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Contravariant Components and Covariant Projections in Gyrodynamics

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The concepts of kinetic energy and complementary kinetic energy permit to distinguish between two different formulations of what happens to be the same quantity in Newtonian mechanics. These formulations turn out to play a significant role in gyrodynamics in that they can be used very effectively to establish fundamental equations. In the following Cartesian body-fixed coordinates and Euler angle coordinates will be used to express kinetic energy and complementary kinetic energy and their partial derivatives of a single rigid gyro.

1 Kinetic Energy and Kinetic Coenergy

For a single rigid gyro the definition for the *kinetic energy* T of rotation (Figure 1) is

$$T = \frac{1}{2}\int \omega_i dH_i \qquad i = x, y, z \tag{1}$$

with

$$\boldsymbol{\omega} = \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \{\mathbf{e}_i\}^T \{\omega_i\} = \{\mathbf{e}\}^T \{\omega\} \tag{2}$$

for the angular velocity vector, and

$$\mathbf{H} = \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \{\mathbf{e}_i\}^T \{H_i\} = \{\mathbf{e}\}^T \{H\} \tag{3}$$

for the angular momentum vector. The $Cxyz$ cooordinate system is gyro-fixed, principal and orthogonal, and has its origin in the mass centre C of the gyro (Figure 2). Since $H_x = A\omega_x$, $H_y = B\omega_y$, and $H_z = C\omega_z$, and with

$$[I] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{4}$$

as the gyro's inertia tensor expressed in $Cxyz$ coordinates, the kinetic energy becomes

$$T = \frac{1}{2}\left(\frac{H_x^2}{A} + \frac{H_y^2}{B} + \frac{H_z^2}{C}\right) \tag{5}$$

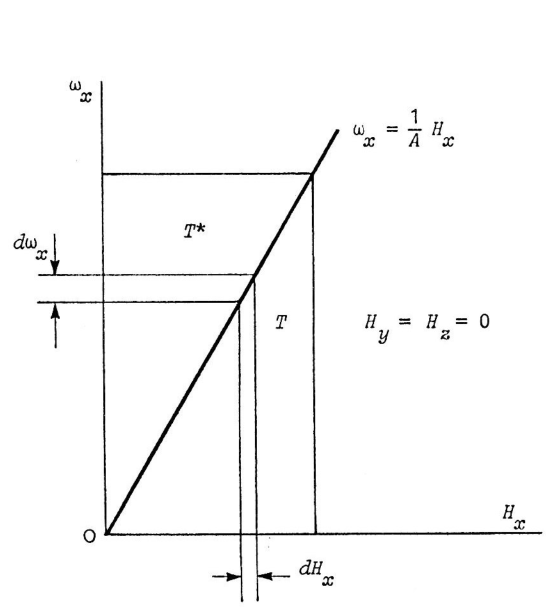

Figure 1. Kinetic Energy T and Kinetic Coenergy T^*

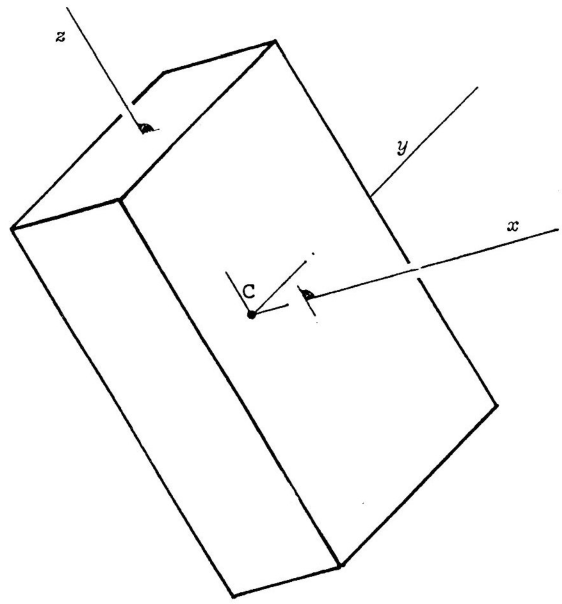

Figure 2. Rectangular Body with Uniformly Distributed Mass, $A > B > C$, Mass Centre C, and Body-fixed **C***xyz* Coordinates

The *complementary kinetic energy* (or *kinetic coenergy*) T^* is defined by

$$T^* = \frac{1}{2}\int H_i d\omega_i \tag{6}$$

Upon integration, it results in the well-known

$$T^* = \frac{1}{2}\left(A\omega_x^2 + B\omega_y^2 + C\omega_z^2\right) \tag{7}$$

In Newtonian mechanics the two quantities $T(H)$ and $T^*(\omega)$ are equal, they merely differ in the variables used (Figure 1). This distinction (Rimrott et al., 1993) is of significance in Analytical Mechanics, where one uses e.g. definitions such as

$$\omega_i = \frac{\partial T}{\partial H_i}$$

in particular

$$\omega_x = \frac{\partial T}{\partial H_x} = \frac{H_x}{A} \tag{8a}$$

$$\omega_y = \frac{\partial T}{\partial H_y} = \frac{H_y}{B} \tag{8b}$$

$$\omega_z = \frac{\partial T}{\partial H_z} = \frac{H_z}{C} \tag{8c}$$

or

$$H_i = \frac{\partial T^*}{\partial \omega_i}$$

in particular

$$H_x = \frac{\partial T^*}{\partial \omega_x} = A\omega_x \tag{9a}$$

$$H_y = \frac{\partial T^*}{\partial \omega_y} = B\omega_y \tag{9b}$$

$$H_z = \frac{\partial T^*}{\partial \omega_z} = C\omega_z \tag{9c}$$

We may write for the kinetic coenergy (7)

$$T^* = \frac{1}{2}\{\omega\}^T[A_0]\{\omega\} \tag{10}$$

with

$$\{\omega\} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{11}$$

and

$$[A_0] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{12}$$

For the angular momentum (9) we may write

$$\{H\} = [A_0]\,\{\omega\} \tag{13}$$

with

$$\{H\} = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \tag{14}$$

For the kinetic energy (5) one may write

$$T = \frac{1}{2}\{H\}\,[A_0]^{-1}\,\{H\} \tag{15}$$

with

$$[A_0]^{-1} = \begin{bmatrix} \frac{1}{A} & 0 & 0 \\ 0 & \frac{1}{B} & 0 \\ 0 & 0 & \frac{1}{C} \end{bmatrix} \tag{16}$$

For the angular velocity (8) one obtains

$$\{\omega\} = [A_0]^{-1}\,\{H\} \tag{17}$$

2 Eulerian Coordinates

In addition for the gyro-fixed Cartesian coordinates $Cxyz$, there are other coordinate systems, such as the Euler angle system, or the Cardan angle systems (Rimrott, 1988), each with distinct advantages - and of course corresponding disadvantages.

Very common is the Eulerian coordinate system consisting of the precession angle ψ , the nutation angle ν , and the spin angle σ , which we shall employ exclusively for the present paper.

The angular velocity vector $\boldsymbol{\omega}$ (Figures 3 and 4) may be consequently expressed by

$$\boldsymbol{\omega} = \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \mathbf{e}_\psi & \mathbf{e}_\nu & \mathbf{e}_\sigma \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\nu} \\ \dot{\sigma} \end{bmatrix} \tag{18}$$

and a relationship between the components can be established.

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \sin\nu\sin\sigma & \cos\sigma & 0 \\ \sin\nu\cos\sigma & -\sin\sigma & 0 \\ \cos\nu & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\nu} \\ \dot{\sigma} \end{bmatrix} \quad \text{or} \quad \{\omega_i\} = [J_1]\{\dot{q}_k\} \qquad k = \psi, \nu, \sigma \tag{19}$$

While the $Cxyz$ coordinate system, with basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, is orthogonal (Figure 3), the Eulerian system, with basis vectors $\mathbf{e}_\psi$, $\mathbf{e}_\nu$, and $\mathbf{e}_\sigma$, is not (Figure 4).

