case when the lower disc is stationary ($g = 0$). It is clearly seen from this figure that as time elapses, L^* increases monotonically. The results also illustrate the fact that L^* increases as the parameter ε or the gap between the discs increases. Further, we notice that L^* remains almost invariant in time for small values of ε.

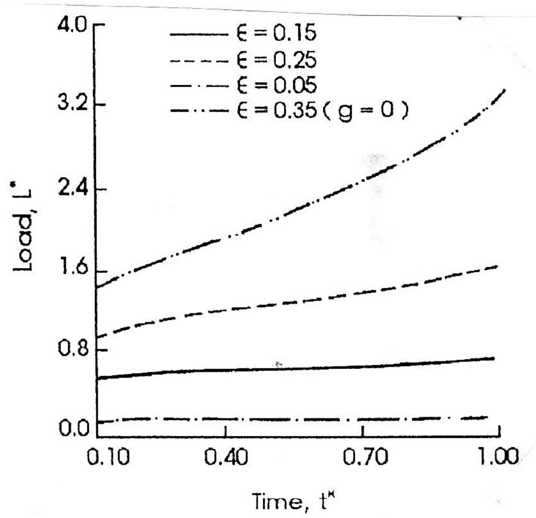

Figure 4. Variation of the non-Dimensional Load for $M = 0$ and $Re = 25$

Figure 5 shows the variation of the non-dimensional load L^* with Re at time $t^* = 1$ for $M = 5$ and different values of ε when $\phi(t^*) = 1 - 0.4t^{*2}$ and $\beta(t^*)$ is given by equation (13). It is evident from this figure that the effect of ε on L^* is much more prominent than the effect due to an increase of Re, particularly for $Re > 50$.

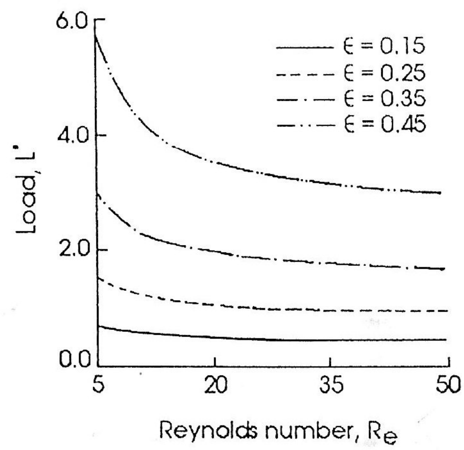

Figure 5. Variation of the non-Dimensional Load at $t^* = 1$ and $M = 5$

Figure 6 presents the variation of the non-dimensional torque T^* on the lower disc with time t^* for $Re = 5$ and different values of M. Graphs are depicted for flow due to an accelerated $\phi(t^*) = 1 - 0.3t^{*2}$ or a decelerated ($\phi(t^*) = 1 - 0.3t^{*2}$) rotation of the disc with $\beta(t^*) = 1 - 0.25t^*$. It is seen that for an accelerated disc, the magnitude of T^* increases monotonically with time, while T^* decreases with time when the rotation of the disc is decelerated. This result is plausible on physical grounds. For an accelerating disc, its angular velocity increases with time and consequently the torque required to maintain such an angular velocity should also increase with time. Just the reverse is true for a decelerating disc. Further, we observe from this figure that the magnetic field produces a huge increment in the magnitude of T^*. This can be explained physically as follows: It is well known that a magnetic field imparts some rigidity to the conducting fluid. Thus, with increase in the magnetic field, greater effort will be necessary to maintain the rotation of the disc and this implies an increase in T^* with an increase of the parameter M.

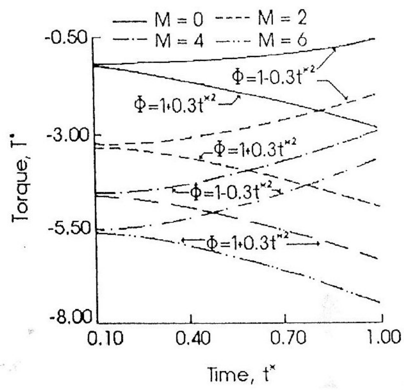

Figure 6. Variation of the Non-dimensional Torque for $Re = 5$

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