TECHNISCHE MECHANIK, Band 16, Heft 4 (1996), 285-296
Manuskripteingang: 23. Januar 1996

Double Asymptotic Method for Nonlinear Forced Oscillations Problem of Mechanical Systems with Time Dependent Parameters

V. Z. Gristchak, V. N. Kabak

The double asymptotic method for the forced oscillations of nonlinear vibration problems of some mechanical systems with time dependent characteristics is an asymptotic procedure developed with perturbation and WKB-theory. The method is illustrated for some coefficient functions and results of calculations are compared with direct numerical solutions.

1 Introduction

The problems of forced oscillations of nonlinear dynamics problems have been one of the most fundamental subjects in the study of the behavior of mechanical systems in modern aerospace, machinery and structural industries. For example, the shock wave interaction with ground objects and earthquake stability of structures are in general nonhomogeneous in space and in time. The specific characteristic of these problems is the complicated character of interaction with the object and evaluation of an external loading connected with the vibration behavior of the structure. The ground object at some distance from the place of an explosion, is exposed to the product of detonation or air-shock wave. With this connected complicated diffraction picture of interaction, the object is subjected to instationary (dependent on the time) pressure. Dynamic pressure is a function of parameters of wave, geometrical and physical characteristics of the object and its displacement and orientation with respect to the space wave front. The pertinent literature in the area of applications of approximate analytical or analytical-numerical methods for nonlinear dynamics problems has been discussed by Adrianov et al. (1994), Gorman (1982), Kobayaski and Sonoda (1991), Timoshenko et al. (1974) and Volmir (1972). In the present paper an approximate analytical method on the basis of double (perturbation and phase-integral or WKB methods) asymptotic expansion in closed form for some forced oscillation nonlinear dynamics problems of mechanical systems with the time dependent parameters is discussed.

2 Description of the Method

Consider the nonlinear dynamics problem for a mechanical system with time dependent characteristics of mass or density. The corresponding differential equation that describes the process of forced oscillations can be taken in the form

$$f''(t) + \omega^2(t)f + \alpha P(t)f^2 + \alpha^2 Q(t)f^3 = \gamma(t) \qquad (1)$$

where $\omega^2(t) = \omega_0^2\,\varphi(t)$, $\varphi(t)$ a given function of time. The parameter of natural frequency of vibration $\omega_0 > 0$ we take into account as a large parameter. The function $\varphi(t) > 0$ is a continuous function, α is a small parameter $0 < \alpha << 1$. For example, for the nonlinear vibration problem of the shallow cylindrical shell the parameter $\alpha = \frac{a_1}{h}$, where a_1 is longitudinal dimension of shell and h is the shell thickness. We assume here that $\varphi(t)$, $P(t)$ and $Q(t)$ are continuously differentiable functions.

In order to obtain an approximate analytical solution of the initial nonlinear differential equation we will use the double asymptotic expansion method that includes two steps of solution (Gristchak and Golovan, 1995). On the first step *(outer perturbation expansion)* the solution of equation (1) is presented as the expansion on the small parameter α.

$$f = f_0 + \alpha f_1 + \alpha^2 f_2 \cdots \qquad (2)$$

We take into account three terms of expansion (2) and after substitution of equation (2) into equation (1) and comparing the coefficients with the same order of parameter α, we obtain a system of equations for the unknown functions $f_0, f_1, \cdots$

$$f_0'' + \omega^2(t) f_0 = \gamma(t) \qquad \text{at } \alpha^0 \tag{3}$$

$$f_1'' + \omega^2(t) f_1 = -P(t) f_0^2 \qquad \text{at } \alpha^1 \tag{4}$$

$$f_2'' + \omega^2(t) f_2 = -2P(t) f_0 f_1 - Q(t) f_0^3 \qquad \text{at } \alpha^2 \tag{5}$$

Using the fact that the natural frequency of vibration parameter ω_0^2 is large in the comparison with unity, the solution of the first homogeneous equation of the system we look for on the basis of the two term phase-integral or WKB-method (Nayfeh, 1981) (*inner asymptotic expansion*). Omitting the details of the simple calculations, we obtain the functions f_{00}, f_{01}, f_{11} in the expansion

$$f_0(t) = \exp \int_0^t \left(\omega_0 f_{00} + f_{01} + \frac{1}{\omega_0} f_{11} + \cdots \right) d\tau \tag{6}$$