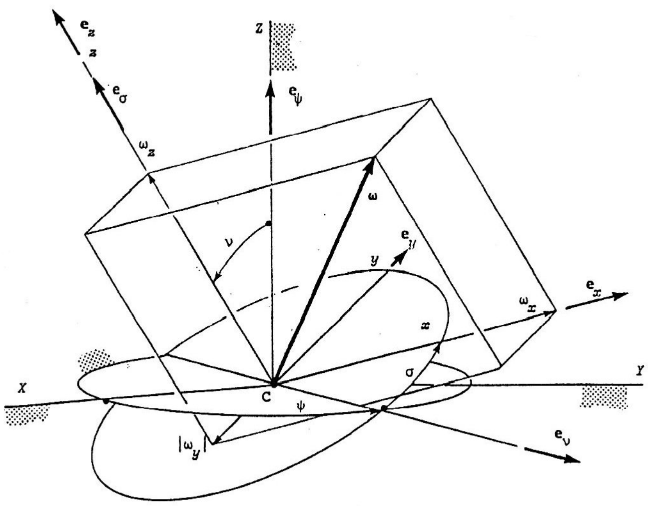

Figure 3. Position of Gyro-fixed $Cxyz$ Coordinates in Euler Angles, and Angular Velocity $\boldsymbol{\omega}$

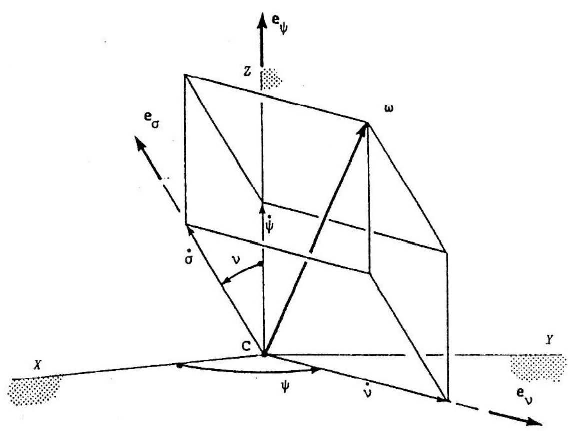

Figure 4. Eulerian Contravariant Components of the Angular Velocity

There is second possibility for describing the angular velocity, and that is to use covariant projections of the angular velocity vector upon the Eulerian axes $\mathbf{e}_\psi$, $\mathbf{e}_\nu$ and $\mathbf{e}_\sigma$. We shall name these projections ω_ψ, ω_ν, and ω_σ . The transformation equation between Cartesian and second Eulerian formulation is

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\sigma}{\sin\nu} & \cos\sigma & -\dfrac{\sin\sigma}{\tan\nu} \\ \dfrac{\cos\sigma}{\sin\nu} & -\sin\sigma & -\dfrac{\cos\sigma}{\tan\nu} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_\psi \\ \omega_\nu \\ \omega_\sigma \end{bmatrix} \quad \text{or} \quad \{\omega_i\} = [J_2]\{\omega_k\} = [J_1]^{-T}\{\omega_k\} \tag{20}$$

since it can readily be shown that the Jacobian matrices $[J_2]$ and $[J_1]$ are related, by $[J_2] = [J_1]^{-T}$.

Other relationship of interest are

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \sin\nu\sin\sigma & \cos\sigma & 0 \\ \sin\nu\cos\sigma & -\sin\sigma & 0 \\ \cos\nu & 0 & 1 \end{bmatrix} \begin{bmatrix} H_\psi \\ H_\nu \\ H_\sigma \end{bmatrix} \quad \text{or} \quad \{H_i\} = [J_1]\{H_k\} \tag{21}$$

$$\begin{bmatrix} H_\psi \\ H_\nu \\ H_\sigma \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\sigma}{\sin\nu} & \dfrac{\cos\sigma}{\sin\nu} & 0 \\ \cos\sigma & -\sin\sigma & 0 \\ -\dfrac{\sin\sigma}{\tan\nu} & -\dfrac{\cos\sigma}{\tan\nu} & 1 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \quad \text{or} \quad \{H_k\} = [J_1]^{-1}\{H_i\} \tag{21a}$$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\sigma}{\sin\nu} & \cos\sigma & -\dfrac{\sin\sigma}{\tan\nu} \\ \dfrac{\cos\sigma}{\sin\nu} & -\sin\sigma & -\dfrac{\cos\sigma}{\tan\nu} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_\psi \\ p_\nu \\ p_\sigma \end{bmatrix} \quad \text{or} \quad \{H_i\} = [J_2]\{p_k\} = [J_1]^{-T}\{p_k\} \tag{22}$$

The inverse of equation (22) is

$$\begin{bmatrix} p_\psi \\ p_\nu \\ p_\sigma \end{bmatrix} = \begin{bmatrix} \sin\nu\sin\sigma & \sin\nu\cos\sigma & \cos\nu \\ \cos\sigma & -\sin\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \quad \text{or} \quad \{p_k\} = [J_1]^T\{H_i\} \tag{22a}$$

The column matrix $\{p_k\}$ will be defined in the section following.
Equating relations (21) and (22) leads to

$$\{H_k\} = [J_2]^T[J_2]\{p_k\} = [J_1]^{-1}[J_1]^{-T}\{p_l\} \quad \text{with} \quad \begin{matrix} k = \psi, \nu, \sigma \\ l = \psi, \nu, \sigma \end{matrix} \tag{23}$$

or

$$\begin{bmatrix} H_\psi \\ H_\nu \\ H_\sigma \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sin^2\nu} & 0 & -\dfrac{\cos\nu}{\sin^2\nu} \\ 0 & 1 & 0 \\ -\dfrac{\cos\nu}{\sin^2\nu} & 0 & \dfrac{1}{\sin^2\nu} \end{bmatrix} \begin{bmatrix} p_\psi \\ p_\nu \\ p_\sigma \end{bmatrix}$$

An inversion of equation (23) results in

$$\{p_l\} = [J_2]^{-1}[J_2]^{-T}\{H_k\} = [J_1]^T[J_1]\{H_k\} \tag{23a}$$

$$\begin{bmatrix} p_\psi \\ p_\nu \\ p_\sigma \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cos\nu \\ 0 & 1 & 0 \\ \cos\nu & 0 & 1 \end{bmatrix} \begin{bmatrix} H_\psi \\ H_\nu \\ H_\sigma \end{bmatrix}$$

An inversion of equation (19) results in

$$\{\dot{q}_k\} = [J_1]^{-1}\{\omega_i\} \tag{24}$$

or

$$\begin{bmatrix} \dot{\psi} \\ \dot{\nu} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\sigma}{\sin\nu} & \dfrac{\cos\sigma}{\sin\nu} & 0 \\ \cos\sigma & -\sin\sigma & 0 \\ -\dfrac{\sin\sigma}{\tan\nu} & -\dfrac{\cos\sigma}{\tan\nu} & 1 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

An inversion of equation (20) results in

$$\{\omega_k\} = [J_2]^{-1}\{\omega_i\} = [J_1]^T\ \{\omega_i\} \tag{24a}$$

or

$$\begin{bmatrix} \omega_\psi \\ \omega_\sigma \\ \omega_\nu \end{bmatrix} = \begin{bmatrix} \sin\nu\sin\sigma & \sin\nu\cos\sigma & \cos\nu \\ \cos\sigma & -\ \sin\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

A multiplication of equations (25) and (19) gives

$$\{\omega_l\} = [J_1]^T\ [J_1]\ \{\dot{q}_k\} \tag{26}$$

or

$$\begin{bmatrix} \omega_\psi \\ \omega_\nu \\ \omega_\sigma \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cos\nu \\ 0 & 1 & 0 \\ \cos\nu & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot\psi \\ \dot\nu \\ \dot\sigma \end{bmatrix}$$

and an inversion leads to

$$\{\dot q_k\} = [J_1]^{-1}[J_1]^{-T}\{\omega_l\} \tag{26a}$$

or

$$\begin{bmatrix} \dot\psi \\ \dot\nu \\ \dot\sigma \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sin^2\nu} & 0 & -\dfrac{\cos\nu}{\sin^2\nu} \\ 0 & 1 & 0 \\ -\dfrac{\cos\nu}{\sin^2\nu} & 0 & \dfrac{1}{\sin^2\nu} \end{bmatrix} \begin{bmatrix} \omega_\psi \\ \omega_\nu \\ \omega_\sigma \end{bmatrix}$$

