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# Buckling of Imperfect Sandwich Cones under Axial Compression - Equivalent-Cylinder Approach. Part II

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- continued from Band 14, Heft 3/4, (1994), 239-248 -

## 5 Another Form of the Above Equations

In some cases, it may be convenient to indroduce the following new functions $\Psi$ and $\Phi$:

$$\beta_s = -\Psi_{,s} + \frac{1}{s\cos\varphi}\Phi_{,\theta} \qquad \beta_\theta = -\frac{1}{s\cos\varphi}\Psi_{,\theta} - \Phi_{,s} \qquad w = \left(1 - \frac{D}{Gc}\nabla^2\right)\Psi \tag{52}$$

The total solution (52) satisfies equation (37). The rest must satisfy the following equations:

$$D\nabla^4\Psi + \frac{\tan\varphi}{s}F_{,ss} = q_n + L\left(F_{,w} + w_0\right) \qquad \frac{1}{A}\nabla^4 F = \frac{1}{2}L\left(w,\ w+2w_0\right) + \frac{\tan\varphi}{s}w_{,ss} \tag{53}$$

$$\left(1 - \frac{1-\nu}{2}\,\frac{D}{Gc}\nabla^2\right)\Phi = 0 \tag{54}$$

In order to simplify the above equations , we check the order of magnitude of the operator (54) in brackets.

$$1 - \frac{1-\nu}{2}\,\frac{D}{Gc}\nabla^2 \sim 1 - \left(\frac{t}{R}\right)^3 \sim 1 \tag{55}$$

Thus we have

$$\Phi = 0 \tag{56}$$

and the total solution (52) becomes

$$\beta_s = -\Psi_{,s} \qquad \beta_\theta = -\frac{1}{\cos\varphi}\Psi_{,\theta} \qquad w = \left(1 - \frac{D}{Gc}\nabla^2\right)\Psi \tag{57}$$

Equations (53) are now reduced to

$$D\nabla^4\psi + \frac{\tan\varphi}{s}F_{,ss} = q_n + L\left(F,\ w+w_0\right) \qquad \frac{1}{A}\nabla^4 F = \frac{1}{2}L\left(w,\ w+2w_0\right) + \frac{\tan\varphi}{s}w_{,ss} \tag{58}$$

The functions $\Psi$ and $\Phi$ may be called displacement function and shear force function, respectively. From the above analysis of order of magnitude, we can accept that the results for a sandwich cone with deformable core will be acceptable for cases of large shear modulus $G$ of core. But it would not be true for a soft core. Thus we conclude that the theory presented here is for stiff cores.

## 6 Governing Equations of „Equivalent-Cylinder“

The above equations for a sandwich cone are partial differential equations with variable coefficients. It is difficult to solve them. In order get some useful information before we attempt even more complicated problems, we introduce the assumption of „equivalent-cylinder“ to simplify these equations. Thus, we use an average value $s_c$, instead of the variable $s$. The assumptions are the following:

(a) The wall thickness of the cylinder is equal to that of the cone, i.e. the quasi-cylinder has the same facing and core thickness as found in the cone

(b) The radius of the cylinder is equal to the finite principal radius of curvature at the middle of the cone

(c) The length of the cylinder is equal to the slant length of the cone

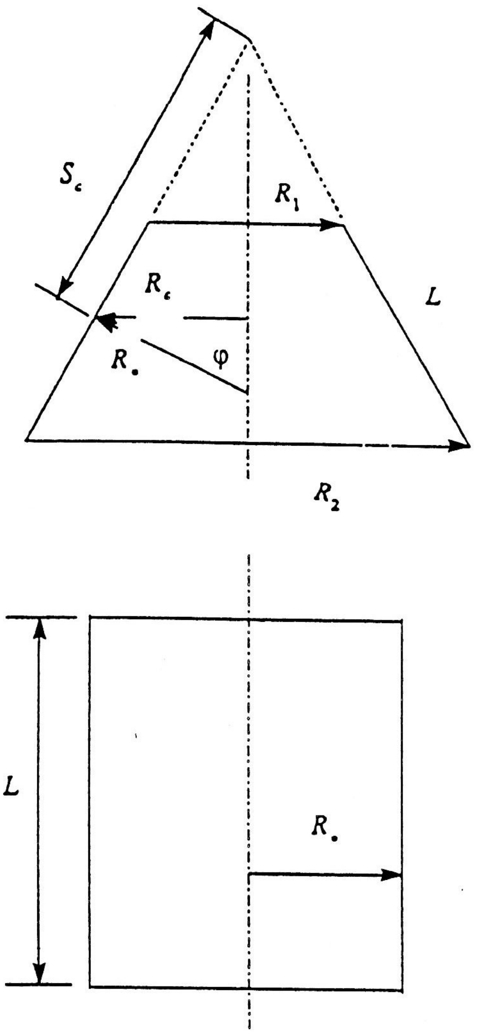


Figure 3. Simplification of cone

Based on the above assumptions, we have the following simplified relations:

(a) Geometry

$$s \rightarrow s_c \qquad s_c = \frac{s_1 + s_2}{2} \qquad R_c = \frac{R_1 + R_2}{2} \qquad R_c = s_c \cos\varphi \qquad R_* = s_c \cot\varphi \tag{59}$$

(b) Axial load

$$N_c = N_0 \sin\varphi = N_0 \cos\alpha \tag{60}$$

Under the above assumptions, we can get the governing equations of the „equivalent-cylinder" as follows:

(a) Equilibrium equations in terms of internal forces and moments

$$N_{x,x} + N_{xy,y} - \frac{1}{s_c}N_y = 0 \qquad N_{xy,x} + N_{y,y} + \frac{1}{s_c}N_{xy} = 0$$

$$Q_{x,x} + Q_{y,y} + N_x(w+w_0)_{,xx} + N_y\left[(w+w_0)_{,yy} + \frac{1}{s_c}(w+w_0)_{,x}\right] + 2N_{xy}\left[(w+w_0)_{,xy} - \frac{1}{s_c}(w+w_0)_{,y}\right]$$
$$- \frac{1}{R_*}N_y + q_n = 0$$

$$Q_x = M_{x,x} + M_{xy,y} - \frac{1}{s_c}M_y \qquad Q_y = M_{y,y} + M_{xy,x} + \frac{1}{s_c}M_{xy} \tag{61}$$

(b) Strains and curvature changes

$$\varepsilon_x = u_{,x} + \frac{1}{2}(w_{,x})^2 + w_{,x}w_{0,x} \qquad \varepsilon_y = v_{,y} + \frac{u}{s_c} + \frac{w}{R_*} + \frac{1}{2}(w_{,y})^2 + w_{,y}w_{0,y}$$

$$\varepsilon_{xy} = v_{,x} - \frac{v}{s_c} + u_{,y} + w_{,x}w_{,y} + w_{,x}w_{0,y} + w_{,y}w_{0,x} \tag{62}$$

$$\kappa_x = \beta_{x,x} \qquad \kappa_y = \beta_{y,y} + \frac{1}{s_c}\beta_x \qquad \kappa_{xy} = \beta_{y,x} - \frac{1}{s_c}\beta_y + \beta_{x,y} \tag{63}$$

(c) Shear forces in terms of shear angles

$$\begin{aligned} Q_x &= D\left[\beta_{x,xx} - \frac{1}{s_c^2}\beta_x + \frac{1-\nu}{2}\beta_{x,yy}\right] + D\left[\frac{1-\nu}{2}\beta_{y,xy} - \frac{3-\nu}{2}\frac{1}{s_c}\beta_{y,y}\right] \\ Q_y &= D\left[\frac{1+\nu}{2}\beta_{x,xy} + \frac{3-\nu}{2}\frac{1}{s_c}\beta_{x,y}\right] + D\left[\beta_{y,yy} + \frac{1-\nu}{2}\left(\beta_{y,xx} - \frac{1}{s_c^2}\beta_y\right)\right] \end{aligned} \tag{64}$$

(d) Stress resultants in terms of stress function $F$

$$N_x = \frac{1}{s_c}F_{,x} + F_{,yy} \qquad N_y = F_{,xx} \qquad N_{xy} = F_{,xy} - \frac{1}{s_c}F_{,y} \tag{65}$$