From the equation (3) it follows that:

$$f_{00} = \frac{c_{1,2}}{\varphi^{\frac{1}{4}}(t)} \exp \int_0^t \pm i\omega_0 \varphi^{\frac{1}{2}}(\tau) d\tau \tag{7}$$

or

$$f_{00} = AG_1 + BG_2 \tag{8}$$

where

$$G_1 = \frac{\cos\left(\int_0^t \omega_0 \varphi(\tau)^{\frac{1}{2}} d\tau \right)}{\varphi^{\frac{1}{4}}(t)} \qquad G_2 = \frac{\sin\left(\int_0^t \omega_0 \varphi(\tau)^{\frac{1}{2}} d\tau \right)}{\varphi^{\frac{1}{4}}(t)} \tag{9}$$

are the G-functions of the first kind (Gristchak and Golovan, 1995). We will introduce the following notations:

$$A = c_1 + c_2 \qquad B = i(c_1 - c_2) \tag{10}$$

Using the method of variation of arbitrary constants for the particular solutions of equations (3) and (4) we have

$$f_{01} = A(t) G_1(t) + B(t) G_2(t) \tag{11}$$

$$f_{11} = M(t) G_1(t) + N(t) G_2(t) \tag{12}$$

$$A'(t) G_1(t) + B'(t) G_2(t) = 0 \tag{13}$$

$$A'(t)\left[-\omega_0 \varphi(t)^{\frac{1}{2}} G_2 - \frac{\varphi'}{4\varphi} G_1 \right] + B(t)\left[\omega_0 \varphi(t)^{\frac{1}{2}} G_1 - \frac{\varphi'}{4\varphi} G_2 \right] = \gamma(t) \tag{14}$$

$$A(t) = \int_0^t -\frac{\gamma(\tau)}{\omega_0} G_2(\tau) d\tau + D_1 \qquad B(t) = \int_0^t \frac{\gamma(\tau)}{\omega_0} G_1(\tau) d\tau + D_2 \tag{15}$$

$$f_0(t) = G_1\left[K - \frac{1}{\omega_0}\int_0^t \gamma(\tau)\, G_2\, d\tau\right] + G_2(t)\left[L + \frac{1}{\omega_0}\int_0^t \gamma(\tau)\, G_1(\tau)\, d\tau\right] \tag{16}$$

Here we denoted

$$K = A + D_1 \qquad\qquad L = B + D_2 \tag{17}$$

The particular solution in the second approximation can be written as

$$M'(t)\, G_1(t) + N'(t)\, G_2(t) = 0 \tag{18}$$

$$M'(t)\left[-\omega_0\, \varphi(t)^{\frac{1}{2}}\, G_2 - \frac{\varphi'}{4\varphi} G_1\right] + N'(t)\left[\omega_0\, \varphi(t)^{\frac{1}{2}}\, G_1 - \frac{\varphi'}{4\varphi} G_2\right] = -p(t) f_0^2 \tag{19}$$

$$M(t) = \int_0^t \frac{p(\tau) f_0^2}{\omega_0} G_2(\tau) d\tau + K_1 \qquad\qquad N(t) = -\int_0^t \frac{p(\tau) f_0^2}{\omega_0} G_1(\tau) d\tau + K_2 \tag{20}$$

Without losing generality we put $K_1 = K_2 = 0$, then

$$f_1(t) = \frac{1}{\omega_0}\left[G_1(\tau)\int_0^t p(\tau) f_0^2\, G_2(\tau) d\tau - G_2(\tau)\int_0^t p(\tau) f_0^2\, G_1(\tau) d\tau\right] \tag{21}$$

Substituting the functions f_0, and f_1 into the expansion (2) we obtain the approximate analytical solution of the following equation

$$\ddot{f}(t) + \omega^2(t) f + \alpha P(t) f^2 = \gamma(t) \tag{1*}$$

in the form

$$\begin{aligned} f(t) = {} & G_1(t)\left[K - \frac{1}{\omega_0}\int_0^t \gamma(\tau)\, G_2(\tau)\, d\tau\right] + G_2(t)\left[L + \frac{1}{\omega_0}\int_0^t \gamma(\tau)\, G_1(\tau)\, d\tau\right] \\ & + \alpha\left\{M G_1(t) + N G_2(t)\, \frac{1}{\omega_0} + \left[G_1(\tau)\int_0^t p(\tau) f_0^2\, G_2(\tau)\, d\tau - G_2(\tau)\int_0^t p(\tau) f_0^2\, G_1(\tau) d\tau\right]\right\} \end{aligned} \tag{22}$$

Finally the solution (22) for two approximations for α and for three approximations for $\left(\frac{1}{\omega_0}\right)$ of the initial equation (1^*) is given in the form

$$f(t)=G_1(t)\left[K-\frac{1}{\omega_0}\gamma_2(t)\right]+G_2(t)\left[L+\frac{1}{\omega_0}\gamma_1(t)\right]$$

$$+\alpha\times\left\{G_1(t)\left[-\frac{1}{\omega_0^3}p_{112}\left(\gamma_2^2\right)+\frac{K^2}{\omega_0}p_{112}+\frac{1}{\omega_0^3}p_{222}\left(\gamma_1^2\right)+\frac{L^2}{\omega_0}p_{222}-\frac{2K}{\omega_0^2}p_{112}(\gamma_2)\right.\right.$$

$$\left.-\frac{2}{\omega_0^3}p_{112}(\gamma_2,\gamma_1)-\frac{2L}{\omega_0^2}p_{221}(\gamma_2)+\frac{2K}{\omega_0^2}p_{221}(\gamma_1)+\frac{2KL}{\omega_0}p_{221}+\frac{2L}{\omega_0^2}p_{222}(\gamma_1)\right] \qquad (23)$$

$$-G_2(t)\left[-\frac{1}{\omega_0^3}p_{111}\left(\gamma_2^2\right)+\frac{K^2}{\omega_0}p_{111}+\frac{1}{\omega_0^3}p_{122}\left(\gamma_1^2\right)+\frac{L^2}{\omega_0}p_{122}-\frac{2K}{\omega_0^2}p_{111}(\gamma_2)\right.$$

$$\left.\left.-\frac{2}{\omega_0^3}p_{112}(\gamma_1,\gamma_2)-\frac{2L}{\omega_0^2}p_{112}(\gamma_2)+\frac{2K}{\omega_0^2}p_{112}(\gamma_1)+\frac{2KL}{\omega_0}p_{112}+\frac{2L}{\omega_0^2}p_{122}(\gamma_1)\right]\right\}$$