3 Kinetic Coenergy $T^*(q,\dot q)$

We begin with equation (7)

$$T^* = \frac{1}{2}\left(A\omega_x^2 + B\omega_y^2 + C\omega_z^2\right)$$

and use equation (19) to obtain the kinetic coenergy in terms of the Euler angles $q_1 = \psi$, $q_2 = \nu$, $q_3 = \sigma$, and their time derivatives.

$$T^*(q,\dot q) = \frac{1}{2}\left(A(\dot\psi\sin\nu\sin\sigma + \dot\nu\cos\sigma)^2 + B(\dot\psi\sin\nu\cos\sigma - \dot\nu\sin\sigma)^2 + C(\dot\psi\cos\nu + \dot\sigma)^2\right) \tag{27}$$

The covariant angular momentum projections (Figure 7) then are defined as

$$p_k = \frac{\partial T^*(q,\dot q)}{\partial \dot q_k}$$

in particular

$$p_\psi = \frac{\partial T^*}{\partial\dot\psi} = \left(\left(A\sin^2\sigma + B\cos^2\sigma\right)\sin^2\nu + C\cos^2\nu\right)\dot\psi + \left((A-B)\sin\nu\sin\sigma\cos\sigma\right)\dot\nu + (C\cos\nu)\dot\sigma \tag{28a}$$

$$p_\nu = \frac{\partial T^*}{\partial\dot\nu} = \left((A-B)\sin\nu\sin\sigma\cos\sigma\right)\dot\psi + \left(A\cos^2\sigma + B\sin^2\sigma\right)\dot\nu \tag{28b}$$

$$p_\sigma = \frac{\partial T^*}{\partial\dot\sigma} = (C\cos\nu)\dot\psi + C\dot\sigma \tag{28c}$$

If a (symmetric) matrix

$$[A_1] = \begin{bmatrix} \left(A\sin^2\sigma + B\cos^2\sigma\right)\sin^2\nu + C\cos^2\nu & (A-B)\sin\nu\sin\sigma\cos\sigma & C\cos\nu \\ (A-B)\sin\nu\sin\sigma\cos\sigma & A\cos^2\sigma + B\sin^2\sigma & 0 \\ C\cos\nu & 0 & C \end{bmatrix} \tag{29}$$

is introduced, the kinetic coenergy (27) can then also be expressed as

$$T^* = \frac{1}{2}\{\dot{q}\}^T[A_1]\{\dot{q}\} \tag{30}$$

An inspection of equations (28) and (29) shows that

$$\{p\} = [A_1]\{\dot{q}\} \tag{31}$$

leading to the conclusion that the generalized (angular) momentum $\{p\}$ can be obtained by a mapping procedure. It is interesting to note, that while $\{\dot{q}\}$ represents the *contravariant* components of the angular velocity vector $\boldsymbol{\omega}$, the quantity $\{p\}$ represents the *covariant* projections of the angular momentum vector $\mathbf{H}$.

4 Kinetic Coenergy $T^*(q,\omega)$

Beginning again with equation (7), and using equations (20), we obtain for the kinetic coenergy

$$T^*(q,\omega) = \frac{1}{2}\left(A\left(\left(\omega_\psi - \omega_\sigma\cos\nu\right)\frac{\sin\sigma}{\sin\nu} + \omega_\nu\cos\sigma\right)^2 + B\left(\left(\omega_\psi - \omega_\sigma\cos\nu\right)\frac{\cos\sigma}{\sin\nu} - \omega_\nu\sin\sigma\right)^2 + C\omega_\sigma^2\right) \tag{32}$$

The contravariant angular momentum components then are defined as

$$H_k = \frac{\partial T^*(q,\omega)}{\partial\omega_k}$$

in particular

$$H_\psi = \frac{\partial T^*}{\partial\omega_\psi} = \frac{1}{\sin^2\nu}\left(A\sin^2\sigma + B\cos^2\sigma\right)\omega_\psi + \frac{\sin\sigma\cos\sigma}{\sin\nu}(A-B)\omega_\nu - \frac{1}{\sin\nu\tan\nu}\left(A\sin^2\sigma + B\cos^2\sigma\right)\omega_\sigma \tag{33a}$$

$$H_\nu = \frac{\partial T^*}{\partial\omega_\nu} = \frac{\sin\sigma\cos\sigma}{\sin\nu}(A-B)\omega_\psi + \left(A\cos^2\sigma + B\sin^2\sigma\right)\omega_\nu - \frac{\sin\sigma\cos\sigma}{\tan\nu}(A-B)\omega_\sigma \tag{33b}$$

$$H_\sigma = \frac{\partial T^*}{\partial\omega_\sigma} = -\frac{1}{\sin\nu\tan\nu}\left(A\sin^2\sigma + B\cos^2\sigma\right)\omega_\psi - \frac{\sin\sigma\cos\sigma}{\tan\nu}(A-B)\omega_\nu + \left(C + \frac{1}{\tan^2\nu}\left(A\sin^2\sigma + B\cos^2\sigma\right)\right)\omega_\sigma \tag{33c}$$

If a (symmetric) matrix

$$[A_2] = \begin{bmatrix} \left(A\sin^2\sigma + B\cos^2\sigma\right)\frac{1}{\sin^2\nu} & (A-B)\frac{\sin\sigma\cos\sigma}{\sin\nu} & -\left(A\sin^2\sigma + B\cos^2\sigma\right)\frac{1}{\sin\nu\tan\nu} \\ (A-B)\frac{\sin\sigma\cos\sigma}{\sin\nu} & A\cos^2\sigma + B\sin^2\sigma & -(A-B)\frac{\sin\sigma\cos\sigma}{\tan\nu} \\ -\left(A\sin^2\sigma + B\cos^2\sigma\right)\frac{1}{\sin\nu\tan\nu} & -(A-B)\frac{\sin\sigma\cos\sigma}{\tan\nu} & C + \left(A\sin^2\sigma + B\cos^2\sigma\right)\frac{1}{\tan^2\nu} \end{bmatrix} \tag{34}$$

is introduced, the kinetic coenergy (18) can then also be expressed as

$$T^* = \frac{1}{2}\{\omega\}^T[A_2]\{\omega\} \tag{35}$$

and an inspection of equations (33) and (34) shows that

$$\{H\} = [A_2]\{\omega\} \tag{36}$$

leading again to the conclusion that the angular momentum $\{H\}$ can be obtained by a mapping procedure. In equation (36) the $\{\omega\}$ column matrix contains the *covariant* projections of the angular velocity vector $\boldsymbol{\omega}$, while the $\{H\}$ column matrix contains the *contravariant* components of the angular momentum vector $\mathbf{H}$.

5 Angular Velocity $\boldsymbol{\omega}$, Angular Momentum H and Torque M

The angular velocity vector $\boldsymbol{\omega}$ may be expressed in gyro-fixed Cartesian coordinates and in Eulerian coordinates (Figure 5) by

$$\boldsymbol{\omega} = \begin{bmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\end{bmatrix}\begin{bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{bmatrix} = \begin{bmatrix}\mathbf{e}_\psi & \mathbf{e}_\nu & \mathbf{e}_\sigma\end{bmatrix}\begin{bmatrix}\dot\psi \\ \dot\nu \\ \dot\sigma\end{bmatrix} \tag{37}$$

Since Eulerian coordinates are non-orthogonal, there is the possibility of representing the angular velouty vector $\boldsymbol{\omega}$ by means of covariant projections ω_ψ, ω_ν, and ω_σ (Figure 6), as given by equation (20). The magnitude ω of the angular velocity is obtainable from

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2 \tag{38a}$$

$$\omega^2 = \dot\psi^2 + \dot\nu^2 + \dot\sigma^2 + 2\dot\psi\dot\sigma\cos\nu \tag{38b}$$

$$\omega^2 = \frac{1}{\sin^2\nu}\left(\omega_\psi^2 + \omega_\nu^2\sin^2\nu + \omega_\sigma^2\right) - \frac{2\cos\nu}{\sin^2\nu}\omega_\psi\omega_\sigma \tag{38c}$$