(e) Governing equations in terms of $F$, $w$, $\beta_x$ and $\beta_y$

$$\begin{aligned} Gc\,(w_{,x}+\beta_x) &= D\left[\beta_{x,xx} - \frac{1}{s_c^2}\beta_x + \frac{1-\nu}{2}\beta_{x,yy}\right] + D\left[\frac{1+\nu}{2}\beta_{y,xy} - \frac{3-\nu}{2}\frac{1}{s_c}\beta_{y,y}\right] \\ Gc\,(w_{,y}+\beta_y) &= D\left[\frac{1+\nu}{2}\beta_{x,xy} + \frac{3-\nu}{2}\frac{1}{s_c}\beta_{x,y}\right] + D\left[\beta_{y,yy} + \frac{1-\nu}{2}\left(\beta_{y,xx} - \frac{1}{s_c^2}\beta_y\right)\right] \end{aligned} \tag{66}$$

$$\begin{aligned} &D\left[\beta_{x,xxx} - \frac{1}{s_c}\beta_{x,xx} + \frac{1}{s_c^2}\beta_{x,x} + \beta_{x,xyy}\right] + D\left[\beta_{y,xxy} - \frac{1}{s_c}\beta_{y,xy} + \frac{1}{s_c^2}\beta_{y,y} + \beta_{y,yyy}\right] \\ &- \frac{1}{R_*}F_{,xx}\left[F_{,yy} + \frac{1}{s_c}F_{,x}\right](w+w_0)_{,xx} + F_{,xx}\left[\frac{1}{s_c}(w+w_0)_{,x} + (w+w_0)_{,yy}\right] - 2\left[F_{,xy} - \frac{1}{s_c}F_{,y}\right] \\ &\left[(w+w_0)_{,xy} - (w+w_0)_{,yy}\right] + q_n = 0 \end{aligned} \tag{67}$$

$$
\frac{1}{A}\left[\frac{\partial^2}{\partial x^2}+\frac{1}{s_c}\frac{\partial}{\partial x}+\frac{\partial^2}{\partial y^2}\right]^2 F = \frac{1}{2}\left[\frac{1}{s_c}w_{,x}+w_{,yy}\right]\left(w+2w_0\right)_{,xx} + \frac{1}{2}w_{,xx}
$$

$$
\left[\frac{1}{s_c}\left(w+2w_0\right)_{,x}+\left(w+2w_0\right)_{,yy}\right] - \left[w_{,xy}-\frac{1}{s_c}w_{,y}\right]\left[\left(w+2w_0\right)_{,xy}-\left(w+2w_0\right)_{,y}\right] + \frac{1}{R_*}w_{,xx} \tag{68}
$$

If the small quantity $\varepsilon = 1/s_c$ aproaches zero, the above equations will reduce to those of a sandwich cylinder. In former papers, this small term was omitted. For a sufficiently long cone, this small term may indeed be omitted, but the shorter the cone, the larger the errors caused by omitting this term.

## 7 Prebuckling Axisymmetric Solution

Assuming an initial axisymmetric shape imperfection, one can write the prebuckling deflection as

$$
w(x,y) = w^*(x) \tag{69}
$$

From $N_{x,x} = 0$, or $N_x = -N_0$, we have the solution of the Airy function in the prebuckling state in the following form:

$$
F(x,y) = -\frac{1}{2}N_0 y^2 + F^*(x) \tag{70}
$$

For the axisymmetric state, the shear angle is

$$
\beta_x(x,y) = \beta_x^*(x) \qquad \beta_y(x,y) = 0 \tag{71}
$$

After substituting these solutions into equations (66), (67) and (68), we have the equations of the prebuckling state.

$$
\begin{aligned}
&\frac{D}{Gc}\left(\beta^*_{x,xx}-\varepsilon^2\beta^*_x\right) = w^*_x + \beta^*_x\\
&\frac{1}{A}\left(F^*_{,xxxx}+2\varepsilon F^*_{,xxx}+\varepsilon^2 F^*_{,xx}\right) = \frac{1}{2}\left[w^*_{,x}\left(w^*+2w_0\right)_{,x}\right]_{,x} + \frac{1}{R_*}w^*_{,xx}\\
&D\left(\beta^*_{x,xxx}-\varepsilon\beta^*_{x,xx}+\varepsilon^2\beta^*_{x,xx}\right) = \frac{1}{R_*}F^*_{,xx} + N_0\left(w^*+w_0\right)_{,xx} - \varepsilon\left[F^*_{,x}\left(w^*+w_0\right)_{,x}\right]_{,x}
\end{aligned} \tag{72}
$$

These equations are nonlinear equations since there is a (small) nonlinear term. For the sandwich cylinder the equations of the prebuckling state are linear equations. As initial axisymmetric shape imperfection can be adopted a Koiter type imperfection

$$
w_0 = -\mu\cos\left(\frac{2\pi x}{l_x}\right) \tag{73}
$$

The order of magnitude of the imperfection amplitude relative to the imperfection wavelength $l_x$, as required by equation (3), is given by

$$
\frac{4\mu\pi^2}{l_x^2} \leq 0\,(1) \tag{74}
$$

The above equations are nonlinear. Two solving methods can be used to get the solution for the prebuckling state.