The integrals that correspond to solution (23) can be written as

$$p_{112}\left(\gamma_2^2\right)=\int_0^t p(\tau)\,G_1^2(\tau)\,G_2(\tau)\left(\int_0^\tau\gamma(z)\,G_2(z)\,dz\right)^2 d\tau$$

$$p_{112}=\int_0^t p(\tau)\,G_1^2(\tau)\,G_2(\tau)\,d\tau$$

$$p_{222}\left(\gamma_1^2\right)=\int_0^t p(\tau)\,G_2^3(\tau)\left(\int_0^\tau\gamma(z)G_1(z)\,dz\right)^2 d\tau$$

$$p_{222}=\int_0^t p(\tau)\,G_2^3(\tau)\,d\tau$$

$$p_{122}(\gamma_2)=\int_0^t p(\tau)\,G_1(\tau)\,G_2^2(\tau)\left(\int_0^\tau\gamma(z)\,G_2(z)\,dz\right)d\tau$$

$$p_{122}(\gamma_2,\gamma_1)=\int_0^t p(\tau)\,G_1(\tau)\,G_2^2(\tau)\left(\int_0^\tau\gamma(z)\,G_1(z)\,dz\right)\left(\int_0^\tau\gamma(z)\,G_2(z)\,dz\right)d\tau$$

$$p_{221}(\gamma_2)=\int_0^t p(\tau)\,G_2^2(\tau)\,G_1(\tau)\left(\int_0^\tau\gamma(z)\,G_2(z)\,dz\right)d\tau$$

$$p_{221}(\gamma_1)=\int_0^t p(\tau)\,G_2^2(\tau)\,G_1(\tau)\left(\int_0^\tau\gamma(z)\,G_1(z)\,dz\right)d\tau$$

$$p_{221}=\int_0^t p(\tau)\,G_2^2(\tau)\,G_1(\tau)d\tau$$

$$p_{222}(\gamma_1)=\int_0^t p(\tau)\,G_2^3(\tau)\left(\int_0^\tau\gamma(z)\,G_1(z)\,dz\right)d\tau$$

$$p_{111}\left(\gamma_2^2\right)=\int_0^t p(\tau)\,G_1^3(\tau)\left(\int_0^\tau\gamma(z)\,G_2(z)\,dz\right)^2 d\tau$$

$$p_{111}=\int_0^t p(\tau)\,G_1^3(\tau)\,d\tau$$

$$p_{122}\left(\gamma_1^2\right) = \int_0^t p(\tau)\, G_1(\tau)\, G_2^2(\tau)\left(\int_0^\tau \gamma(z)\, G_1(z)\, dz\right)^2 d\tau$$

$$p_{122} = \int_0^t p(\tau)\, G_1(\tau)\, G_2^2(\tau) d\tau \tag{24}$$

$$p_{111}(\gamma_2) = \int_0^t p(\tau)\, G_1^3(\tau)\left(\int_0^\tau \gamma(z)\, G_2(z)\, dz\right) d\tau$$

$$p_{122}(\gamma_2, \gamma_1) = \int_0^t p(\tau)\, G_1^2(\tau)\, G_2(\tau)\left(\int_0^\tau \gamma(z)\, G_2(z)\, dz\right)\left(\int_0^\tau \gamma(z)\, G_1(z)\, dz\right) d\tau$$

$$p_{221}(\gamma_2) = \int_0^t p(\tau)\, G_2^2(\tau)\, G_1(\tau)\left(\int_0^\tau \gamma(z)\, G_2(z)\, dz\right) d\tau$$

$$p_{112}(\gamma_1) = \int_0^t p(\tau)\, G_1^2(\tau)\, G_2(\tau)\left(\int_0^\tau \gamma(z)\, G_1(z)\, dz\right) d\tau$$

$$p_{122}(\gamma_1) = \int_0^t p(\tau)\, G_1(\tau)\, G_2^2(\tau)\left(\int_0^t \gamma(z)\, G_1(z)\, dz\right) d\tau$$

From the general solution (23) of the initial nonlinear dynamics equation (1^*) with quadratic nonlinearity follows that we can neglect the terms of an order ε^1 (we take into account terms of an order ε^{-1} and ε^0). In this simplification our approximate analytical solution of equation (1^*) becomes

$$f(t) = G_1(t)\left[K - \frac{1}{\omega_0}\gamma_2(t)\right] + G_2(t)\left[L + \frac{1}{\omega_0}\gamma_1(t)\right] + \alpha\left\{G_1(t)\left[\frac{K^2}{\omega_0}p_{112} + \frac{L^2}{\omega_0}p_{222} + \frac{2KL}{\omega_0}p_{221}\right]\right.$$
$$\left. - G_2(t)\left[\frac{K^2}{\omega_0}p_{111} + \frac{L^2}{\omega_0}p_{122} + \frac{2KL}{\omega_0}p_{112}\right]\right\} \tag{25}$$

The constants K and L can be obtained from the initial conditions, for example,

$$f(0) = 0 \tag{26}$$

$$f'(0) = \varphi_0 \tag{27}$$

In the following sections we shall discuss some specific problems on the basis of given approximate analytical solutions.