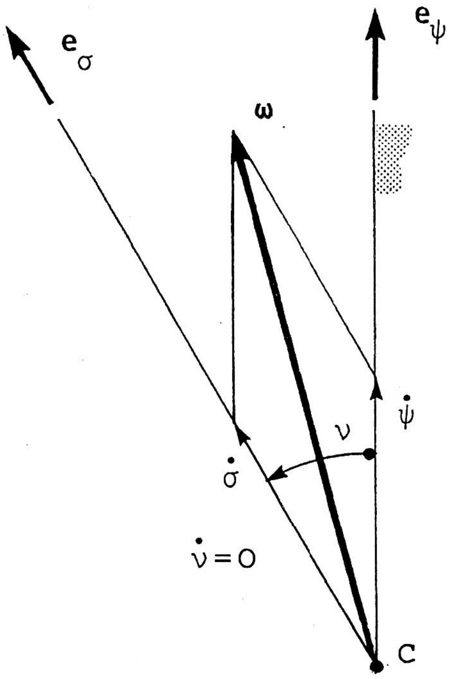

Figure 5. Contravariant Components of the Angular Velocity

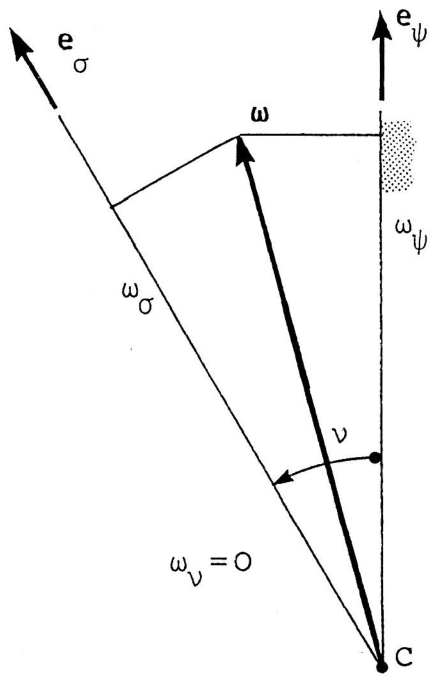

Figure 6. Covariant Projections of the Angular Velocity

The angular momentum (Figure 7) is

$$\mathbf{H} = \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \mathbf{e}_\psi & \mathbf{e}_\nu & \mathbf{e}_\sigma \end{bmatrix} \begin{bmatrix} H_\psi \\ H_\nu \\ H_\sigma \end{bmatrix} \tag{39}$$

There are also covariant projections p_ψ, p_ν, and p_σ (Figure 8) of the angular momentum, as given by equations (22). The magnitude H of the angular momentum is obtainable from

$$H^2 = H_x^2 + H_y^2 + H_z^2 \tag{40a}$$

$$H^2 = H_\psi^2 + H_\nu^2 + H_\sigma^2 + 2H_\psi H_\sigma \cos\nu \tag{40b}$$

$$H^2 = \frac{1}{\sin^2\nu}\left(p_\psi^2 + p_\nu^2 \sin^2\nu + p_\sigma^2\right) - \frac{2\cos\nu}{\sin^2\nu} p_\psi p_\sigma \tag{40c}$$

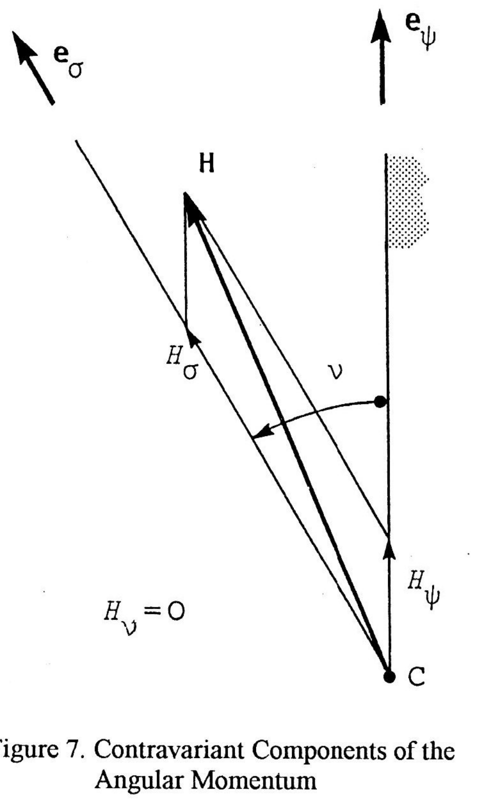

Figure 7. Contravariant Components of the Angular Momentum

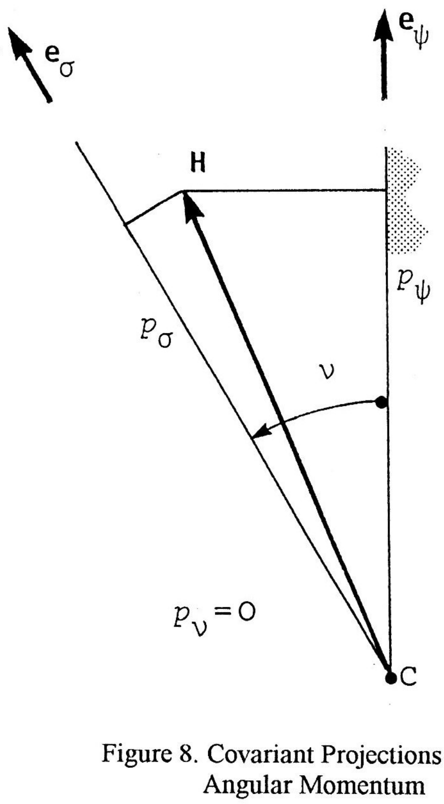

Figure 8. Covariant Projections of the Angular Momentum

The torque **M** (Figure 9) acting on the gyro can be expressed by

$$\mathbf{M} = \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} \mathbf{e}_\psi & \mathbf{e}_\nu & \mathbf{e}_\sigma \end{bmatrix} \begin{bmatrix} M_\psi \\ M_\nu \\ M_\sigma \end{bmatrix} \tag{41}$$

It may also be expressed in terms of the covariant projections Q_ψ, Q_ν, and Q_σ (Figure 10). The magnitude M of the torque is obtainable from

$$M^2 = M_x^2 + M_y^2 + M_z^2 \tag{42a}$$

$$M^2 = M_\psi^2 + M_\nu^2 + M_\sigma^2 + 2M_\psi M_\sigma \cos\nu \tag{42b}$$

$$M^2 = \frac{1}{\sin^2\nu}\left(Q_\psi^2 + Q_\nu^2 \sin^2\nu + Q_\sigma^2\right) - \frac{2\cos\nu}{\sin^2\nu} Q_\psi Q_\sigma \tag{42c}$$

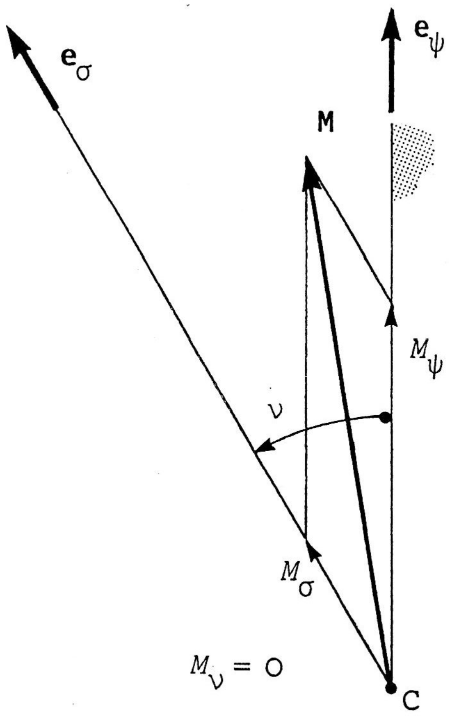

Figure 9. Contravariant Components of the Torque

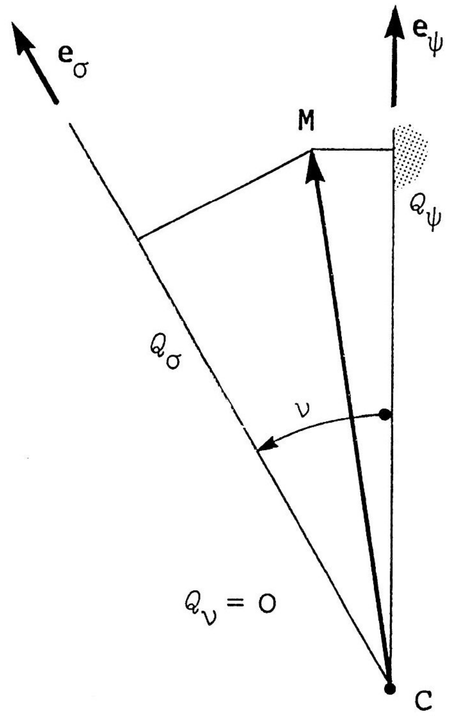

Figure 10. Covariant Projections of the Torque

6 The Sum of Kinetic Energy and Kinetic Coenergy

Using vector representation, we have (Figure 1)

$$T + T^* = \mathbf{H} \cdot \boldsymbol{\omega} \tag{43}$$

Using gyro-fixed Cartesian coordinates we may write

$$T + T^* = H_i \omega_i$$

For subscripts i and k the summation convention is to apply, thus

$$T + T^* = H_x \omega_x + H_y \omega_y + H_z \omega_z \tag{44}$$

Using Eulerian coordinates, we have either

$$T + T^* = p_k \dot{q}_k$$

i.e.