Perturbation method

The Solution can be expanded as follows:

$$w^*(x) = w_0^* + \varepsilon w_1^* + \varepsilon w_2^* + \cdots \quad \beta_x^*(x) = \beta_{x0}^* + \varepsilon^2\beta_{x1}^* + \varepsilon^2\beta_{x2}^* + \cdots$$
$$F^*(x) = F_0^* + \varepsilon F_1^* + \varepsilon^2 F_2^* + \cdots \tag{75}$$

(a) $\varepsilon^0$

$$\frac{D}{Gc}\beta^*_{x0,xx} = w^*_{0,x} + \beta^*_{x0} \qquad F^*_{0,xxxx} = \frac{A}{R_*}w^*_{0,xx} \tag{76}$$

$$D\beta^*_{x0,xxx} = \frac{1}{R_*}F_{0,xx} + N_0\left(w_0^* + w_0\right)_{,xx}$$

(b) $\varepsilon^1$

$$\frac{D}{Gc}\beta^*_{x1,xx} = w^*_{1,x} + \beta_{x1}$$
$$\frac{1}{A}\left(F^*_{1,xxxx} + F^*_{0,xxx}\right) = \frac{1}{R_*}w^*_{1,xx} + \frac{1}{2}\left[w^*_{0,x}\left(w_0^* + 2w_0\right)_{,x}\right] \tag{77}$$
$$D\left[\beta^*_{x1,xxx} - \beta^*_{x0,xx}\right] = \frac{1}{R_*}F^*_{1,xx} + N_0 w^*_{1,xx} - \left[F^*_{0,x}\left(w_0^* + w_0\right)_{,x}\right]_{,x}$$

From physical considerations, particular solutions are taken to be of the form

$$F_0^*(x) = N\cos\left(\frac{2\pi x}{l_x}\right) \quad w_0^*(x) = K + M\cos\left(\frac{2\pi x}{l_x}\right) \quad \beta^*_{x0}(x) = P\sin\left(\frac{2\pi x}{l_x}\right)$$
$$F_1^*(x) = N_1\cos\left(\frac{2\pi x}{l_x}\right) \quad w_1^*(x) = M_1\cos\left(\frac{2\pi x}{l_x}\right) \quad \beta^*_{x1}(x) = P_1\sin\left(\frac{2\pi x}{l_x}\right) \tag{78}$$

Galerkin's procedure can be applied to obtain an approximate solution for the prebuckling state, and we have then

$$P = \frac{2\mu\lambda\pi}{\left(1+2\rho^2\chi_c\right)\left(\lambda_0-\lambda\right)l_x} \qquad M = -\frac{\mu\lambda}{\lambda_0-\lambda} \qquad N = \frac{\alpha\mu\lambda}{2\rho^2\left(\lambda_0-\lambda\right)} \tag{79}$$

$$P_1 = -\frac{\gamma}{3\pi}\,\frac{1}{1+2\chi_c\rho^2}\,\frac{\mu^2\gamma^2}{\left(\lambda_0-\lambda\right)^2} \qquad M_1 = \frac{l_x}{6\pi^2}\,\frac{\mu^2\gamma^2}{\left(\lambda_0-\lambda\right)^2} \tag{80}$$

$$N_1 = -\frac{1}{6\pi^2}\,\frac{\alpha}{2\rho^2}\,\frac{\mu^2\lambda^2}{\left(\lambda_0-\lambda\right)^2}\left\{1 - \frac{4\rho^2\gamma}{\lambda}\left[\gamma + 2(\lambda_0-\lambda)\right]\right\}$$