3 Nonlinear Oscillations with Constant Force

The integrals for the nonlinear solution (25) with constant right hand side of initial differential equation (1^*) of motion can be evaluated as

$$
\begin{aligned}
p_{112} &= -\frac{1}{3}\,\frac{P}{\omega_0}\cos^3(\omega_0 t) \\
p_{222} &= \frac{1}{12}P\,\frac{1}{\omega_0}\left[-9\cos\left(\omega_o t\right) + \cos(3\omega_0 t)\right] \\
p_{221} &= \frac{1}{3}\,\frac{P}{\omega_0}\,\sin^3(\omega_0 t) \\
p_{111} &= \frac{1}{12}\,\frac{P}{\omega_0}\left[-9\,\sin(\omega_0 t) + \sin(3\omega_0 t)\right] \\
p_{122} &= \frac{1}{3}\,\frac{P}{\omega_0}\sin^3(\omega_0 t)
\end{aligned}
\tag{28}
$$

Finally the solution of the nonlinear problem can be written as

$$
\begin{aligned}
f(\tau) &= KG_1 + LG_2 - G_1\frac{\Omega}{m\omega_0}\int_0^t G_2(\tau)\,d\tau + G_2\frac{\Omega}{m\omega_0}\int_0^t G_1(\tau)\,d\tau \\
&+ \frac{\alpha}{\omega_0}\left\{K^2\left(G_1\,p_{112} - G_2\,p_{111}\right) + 2KL\left(G_1\,p_{221} - G_2\,p_{112}\right) + L^2\left(G_1\,p_{222} - G_2\,p_{221}\right)\right\}
\end{aligned}
\tag{29}
$$

Some numerical calculations for this specific case are given in the following Figures 1 to 4.

Figure 1. Comparison of Linear and Nonlinear Solutions

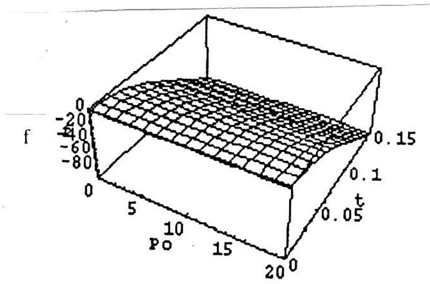

Figure 2. Solution for Small Values of P_0

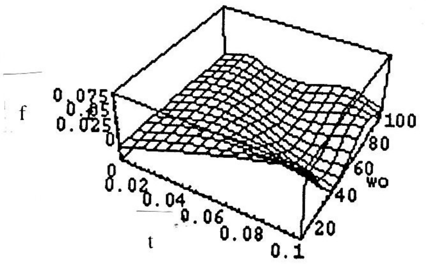

Figure 3. Linear Solution

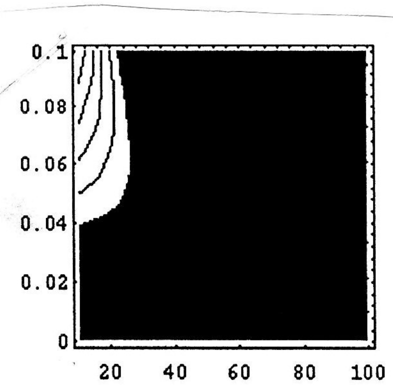

Figure 4. Contour Plot of Linear Solution

4 Oscillation Function for Nonlinear Term with Constant Force

The nonlinear function $P(t)$, the function $\omega(t)$ of the natural frequency, and the forcing function $\gamma(t)$ are considered as

$$P(t) = P_0 \sin(b\,t) \qquad \gamma(t) = \frac{\Omega}{m} = const \qquad \omega(t) = \omega_0 = const \tag{30}$$