$$T + T^* = p_\psi \dot{\psi} + p_\nu \dot{\nu} + p_\sigma \dot{\sigma} \tag{45}$$

or

$$T + T^* = H_k \omega_k$$

i.e.

$$T + T^* = H_\psi \omega_\psi + H_\nu \omega_\nu + H_\sigma \omega_\sigma \tag{46}$$

7 The Lagrange Equation

The celebrated Lagrange equation is associated with the kinetic coenergy in the form $T^*(q, \dot{q})$. The coordinates $q_1 = \psi$, $q_2 = \nu$, and $q_3 = \sigma$ are the three generalized coordinates required for the description of the angular motion of a single rigid gyro. Lagrange's equation in its fundamental form is typically written

$$\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_k} - \frac{\partial T^*}{\partial q_k} = Q_k$$

or, in particular

$$\frac{d}{dt} \frac{\partial T^*}{\partial \dot{\psi}} - \frac{\partial T^*}{\partial \psi} = Q_\psi \tag{46a}$$

$$\frac{d}{dt} \frac{\partial T^*}{\partial \dot{\nu}} - \frac{\partial T^*}{\partial \nu} = Q_\nu \tag{46b}$$

$$\frac{d}{dt} \frac{\partial T^*}{\partial \dot{\sigma}} - \frac{\partial T^*}{\partial \sigma} = Q_\sigma \tag{46c}$$

where, according to equations (28)

$$\frac{\partial T^*}{\partial \dot{\psi}} = p_\psi$$

$$\frac{\partial T^*}{\partial \dot{\nu}} = p_\nu$$

$$\frac{\partial T^*}{\partial \dot{\sigma}} = p_\sigma$$

are the three generalized momenta. The represent covariant projections of the angular momentum vector $\mathbf{H}$.

The terms Q_ψ, Q_ν, and Q_σ are the three generalized forces. They represent covariant projections of the torque $\mathbf{M}$ (Figure 10).

8 Origin of the Mapping Matrices

The mapping matrix $[A_0]$, equation (12), is composed of

$$[A_0] = [J_0]^T [I] [J_0] \tag{47}$$

with

$$[J_0] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{48}$$

and $[I]$ from equation (4).

The mapping matrix $[A_1]$, equation (29), is composed of

$$[A_1] = [J_1]^T [I][J_1] \tag{49}$$

with $[J_1]$ from equation (19) and $[I]$ from equation (4).

The mapping matrix $[A_2]$, equation (34), is composed of

$$[A_2] = [J_2]^T [I][J_2] \tag{50}$$

with $[J_2]$ from equation (12) and $[I]$ from equation (4). Since $[J_2] = [J_1]^{-T}$, one may also write

$$[A_2] = [J_1]^{-1} [I] [J_1]^{-T} \tag{51}$$

9 Contravariant Components and Covariant Projections

Contravariant components are vector components. Covariant projections are not.

The sum of kinetic energy and coenergy represents an *invariant* quantity, i.e. independent of the coordinate system used to compute it. We shall use this property to establish covariance and contravariance.

Let us now change from the Cartesian formulation (44) to the Eulerian formulation (45). First we observe that

$$\omega_i = \frac{\partial \omega_i}{\partial \dot{q}_k} \dot{q}_k \tag{52}$$

from equation (19). Thus, from equations (44) and (45),

$$T + T^* = H_i \omega_i = H_i \frac{\partial \omega_i}{\partial \dot{q}_k} \dot{q}_k = p_k \dot{q}_k \tag{53}$$

Consequently

$$p_k = \frac{\partial \omega_i}{\partial \dot{q}_k} H_i \tag{54}$$

To demonstrate the contravariant transformation (52) in more detail we write

$$\omega_x = \frac{\partial\omega_x}{\partial\dot{\psi}}\dot{\psi} + \frac{\partial\omega_x}{\partial\dot{\nu}}\dot{\nu} + \frac{\partial\omega_x}{\partial\dot{\sigma}}\dot{\sigma} \tag{55a}$$

$$\omega_y = \frac{\partial\omega_y}{\partial\dot{\psi}}\dot{\psi} + \frac{\partial\omega_y}{\partial\dot{\nu}}\dot{\nu} + \frac{\partial\omega_y}{\partial\dot{\sigma}}\dot{\sigma} \tag{55b}$$

$$\omega_z = \frac{\partial\omega_z}{\partial\dot{\psi}}\dot{\psi} + \frac{\partial\omega_z}{\partial\dot{\nu}}\dot{\nu} + \frac{\partial\omega_z}{\partial\dot{\sigma}}\dot{\sigma} \tag{55c}$$

Relationship (19) supplies the partial derivatives appearing in equations (55), leading to

$$\omega_x = \dot{\psi}\sin\nu\sin\sigma + \dot{\nu}\cos\sigma \tag{56a}$$

$$\omega_y = \dot{\psi}\sin\nu\cos\sigma - \dot{\nu}\sin\sigma \tag{56b}$$

$$\omega_z = \dot{\psi}\cos\nu + \dot{\sigma} \tag{56c}$$

Figure 11 depicts the situation, for $\sigma = 90°$ and $\omega_y = 0$. Note that $\boldsymbol{\omega}$ is the diagonal of a parallelogram of sides $\dot{\psi}$ and $\dot{\sigma}$, i.e. we deal with vector *components.*

As is well known a transformation where

$$y_m = \frac{\partial y_m}{\partial x_n} x_n \tag{57}$$

is classified as *contravariant* in tensor theory, where as transformations of the kind

$$y_m = \frac{\partial y_n}{\partial x_m} x_n \tag{58}$$

are cassified as *covariant.* Thus we conclude that equation (52) represents a contravariant transformation, while equation (54) represents a covariant transformation.

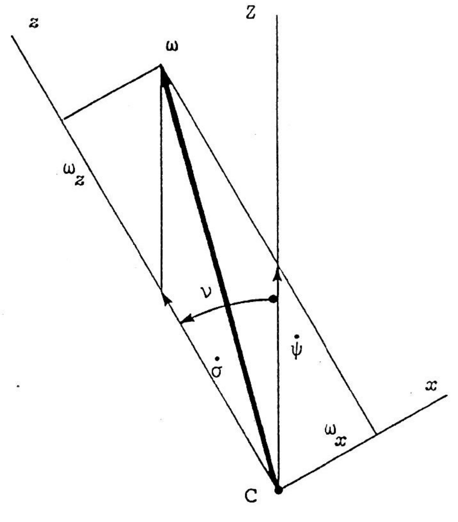

Figure 11. Angular Velocity at the moment where $\omega_y = 0$

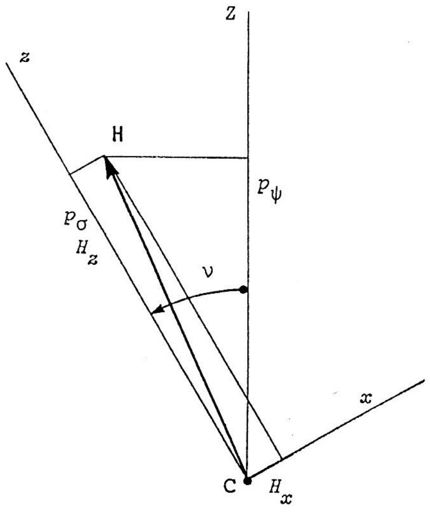

Figure 12. Angular Momentum at the moment where $H_y = 0$

Coordinates	Cartesian Coordinates	Eulerian Coordinates	
Decomposition	No Distinction	Contravariant Components	Covariant Projections
Angles *	$d\theta_x,\ d\theta_y,\ d\theta_z$	$d\psi,\ dv,\ d\sigma$	$d\theta_\psi,\ d\theta_v,\ d\theta_\sigma$
Angular velocity vector $\boldsymbol{\omega}$	$\omega_x,\ \omega_y,\ \omega_z$	$\dot{q}_\psi = \dot{\psi},\ \dot{q}_v = \dot{v},\ \dot{q}_\sigma = \dot{\sigma}$	$\omega_\psi,\ \omega_v,\ \omega_\sigma$
Angular momentum vector $\mathbf{H}$	$H_x,\ H_y,\ H_z$	$H_\psi,\ H_v,\ H_\sigma$	$p_\psi,\ p_v,\ p_\sigma$
Torque vector $\mathbf{M}$	$M_x,\ M_y,\ M_z$	$M_\psi,\ M_v,\ M_\sigma$	$Q_\psi,\ Q_v,\ Q_\sigma$