So we have following deflection:

$$W^*(x) = K - \frac{\mu\lambda}{\lambda_0-\lambda}\left[1 + \frac{\mu\lambda}{\lambda_0-\lambda}\,\frac{\varepsilon l_x}{6\pi^2}\right]\cos\left(\frac{2\pi x}{l_x}\right) + \cdots \tag{81}$$

where

$$\alpha = \sqrt{AD} \qquad \gamma = \sqrt{A/D} \qquad \chi_c = \alpha / R_* Gc \qquad \rho = l/\left(\sqrt{2}l_x\right) \tag{82}$$

$$l = 2\pi\sqrt{R_*/\gamma} \qquad \lambda_0 = 1/\left(4\rho^2\right) + \rho^2/\left(1+2\chi_c\rho^2\right)$$

and

$$N_0 = \frac{2\alpha\lambda}{R_*} \quad \text{i.e.} \quad \lambda = \frac{R_*}{2\alpha}N_0 \tag{83}$$

From the above analysis, we can see that the deflection of a sandwich cone is larger than that of a sandwich cylinder.

Direct analysis

Suppose the solution for prebuckling is of the following form:

$$w^*(x) = K + M\cos\left(\frac{2\pi x}{l_x}\right) \qquad \beta_x^*(x) = -P\sin\left(\frac{2\pi x}{l_x}\right) \qquad F^*(x) = N\cos\left(\frac{2\pi x}{l_x}\right) \tag{84}$$

Substituting equations (84) into equations (72) and using Galerkin's method, we have

$$P = -\frac{2\pi}{l_x}\frac{1}{1+2\chi_c\rho^2\left(1+\dfrac{\varepsilon_0^2}{4\pi^2}\right)}M \qquad N = \frac{1}{1-\dfrac{\varepsilon_0^2}{4\pi^2}}\left[-\frac{\alpha}{2\rho^2}M + \varepsilon_0\frac{\alpha\gamma}{3\pi^2}M(M-2\mu)\right] \tag{85}$$

$$\begin{aligned} &\varepsilon_0^2\frac{4\rho^2\gamma^2}{9\pi^4}\left(1-\frac{\varepsilon_0^2}{4\pi^2}\right)^{-1}M^3 - \varepsilon_0\frac{\gamma}{3\pi^2}\left(1-\frac{\varepsilon_0^2}{4\pi^2}\right)^{-1}\left[1-\varepsilon_0\frac{4\rho^2\gamma\mu}{\pi^2}\right]M^2 \\ &+ \varepsilon_0\frac{2\mu\gamma}{3\pi^3}\left(1-\frac{\varepsilon_0^2}{4\pi^2}\right)^{-1}\left[1+\varepsilon_0\frac{2\rho\gamma\mu}{3\pi^2}\right]M + 2(\lambda-\lambda_c)M = 2\lambda\mu \end{aligned} \tag{86}$$

where

$$\lambda_c = \frac{1}{4\rho^2}\left(1-\frac{\varepsilon_0^2}{4\pi^2}\right)^{-1} + \rho^2\left(1-\frac{\varepsilon_0^2}{4\pi^2}\right)\left[1+2\chi_c\rho^2\left(1+\frac{\varepsilon_0^2}{4\pi^2}\right)\right]^{-1} \qquad \varepsilon_0 = \varepsilon l_x = \frac{l_x}{s_c} \tag{87}$$

Axisymmetric buckling of perfect cone

The solutions for a perfect cone can be obtained when the imperfection amplitude vanishes. In this case, the axisymmetric buckling coefficient $(\lambda_a)$ for the perfect cone is obtained by minimizing $\lambda$ with respect to the axial wave number $\rho^2$.

$$\frac{\partial\lambda_c}{\partial\rho^2} = 0 \qquad \rho^2 \to \infty \qquad \rho^2 = \left(2\left[1-\chi_c-(1+\chi_c)\frac{\varepsilon_0^2}{4\pi^2}\right]\right)^{-1} \tag{88}$$

When the axial core shear flexibility coefficient $\lambda_a$ is the only real root, we have

$$\begin{aligned} \lambda_a &= 1 - \frac{\chi_c}{2}\frac{1+\varepsilon_0^2/\left(4\pi^2\right)}{1-\varepsilon_0^2/\left(4\pi^2\right)} \quad \left(< 1-\frac{\chi_c}{2}\right) \quad \text{when} \quad \chi_c \le 1 \\ \lambda_a &= \frac{1}{2\chi_c}\frac{1-\varepsilon_0^2/\left(4\pi^2\right)}{1+\varepsilon_0^2/\left(4\pi^2\right)} \quad \left(< \frac{1}{2\chi_c}\right) \quad \text{when} \quad \chi_c \ge 1 \end{aligned} \tag{89}$$

The effect of imperfections on buckling

The above analysis can given some important information about the effect of imperfections on the buckling coefficient.