For this specific case the integrals in the solution (25) are

$$\begin{aligned}
p_{112} &= P_0 \int \left[\cos(\omega_0 t)\right]^2 \sin(\omega_0 t)\sin(bt)dt \\
&= \frac{P_0}{8\left(9\omega_0^2 - b^2\right)\left(\omega_0^2 - b^2\right)}\Big\{\sin\left[(b - 3\omega_0)t\right]\left(-b^3 - 3b^2\omega_0 + b\omega_0^2 + 3\omega_0^3\right) \\
&\quad +\sin\left[(b - \omega_0)t\right]\left(-b^3 - b^2\omega_0 + 9b\omega_0^2 + 9\omega_0^3\right) \\
&\quad +\sin\left[(b + \omega_0)t\right]\left(b^3 - b^2\omega_0 - 9b\omega_0^2 + 9\omega_0^3\right) \\
&\quad +\sin\left[(b + 3\omega_0)t\right]\left(b^3 - 3b^2\omega_0 - b\omega_0^2 + 3\omega_0^3\right)\Big\} \\
p_{221} &= P_0 \int \left[\sin(\omega_0 t)\right]^2 \cos(\omega_0 t)\sin(bt)dt \\
&= \frac{P_0}{8\left(9\omega_0^2 - b^2\right)\left(\omega_0^2 - b^2\right)}\Big\{\cos\left[(b - 3\omega_0)t\right]\left(b^3 + 3b^2\omega_0 - b\omega_0^2 - 3\omega_0^3\right) \\
&\quad +\cos\left[(b - \omega_0)t\right]\left(b^3 - b^2\omega_0 + 9b\omega_0^2 + 9\omega_0^3\right) \\
&\quad +\cos\left[(b + \omega_0)t\right]\left(b^3 + b^2\omega_0 + 9b\omega_0^2 - 9\omega_0^3\right) \\
&\quad +\cos\left[(b + 3\omega_0)t\right]\left(b^3 - 3b^2\omega_0 - b\omega_0^2 + 3\omega_0^3\right)\Big\} \\
p_{222} &= P_0 \int \left[\sin(\omega_0 t)\right]^3 \sin(bt)dt \\
&= \frac{P_0}{8\left(9\omega_0^2 - b^2\right)\left(\omega_0^2 - b^2\right)}\Big\{\sin\left[(b - 3\omega_0)t\right]\left(-b^3 - 3b^2\omega_0 + b\omega_0^2 + 3\omega_0^3\right) \\
&\quad +\sin\left[(b - \omega_0)t\right]\left(3b^3 + 3b^2\omega_0 - 27b\omega_0^2 - 27\omega_0^3\right) \\
&\quad +\sin\left[(b + \omega_0)t\right]\left(-3b^3 + 3b^2\omega_0 + 27b\omega_0^2 - 27\omega_0^3\right) \\
&\quad +\sin\left[(b + 3\omega_0)t\right]\left(b^3 - 3b^2\omega_0 - b\omega_0^2 + 3\omega_0^3\right)\Big\} \\
p_{111} &= P_0 \int \left[\cos(\omega_o t)\right]^3 \sin(bt)dt \\
&= \frac{P_0}{8\left(9\omega_0^2 - b^2\right)\left(\omega_0^2 - b\right)}\Big\{\cos\left[(b - 3\omega_0)t\right]\left(-b^3 - 3b^2\omega_0 + b\omega_0^2 + 3\omega_0^3\right) \\
&\quad +\cos\left[(b - \omega_0)t\right]\left(-3b^3 - 3b^2\omega_0 + 27b\omega_0^2 + 27\omega_0^3\right) \\
&\quad +\cos\left[(b + \omega_0)t\right]\left(-3b^3 + 3b^2\omega_0 + 27b\omega_0^2 - 27\omega_0^3\right) \\
&\quad +\cos\left[(b + 3\omega_0)t\right]\left(-b^3 + 3b^2\omega_0 + b\omega_0^2 - 3\omega_0^3\right)\Big\}
\end{aligned} \tag{31}$$

Taking into account the boundary conditions (26) the constants K and L are

$$K = 0 \quad \text{and} \quad L = \frac{\varphi_0}{\omega_0} \tag{32}$$

Numerical calculation for this example and corresponding figures are given for the parameters

$$t = 0.1 \qquad \omega_0 = 10 \qquad P_0 = 10 \qquad \alpha = 0.1 \qquad W = \frac{\Omega}{m} \tag{33}$$

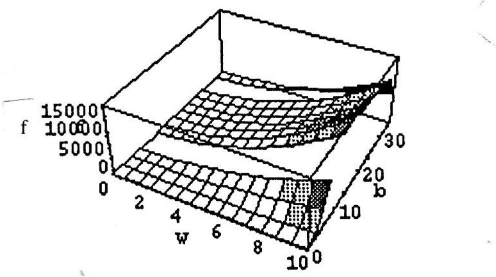

Figure 5. Discontinuity Effects at Constant Forced Oscillations with Periodic $P(t)$