* Only an infinitesimal angle $d\theta$ has vector characteristics, with

$$d\theta^2 = d\theta_x^2 + d\theta_y^2 + d\theta_z^2$$

$$d\theta^2 = d\psi^2 + dv^2 + d\sigma^2 + 2d\psi d\sigma \cos v$$

$$d\theta^2 = \frac{1}{\sin^2 v}\left(d\theta_\psi^2 + d\theta_v^2 \sin^2 v + d\theta_\sigma^2\right) - \frac{2\cos v}{\sin^2 v} d\theta_\psi d\theta_\sigma$$

Table 1. Contravariant Components and Covariant Projections for Cartesian Coordinates and for Eulerian Coordinates

Now let us have a look at equations (54). With the help of equation (52) we find for the covariant projections

$$p_\psi = \frac{\partial\omega_x}{\partial\dot{\psi}} H_x + \frac{\partial\omega_y}{\partial\dot{\psi}} H_y + \frac{\partial\omega_z}{\partial\dot{\psi}} H_z \tag{59a}$$

$$p_v = \frac{\partial\omega_x}{\partial\dot{v}} H_x + \frac{\partial\omega_y}{\partial\dot{v}} H_y + \frac{\partial\omega_z}{\partial\dot{v}} H_z \tag{59b}$$

$$p_\sigma = \frac{\partial\omega_x}{\partial\dot{\sigma}} H_x + \frac{\partial\omega_y}{\partial\dot{\sigma}} H_y + \frac{\partial\omega_z}{\partial\dot{\sigma}} H_z \tag{59c}$$

that is, using equation (22a),

$$p_\psi = H_x \sin v \sin\sigma + H_y \sin v \cos v + H_z \cos v \tag{60a}$$

$$p_v = H_x \cos\sigma - H_y \sin\sigma \tag{60b}$$

$$p_\sigma = H_z \tag{60c}$$

a situation depicted in Figure 12, for $\sigma = 90°$ and $H_y = 0$. Note that p_ψ and p_σ are *projections* of the angular momentum vector $\mathbf{H}$.

In order to interpret the transformations involved in using $\{\omega_k\}$ and $\{H_k\}$, we realize that now $\{H_k\}$ is the key variable. To change from the Cartesian formulation (24) to the second Eulerian formulation (21) is

$$H_i = \frac{\partial H_i}{\partial H_k} H_k = \frac{\partial\omega_i}{\partial\dot{q}_k} H_k \tag{61}$$

where the partial derivatives are the elements of the Jacobi matrix $\left[J_1\right]$, equation (19). Equation (61) represents a contravariant transformation. From equations (44) and (46) we have

$$T + T^* = H_i\omega_i = \frac{\partial\omega_i}{\partial\dot{q}_k}H_k\omega_i = H_k\omega_k \tag{62}$$

Consequently

$$\omega_k = \frac{\partial\theta_i}{\partial q_k}\omega_i = \frac{\partial\omega_i}{\partial\dot{q}_k}\omega_i \tag{63}$$

which represents a covariant transformation (Table 1).

10 Kinetic Energy $T(q,p)$

We make use of the fact that in Newtonian mechanics $T = T^*$, and use various substitutions to obtain for the kinetic energy (1) of the gyro

$$T(q,p) = \frac{1}{2}\left\{\frac{1}{A}\left(\left(p_\psi - p_\sigma\cos\nu\right)\frac{\sin\sigma}{\sin\nu} + p_\nu\cos\sigma\right)^2 + \frac{1}{B}\left(\left(p_\psi - p_\sigma\cos\nu\right)\frac{\cos\sigma}{\sin\nu} - p_\nu\sin\sigma\right)^2 + \frac{1}{C}p_\sigma^2\right\} \tag{64}$$

The contravariant angular velocity components are then defined as

$$\dot{q}_k = \frac{\partial T(q,p)}{\partial p_k}$$

in particular

$$\dot{\psi} = \frac{\partial T}{\partial p_\psi} = \frac{1}{\sin^2\nu}\left(\frac{\sin^2\sigma}{A} + \frac{\cos^2\sigma}{B}\right)p_\psi + \frac{\sin\sigma\cos\sigma}{\sin\nu}\left(\frac{1}{A} - \frac{1}{B}\right)p_\nu - \frac{1}{\sin\nu\tan\nu}\left(\frac{\sin^2\sigma}{A} + \frac{\cos^2\sigma}{B}\right)p_\sigma \tag{65a}$$

$$\dot{\nu} = \frac{\partial T}{\partial p_\nu} = \frac{\sin\sigma\cos\sigma}{\sin\nu}\left(\frac{1}{A} - \frac{1}{B}\right)p_\psi + \left(\frac{\cos^2\sigma}{A} + \frac{\sin^2\sigma}{B}\right)p_\nu - \frac{\sin\sigma\cos\sigma}{\tan\nu}\left(\frac{1}{A} - \frac{1}{B}\right)p_\sigma \tag{65b}$$

$$\dot{\sigma} = \frac{\partial T}{\partial p_\sigma} = -\frac{1}{\sin\nu\tan\nu}\left(\frac{\sin^2\sigma}{A} + \frac{\cos^2\sigma}{B}\right)p_\psi - \frac{\sin\sigma\cos\sigma}{\tan\nu}\left(\frac{1}{A} - \frac{1}{B}\right)p_\nu + \left(\frac{1}{C} + \frac{1}{\tan^2\nu}\left(\frac{\sin^2\sigma}{A} + \frac{\cos^2\sigma}{B}\right)\right)p_\sigma \tag{65c}$$

Equations (65) may also be written in matrix form

$$\begin{bmatrix}\dot{\psi}\\ \dot{\nu}\\ \dot{\sigma}\end{bmatrix} = \begin{bmatrix}\left(\frac{\sin^2\sigma}{A} + \frac{\cos^2\sigma}{B}\right)\frac{1}{\sin^2\nu} & \left(\frac{1}{A} - \frac{1}{B}\right)\frac{\sin\sigma\cos\sigma}{\sin\nu} & -\left(\frac{\sin^2\sigma}{A} + \frac{\cos^2\sigma}{B}\right)\frac{1}{\sin\nu\tan\nu}\\ \left(\frac{1}{A} - \frac{1}{B}\right)\frac{\sin\sigma\cos\sigma}{\sin\nu} & \frac{\cos^2\sigma}{A} + \frac{\sin^2\sigma}{B} & -\left(\frac{1}{A} - \frac{1}{B}\right)\frac{\sin\sigma\cos\sigma}{\tan\nu}\\ -\left(\frac{\sin^2\sigma}{A} + \frac{\cos^2\sigma}{B}\right)\frac{1}{\sin\nu\tan\nu} & -\left(\frac{1}{A} - \frac{1}{B}\right)\frac{\sin\sigma\cos\sigma}{\tan\nu} & \frac{1}{C} + \left(\frac{\sin^2\sigma}{A} + \frac{\cos^2\sigma}{B}\right)\frac{1}{\tan^2\nu}\end{bmatrix}\begin{bmatrix}p_\psi\\ p_\nu\\ p_\sigma\end{bmatrix} \tag{66}$$

or, in shorthand notation,

$$\{\dot{q}\} = \left[A_1\right]^{-1}\{p\} \tag{67}$$

The (symmetric) matrix $[A_1]^{-1}$ is the inverse of the transformation matrix of equation (29) and maps the covariant generalized momentum projections p_k into the contravariant generalized velocity components $\dot{q}_k$.