(a) If we have an approximate relation from equation (86),

$$(\lambda_c - \lambda)M + \varepsilon_0 \frac{\gamma}{6\pi^2}\left[1-\varepsilon_0^2/\left(4\pi^2\right)\right]^{-1} M^2 = \lambda\,(-\mu) \tag{90}$$

one might expect that a calculation of the stationary buckling value of $\lambda_s$ on the basis of equation (90) might be more reliable than the general relation (86). The maximization of $\lambda$ by use of equation (90) leads to

$$\left(1-\frac{\lambda_s}{\lambda_c}\right)^2 = \frac{2\varepsilon_0\gamma}{3\pi^2}\frac{1}{\lambda_c\left[1-\varepsilon_0^2/\left(4\pi^2\right)\right]}\frac{\lambda_s}{\lambda_c}\mu \tag{91}$$

This provides the mathematically palatable result $\lambda_s \to 0$ for $\mu \to \infty$.

(b) If we have an approximate relation

$$(\lambda_c - \lambda)M - \varepsilon_0^2 \frac{2\rho^2\gamma^2}{9\pi^4}\left(1-\frac{\varepsilon_0^2}{4\pi^2}\right)^{-1} M^3 = -\lambda\mu \tag{92}$$

the calculation for stationary buckling coefficients now gives

$$\left(1-\frac{\lambda_s}{\lambda_c}\right)^{3/2} - \frac{3}{2}\sqrt{3}\left(\varepsilon_0^2\frac{2\rho^2\gamma^2}{9\pi^4}\frac{1}{\lambda_c\left[1-\varepsilon_0^2/\left(4\pi^2\right)\right]}\right)^{1/2}\frac{\lambda_s}{\lambda_c}|\mu| = 0 \tag{93}$$

The above Koiter relations (91) and (93) will be modified after we consider the postbuckling effect.

## 8 Bifurcation and Postbuckling Analysis

Under increasing load the amplitude of the lateral deflection $M$ will grow in hyperbolic fashion until the stationary point and/or bifurcation point is reached for imperfect cone and perfect cone, respectively. Let us define the terms $w^p(x, y)$, $f(x,y)$, $b_x(x,y)$ and $b_y(x,y)$ as the second path solution, and write

$$F = -\frac{1}{2}N_0 y^2 + F^*(x) + f(x,y) \qquad w = w^*(x) + w^p(x,y)$$

$$\beta_x = \beta_x^*(x) + b_x(x,y) \qquad \beta_y = b_y(x,y) \tag{94}$$

Substituting the above relations (94) into equations (66), (67) and (68), we obtain the nonlinear approximate equations of neutral equilibrium.
(a) Compatibility equation

$$\frac{1}{A}\left(\frac{\partial^2}{\partial x^2}+\varepsilon\frac{\partial}{\partial x}+\frac{\partial^2}{\partial y^2}\right)^2 f(x,y) = w^p{}_{,yy}\left(w^*+w_0\right)_{,xx} + \frac{1}{R_*}w^p{}_{,xx} + w^p{}_{,xx}w^p{}_{,yy}$$
$$-\left[w^p{}_{,xy}\right]^2 + \varepsilon\left[w^p{}_{,x}\left(w^*+w_0\right)_{,xx} + w^p{}_{,xx}\left(w^*+w_0\right)_{,x} + w^p{}_{,x}w^p{}_{,xx} + 2w^p{}_{,y}w^p{}_{,xy}\right] \tag{95}$$

(b) Equilibrium equations

$$Gc\left(w^P{}_{,x}+b_x\right) = D\left[b_{x,xx} - \varepsilon^2 b_x + \frac{1-\nu}{2}b_{y,yy}\right] + D\left[\frac{1+\nu}{2}b_{y,xy} - \frac{3-\nu}{2}\varepsilon b_{y,y}\right]$$
$$Gc\left(w^P{}_{,y}+b_y\right) = D\left[\frac{1+\nu}{2}b_{x,xy} + \frac{3-\nu}{2}\varepsilon b_{x,y}\right] + D\left[\beta_{y,yy} + \frac{1-\nu}{2}\left(b_{y,xx} - \varepsilon^2 b_y\right)\right] \tag{96}$$