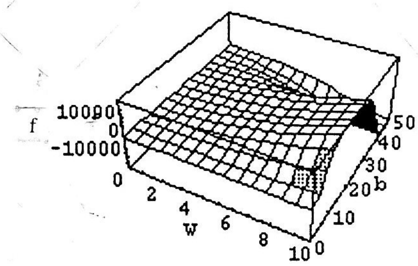

Figure 6. Dependence of Amplitude of Oscillations upon W, b

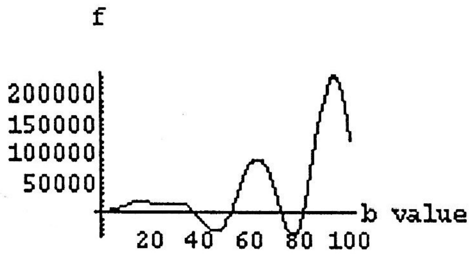

Figure 7. Amplitude of Oscillation at $W = 10$ for Periodic $P(t)$

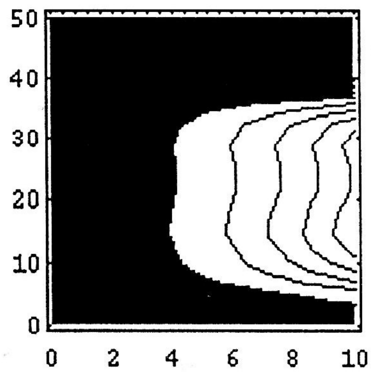

Figure 8. Contour Plot $f = f(W, b)$

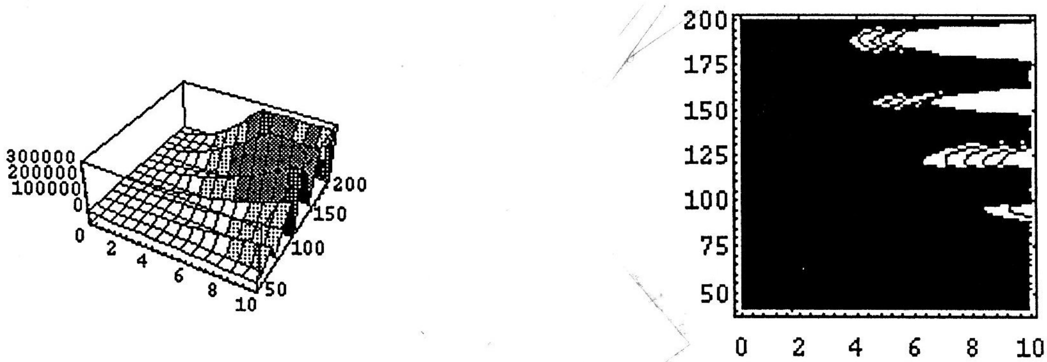

Figure 9. Amplitude Factor and Contour Plot for High Level External Force

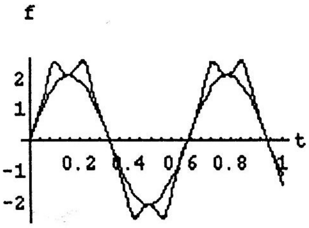

Figure 10. Comparison of Linear and Nonlinear Models for

$$\alpha = 0.1,\ b = 20,\ \frac{\Omega}{m} = 1,\ P_0 = 0.1,\ \omega_0 = 10$$

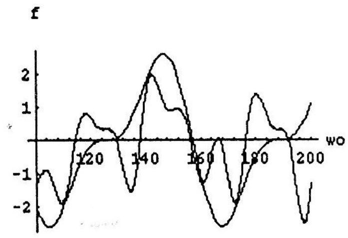

Figure 11. Comparison of Linear and Nonlinear Solutions

$$f = f(\omega_0)$$

5 Power Function for Nonlinear Term with Constant Force

For comparison we shall explore the function

$$P(t) = P_0 t^2 \tag{34}$$

The corresponding coefficient functions in the solution (25) are

$$\begin{aligned}
p_{112} &= P_0\int t^2[\cos(\omega_o t)]^2 \sin(\omega_0 t)dt = \frac{P_0}{108\omega_0^3}\Big[\cos(\omega_0 t)(54 - 27t^2\omega_0^2) \\
&\quad + \cos(3\omega_0 t)(2 - 9t^2\omega_0^2) + \sin(\omega_0 t)(54t\omega_0 + 6t\omega_0)\Big] \\
p_{222} &= P_0\int t^2[\sin(\omega_0 t)]^3 dt = \frac{P_0}{108\omega_0^3}\Big[\cos(\omega_0 t)(162 - 81t^2\omega_0^2) \\
&\quad + \cos(3\omega_0 t)(-2 + 9t^2\omega_0^2) + \sin(\omega_0 t)(162t\omega_0 - 6t\omega_0) \\
p_{221} &= P_0\int t^2[\sin(\omega_0 t)]^2 \cos(\omega_0 t)dt = \frac{P_0}{108\omega_0^3}\Big[54t\omega_0\cos(\omega_0 t) - 6t\omega_0\cos(3\omega_0 t) \\
&\quad + \sin(\omega_0 t)(-54t\omega_0 + 27t^2\omega_0^2) + \sin(3\omega_0 t)(2 - 9t^2\omega_0^2)\Big] \\
p_{111} &= P_0\int t^2[\cos(\omega_0 t)]^3 dt = \frac{P_0}{108\omega_0^3}\Big[162t\omega_0\cos(\omega_0 t) + 6t\omega_0\cos(3\omega_0 t) \\
&\quad + \sin(\omega_0 t)(-162t\omega_0 + 81t^2\omega_0^2) + \sin(3\omega_0 t)(9t^2\omega_0^2 - 2)\Big]
\end{aligned} \tag{35}$$