11 Kinetic Energy $T(q,H)$

The kinetic energy (1) of the gyro may also be expressed as

$$T(q,H) = \frac{1}{2}\left(\frac{1}{A}\left(H_\psi \sin\nu\sin\sigma + H_\nu\cos\sigma\right)^2 + \frac{1}{B}\left(H_\psi \sin\nu\cos\sigma - H_\nu\sin\sigma\right)^2 + \frac{1}{C}\left(H_\psi\cos\nu + H_\sigma\right)^2\right) \tag{68}$$

The covariant angular velocity projections are then defined as

$$\omega_k = \frac{\partial T(q,H)}{\partial H_k}$$

in particular

$$\omega_\psi = \frac{\partial T(q,H)}{\partial H_\psi} = \left(\left(\frac{\sin^2\sigma}{A} + \frac{\cos^2\sigma}{B}\right)\sin^2\nu + \frac{\cos^2\sigma}{C}\right)H_\psi + \left(\left(\frac{1}{A}-\frac{1}{B}\right)\sin\nu\sin\sigma\cos\sigma\right)H_\nu + \frac{\cos\nu}{C}H_\sigma \tag{69a}$$

$$\omega_\nu = \frac{\partial T(q,H)}{\partial H_\nu} = \left(\left(\frac{1}{A}-\frac{1}{B}\right)\sin\nu\sin\sigma\cos\sigma\right)H_\psi + \left(\frac{\cos^2\sigma}{A} + \frac{\sin^2\sigma}{B}\right)H_\nu \tag{69b}$$

$$\omega_\sigma = \frac{\partial T(q,H)}{\partial H_\sigma} = \frac{H_\sigma}{C} \tag{69c}$$

Equations (69) may also be written in matrix form

$$\begin{bmatrix}\omega_\psi\\ \omega_\nu\\ \omega_\sigma\end{bmatrix} = \begin{bmatrix}\left(\frac{\sin^2\sigma}{A}+\frac{\cos^2\sigma}{B}\right)\sin^2\nu+\frac{\cos^2\nu}{C} & \left(\frac{1}{A}-\frac{1}{B}\right)\sin\nu\sin\sigma\cos\sigma & \frac{\cos\nu}{C}\\ \left(\frac{1}{A}-\frac{1}{B}\right)\sin\nu\sin\sigma\cos\sigma & \frac{\cos^2\sigma}{A}+\frac{\sin^2\sigma}{B} & 0\\ \frac{\cos\nu}{C} & 0 & \frac{1}{C}\end{bmatrix}\begin{bmatrix}H_\psi\\ H_\nu\\ H_\sigma\end{bmatrix} \tag{70}$$

or, in shorthand notation

$$\{\omega\} = [A_2]^{-1}\{H\} \tag{71}$$

The (symmetric) matrix $[A_2]^{-1}$ is the inverse of the mapping matrix (34) and maps the contravariant angular momentum components H_k into the covarient angular velocity components ω_k.

12 Canonical Equations

Hamilton's canonical equations involve the *Hamiltonian* (see e.g. Tabarrok, 1994)

$$\mathcal{H} = T(q,p) + V(q) \tag{72}$$

Since the present paper does not use the concept of potential energy V, the Hamiltonian becomes simply

$$\mathcal{H} = T(q,p) \tag{73}$$

with the kinetic energy in the form of equation (64).
Hamilton's canonical equations are then

$$-\dot{p}_k = \frac{\partial \mathcal{H}}{\partial q_k} - Q_k \tag{74a}$$

$$\dot{q}_k = \frac{\partial \mathcal{H}}{\partial p_k} \tag{74b}$$

Taking equation (73) into consideration, an inspection of equations (65) shows that they, in effect, represent equation (74b). With equation (73), the first canonical equation (74a) assumes the form

$$\dot{p}_k + \frac{\partial T}{\partial q_k} = Q_k \tag{75}$$

An inspection of the Lagrange equations (46) shows that they may be written in the form

$$\dot{p}_k - \frac{\partial T^*}{\partial q_k} = Q_k \tag{76}$$

from which we conclude that

$$\frac{\partial T(q,p)}{\partial q_k} = -\frac{\partial T^*(q,\dot{q})}{\partial q_k} \tag{77}$$

The reader is invited to show that thus is indeed the case, e.g. by realizing that equation (45) is not a function of the q_k. Thus

$$\frac{\partial}{\partial q_k}\left(T + T^*\right) = 0$$

giving us again equation (77)

13 Axisymmetric Gyros

Axisymmetric gyros occur very frequently, and the equations presented in the preceding chaptes are then considerably simplified. Axial symmetry can be represented by the inertia moments

$$B = A \tag{78}$$

The mapping matrices $[A_0]$, $[A_1]$ and $[A_2]$ are affected and appear in simpler form, in that they are no longer a function of the spin angle σ.

From equations (4) and (12) we obtain

$$[A_0] = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \tag{79a}$$

and

$$[A_0]^{-1} = \begin{bmatrix} \dfrac{1}{A} & 0 & 0 \\ 0 & \dfrac{1}{A} & 0 \\ 0 & 0 & \dfrac{1}{C} \end{bmatrix} \tag{79b}$$

From equations (29) and (66)

$$[A_1] = \begin{bmatrix} A\sin^2 \nu + C\cos^2 \nu & 0 & C\cos\nu \\ 0 & A & 0 \\ C\cos\nu & 0 & C \end{bmatrix} \tag{80a}$$

and

$$[A_1]^{-1} = \begin{bmatrix} \dfrac{1}{A\sin^2 \nu} & 0 & -\dfrac{\cos\nu}{A\sin^2 \nu} \\ 0 & \dfrac{1}{A} & 0 \\ \dfrac{\cos\nu}{A\sin^2 \nu} & 0 & \dfrac{1}{C}+\dfrac{\cos^2 \nu}{A\sin^2 \nu} \end{bmatrix} \tag{80b}$$

From equations (34) and (70)

$$[A_2] = \begin{bmatrix} \dfrac{A}{\sin^2 \nu} & 0 & -\dfrac{A\cos\nu}{\sin^2 \nu} \\ 0 & A & 0 \\ -\dfrac{A\cos\nu}{\sin^2 \nu} & 0 & C+\dfrac{A\cos^2 \nu}{\sin^2 \nu} \end{bmatrix} \tag{81a}$$

and

$$[A_2]^{-1} = \begin{bmatrix} \dfrac{\sin^2 \nu}{A}+\dfrac{\cos^2 \nu}{C} & 0 & \dfrac{\cos\nu}{C} \\ 0 & \dfrac{1}{A} & 0 \\ \dfrac{\cos\nu}{C} & 0 & \dfrac{1}{C} \end{bmatrix} \tag{81b}$$

14 On Notation

The reader's attention is drawn to a somewhat inconsistent notation. The symbol p_k for the covariant projections of the angular momentum does not fit too well (a capital letter would be more suitable). However its widerspread use in the literature leaves little choice. The symbol ω_k for the covarient projections of the angular velocity fits well into equation (46), it would fit just as well for the contravariant components since it is common practice to use the same symbol for the contravariant components of a vector as for the vector itself. We have opted for the former approach.

A minor irritant is the same letter for the principal inertia moment A and the mapping matrices $[A_0]$, $[A_1]$ and $[A_2]$, a problem also engendered by common practice. We have attempted to alleviate it by using subscripts for the latter.

For the angular momentum we have used the symbols $\mathbf{H}$ and H, for the Hamiltonian the symbol $\mathcal{H}$.