Taking into account the boundary conditions (26) and (27), the constants K and L are

$$K = -\frac{\Omega}{m\,\omega_0^2}$$

$$\left\{L + \frac{1}{\omega_0^4}\left[\alpha(0.5185K^2P + 1.5185L^2P) - 1.03704\,K\,L\,P\right]\right\}\omega_0 = \dot{\varphi}_0 \tag{36}$$

$$L \cong \frac{\dot{\varphi}_0}{\omega_0}$$

Numerical results of calculations for $\dot{\varphi}_0 = 1$ and the corresponding figures are given for the parameters

$$t = 0.1,\ \omega_0 = 10,\ \alpha = 0.1,\ W = \frac{\Omega}{m} \tag{37}$$

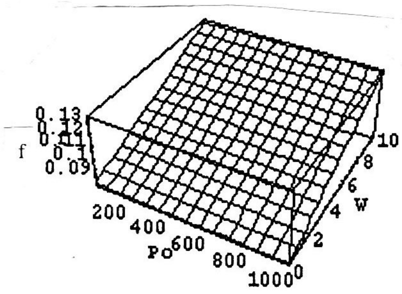

Figure 12. Nonlinear Amplitude Function

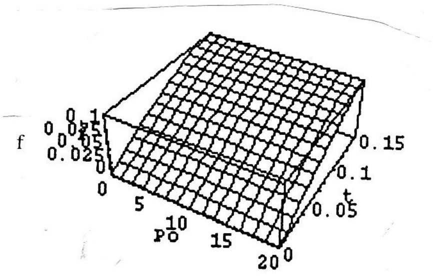

Figure 13. Shape of Nonlinear Vibrations

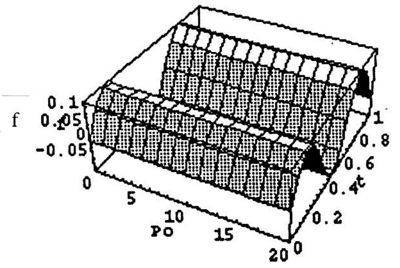

Figure 14. Shape of Nonlinear Solutions

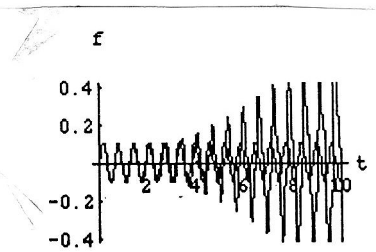

Figure 15. Comparison of Linear and Nonlinear Modes for

$$\alpha = 0.1,\ \frac{\Omega}{m} = 1,\ P_0 = 100,\ \omega_0 = 10$$

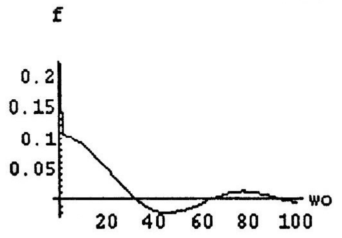

Figure 16. Nonlinear Solution

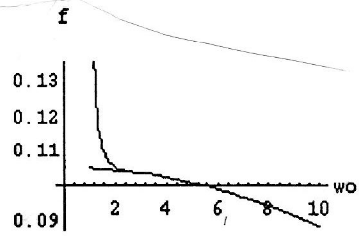

Figure 17. Comparison of Linear and Nonlinear Solutions for t=0.1, P=0.1, W=1, α=0.1

6 Nonlinear Effects for Periodic Forced Oscillations

Taking into account the solution of an initial equation (1^*) we suppose that the forced oscillation is periodic and given by

$$\gamma(t) = \Gamma_0 \sin(a\,t) \tag{38}$$

For this specific case we obtain

$$\begin{aligned} \gamma_1(t) &= \frac{\Gamma_0}{2\left(\omega_0^2 - a^2\right)}\left\{(a + \omega_0)\cos[(a - \omega_0)t] + (a - \omega_0)\cos[(a + \omega_0)t]\right\} \\ \gamma_2(t) &= -\frac{\Gamma_0}{2\left(\omega_0^2 - a^2\right)}\left\{(a + \omega_0)\sin[(a - \omega_0)t] - (a - \omega_0)\sin[(a + \omega_0)t]\right\} \end{aligned} \tag{39}$$

Let us consider as initial conditions for the problem those given by equations (26) and (27). From these equations it follows that

$$K = 0 \quad \text{and} \quad L = \frac{1}{\omega_0}\dot{\varphi}_0 \tag{40}$$

For the case

$$\dot{\varphi}_0 = 1 \quad \text{and} \quad L = \frac{1}{\omega_0} \tag{41}$$

With an assumption $P(t) = P_0 \sin(b\,t)$ the solution for the periodic forced oscillation is

$$\begin{aligned} f(t) = {} & \cos(\omega_0 t)\left\{\frac{\Gamma_0}{2\omega_0\left(\omega_0^2 - a^2\right)}\left[(a+\omega_0)\sin(a-\omega_0)t - (a-\omega_0)\sin(a+\omega_0)t\right]\right\} \\ & + \sin(\omega_0 t)\left\{L + \frac{\Gamma_0}{2\omega_0\left(\omega_0^2 - a^2\right)}\left[(a+\omega_0)\cos(a-\omega_0)t + (a-\omega_0)\cos(a+\omega_0)t\right]\right\} \\ & + \alpha L^2 \frac{1}{\omega_0}\left[\cos(\omega_0 t)\,p_{222} - \sin(\omega_0 t)\,p_{221}\right] \end{aligned} \tag{42}$$

where the integral coefficients p_{ijl} are as given in equations (31). Results of numerical calculations are given in Figures 18 and 19 for

$$P_0 = 1,\ \omega_0 = 10,\ \alpha = 0.1,\ \Gamma_0 = 1,\ b = 20$$

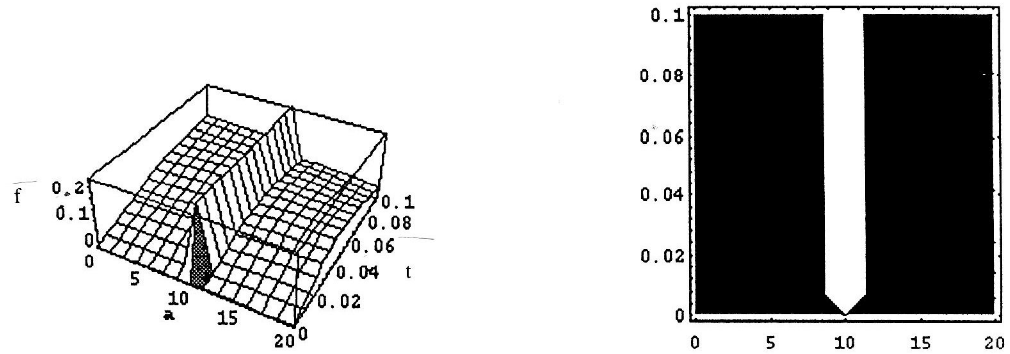

Figure 18 a) Resonance effect

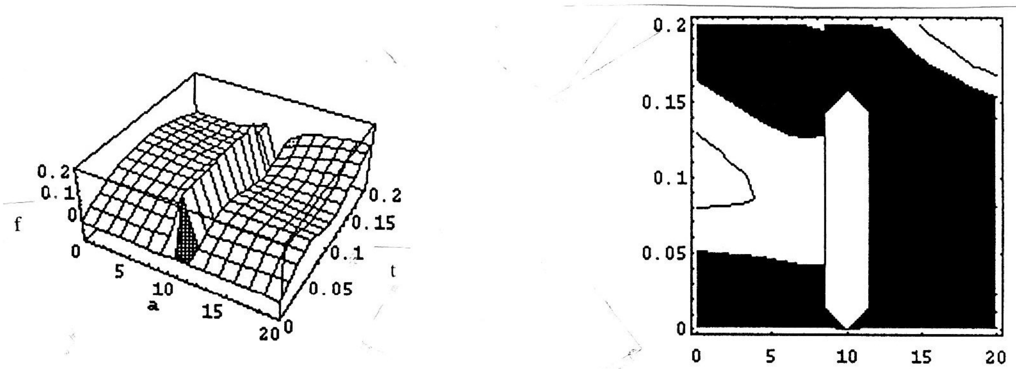

Figure 18 b) Resonance effect

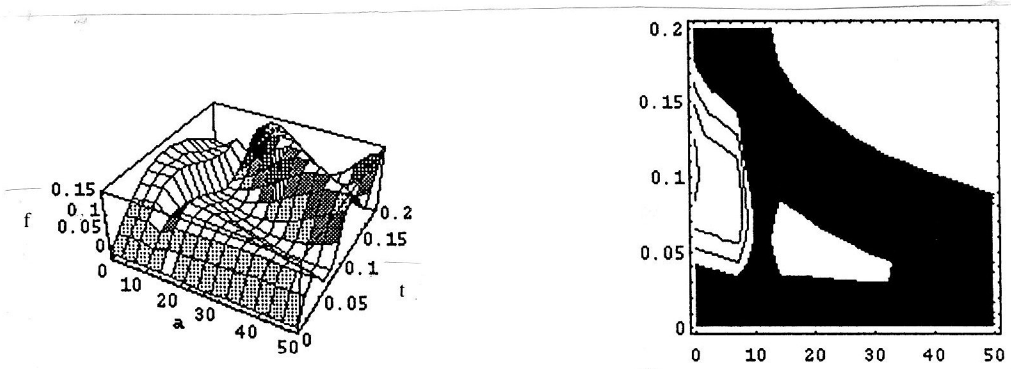

Figure 19. Shape of Vibration

7 Concluding Remarks

The ultimate goal of the proposed double expansion approach is to develop an algorithm for an approximate asymptotic solution of some forced oscillation problems for mechanical systems with time dependent characteristics. An effect of interaction between structural properties of mechanical systems and forcing functions will have to be investigated in more detail especially near the resonance zones.

Acknowledgment

This publication was supported in part by the Ukrainian Committee on Science and Technology under Grant N 63/94 K.

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Addresses: Dr.Sci. Professor V. Z. Gristchak, Vice-President and Chairman of Mathematical Modeling and Information Technology Department; V. N. Kabak, PhD Student, Zaporozhye State University, 66 Zhukovskogo str., UA-330600 Zaporozhye