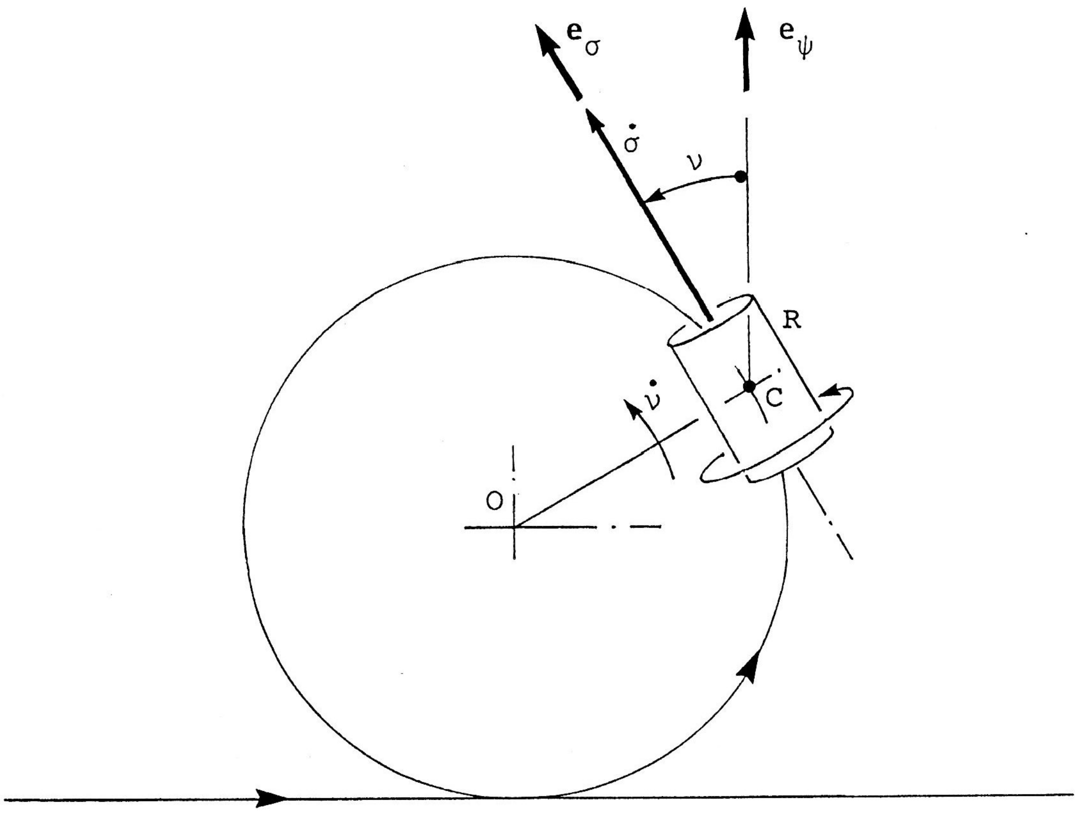

Figure 13. Airplane Engine Rotor R during a Looping Manoeuvre

15 Example

The case of torque-free axisymmetric gyros is of particular importance in practical applications. Take e. g. a fast spinning $(\omega_1 = \dot\sigma >> \dot\nu)$, axisymmetric $(B = A)$ engine rotor R mounted in are airplane executing a looping manoeuvre (Figure 13). The rotor is constrained such that $\psi = \dot\psi = 0$, and $\dot\nu = \omega_2$ = constant. Further $\ddot\sigma = 0$. The torques required to maintain this motion are to be determined.

From Lagrange's equations (46) we have

$$Q_\psi = \dot{p}_\psi - \frac{\partial T^*}{\partial \psi} \qquad \text{(a)}$$

$$Q_\nu = \dot{p}_\nu - \frac{\partial T^*}{\partial \nu} \qquad \text{(b)}$$

$$Q_\sigma = \dot{p}_\sigma - \frac{\partial T^*}{\partial \sigma} \qquad \text{(c)}$$

With $\frac{\partial T^*}{\partial \psi} = 0$, and

$$p_\psi = \frac{\partial T^*}{\partial \dot{\psi}} = A\ \dot{\nu}\ \sin\nu\ \sin\sigma\ \cos\sigma - A\ \dot{\nu}\ \sin\nu\ \sin\sigma\ \cos\sigma + C\ \dot{\sigma}\ \cos\nu \qquad \text{(d)}$$

$$p_\psi = C\ \dot{\sigma}\ \cos\nu \qquad \text{(e)}$$

$$\dot{p}_\psi = \frac{d}{dt} p_\psi = C(\ddot{\sigma}\ \cos\nu - \dot{\nu}\dot{\sigma}\ \sin\nu) = -C\,\dot{\nu}\dot{\sigma}\ \sin\nu \qquad \text{(f)}$$

Thus, from equation (a), the required torque becomes

$$Q_\psi = -C\,\omega_1\,\omega_2\ \sin\nu \qquad \text{(g)}$$

with

$$p_\nu = \frac{\partial T^*}{\partial \dot{\nu}} = A\,\dot{\nu}\ \cos^2\sigma + A\,\dot{\nu}\ \sin^2\sigma = A\,\dot{\nu} \qquad \text{(h)}$$

$$\dot{p}_\nu = A\,\ddot{\nu} = 0 \qquad \text{(i)}$$

$$\frac{\partial T^*}{\partial \nu} = -C(\dot{\psi}\ \cos\nu + \dot{\sigma})\dot{\psi}\ \sin\nu = 0 \qquad \text{(j)}$$

equation (b) gives

$$Q_\nu = 0 \qquad \text{(k)}$$

and with

$$p_\sigma = \frac{\partial T^*}{\partial \dot{\sigma}} = C(\dot{\psi}\ \cos\nu + \dot{\sigma}) = C\,\dot{\sigma} \qquad \text{(l)}$$

$$\dot{p}_\sigma = C\,\ddot{\sigma} = 0 \qquad \text{(m)}$$

$$\frac{\partial T^*}{\partial \sigma} = 0 \qquad \text{(n)}$$

from equation (c)

$$Q_\sigma = 0$$

The magnitude of the applied torque is, from equation (42c),

$$
\begin{aligned}
M^2 &= \frac{1}{\sin^2 \nu} Q_\nu^2 = \frac{1}{\sin^2 \nu} C^2 \omega_1^2 \omega_2^2 \sin^2 \nu \\
M &= C \omega_1 \omega_2
\end{aligned}
\tag{p}
$$

The contravariant components of the angular momentum vector **H** are, from equation (25),

$$
H_\psi = \frac{1}{\sin^2 \nu} p_\psi - \frac{\cos \nu}{\sin^2 \nu} p_\sigma = C\dot{\sigma}\frac{\cos \nu}{\sin^2 \nu} - \frac{\cos \nu}{\sin^2 \nu} C\dot{\sigma} = 0 \tag{q}
$$

$$
H_\nu = p_\nu = A\dot{\nu} = A\omega_2 \tag{r}
$$

$$
H_\sigma = -\frac{\cos \nu}{\sin^2 \nu} p_\psi + \frac{1}{\sin^2 \nu} p_\sigma = -\frac{\cos^2 \nu}{\sin^2 \nu} C\dot{\sigma} + \frac{1}{\sin^2 \nu} C\dot{\sigma} = C\dot{\sigma} = C\omega_1 \tag{s}
$$

The magnitude H of the angular momentum is obtained by using equations (40).

$$
\begin{aligned}
H^2 &= H_\nu^2 + H_\sigma^2 = \frac{1}{\sin^2 \nu}\left(p_\psi^2 + p_\nu^2 \sin^2 \nu + p_\sigma^2\right) - \frac{2\cos \nu}{\sin^2 \nu} = A^2\omega_2^2 + C^2\omega_1^2 \\
H &= \sqrt{A^2\omega_2^2 + C^2\omega_1^2}
\end{aligned}
\tag{t}
$$

The covariant angular velocity projections are, for our example,

$$
\omega_\psi = \dot{\sigma} \cos \nu = \omega_1 \cos \nu \tag{u}
$$

$$
\omega_\nu = \dot{\nu} = \omega_2 \tag{v}
$$

$$
\omega_\sigma = \dot{\sigma} = \omega_1 \tag{w}
$$

The magnitude ω of the angular veloity is obtained by using equations (38).

$$
\begin{aligned}
\omega^2 &= \dot{\nu}^2 + \dot{\sigma}^2 = \frac{1}{\sin^2 \nu}\left(\omega_\psi^2 + \omega_\nu^2 \sin^2 \nu + \omega_\sigma^2\right) - \frac{2\cos \nu}{\sin^2 \nu}\omega_\psi \omega_\sigma \\
\omega &= \sqrt{\omega_2^2 + \omega_1^2}
\end{aligned}
\tag{x}
$$

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