$$D\left[b_{x,xxx} - \varepsilon b_{x,xx} + \varepsilon^2 b_x + b_{x,yy}\right] + D\left[b_{y,xxy} - \varepsilon b_{y,xy} + \varepsilon^2 b_{y,y} + b_{y,yyy}\right] = f_{,xx}/R_* + N_0 w^P{}_{,xx}$$
$$-\left[w^* + w_0\right]_{,xx} f_{,yy} - F^*{}_{,xx} w^P{}_{,yy} + f_{,xx} w^P{}_{,yy} + f_{,yy} w^P{}_{,xx} - 2 f_{,xy} w^P{}_{,xy}$$
$$+\varepsilon\left[f_{,x} w^P{}_{,xx} + f_{,xx} w^P{}_{,x}\right] - \varepsilon\left[F^*{}_{,x} w^P{}_{,xx} + F^*{}_{,xx} w^P{}_{,x} + f_{,x}\left[w^* + w_0\right]_{,xx} + f_{,xx}\left[w^* + w_0\right]_{,x}\right] \tag{97}$$

From the above analysis, we have the approximate solution

$$w^* \approx K - \frac{\mu\lambda}{\lambda_c - \lambda}\cos\frac{2\pi x}{l_x} \qquad F^* \approx \frac{\alpha\mu\lambda}{2\rho^2\left(\lambda_c - \lambda\right)}\cos\frac{2\pi x}{l_x} \tag{98}$$

Substituting (98) into the compatibility and equilibrium equations (95) - (97), we have

(a) Compatibility equation

$$\left(\frac{\partial^2}{\partial x^2} + \varepsilon\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2}\right) f = \frac{A}{R_*}\left[\frac{2\rho^2\mu\gamma\lambda_c}{\lambda_c - \lambda}\cos\frac{2\pi x}{l_x} w^P{}_{,yy} + w^P{}_{,xx}\right] + A\left[w^P{}_{,xx} w^P{}_{,yy} - \left(w^P{}_{,xy}\right)^2\right]$$
$$+A\varepsilon\left[w^P{}_{,x} w^P{}_{,xx} + 2 w^P{}_{,y} w^P{}_{,xy}\right] + \varepsilon\frac{2\rho^2\gamma\mu\lambda_c}{\lambda_c - \lambda}\frac{A}{R_*}\left[\frac{l_x}{2\pi}\sin\frac{2\pi x}{l_x} w^P{}_{,xx} + \cos\frac{2\pi x}{l_x} w^P{}_{,x}\right] \tag{99}$$

(b) Equilibrium equations

$$D\left[b_{x,xxx} - \varepsilon b_{x,xx} + \varepsilon^2 b_{x,x} + b_{x,xyy}\right] + D\left[b_{y,xxy} - \varepsilon b_{y,xy} + \varepsilon^2 b_{y,y} + b_{y,yyy}\right] = \frac{1}{R_*}\left[f_{,xx} + 2\alpha\lambda w^P{}_{,xx}\right]$$
$$-\frac{1}{R_*}\frac{\gamma\mu}{\lambda_c - \lambda}\cos\frac{2\pi x}{l_x}\left[2\rho^2\lambda_c f_{,yy} - \alpha\lambda w^P{}_{,yy}\right] + f_{,xx} w^P{}_{,yy} + f_{,yy} w^P{}_{,xx} + 2 f_{,xy} w^P{}_{,xy}$$
$$+\varepsilon\left[f_{,x} w^P{}_{,xx} + f_{,xx} w^P{}_{,x}\right] - \frac{1}{R_*}\varepsilon\frac{l_x}{2\pi}\frac{\gamma\mu}{\lambda_c - \lambda}\sin\frac{2\pi x}{l_x}\left(2\rho^2\lambda_c f_{,xx} - \alpha\lambda w^P{}_{,xx}\right) - \frac{1}{R_*}\varepsilon\frac{\mu\lambda}{\lambda_c - \lambda}$$
$$\cos\frac{2\pi x}{l_x}\left(2\rho^2\lambda_c f_{,x} - \alpha\lambda w^P{}_{,x}\right) \tag{100}$$

A rigorous solution of the above coupled equations of neutral equilibrium with given boundary conditions is difficult. An approximate solution of the nonlinear Donnell type equations is obtained as follows: First, the compatibility equation (99) and equilibrium equation (96) are solved approximately for the stress function $f(x,y)$ and shear angles $b_x(x,y)$, $b_y(x,y)$ in terms of the following assumed radial displacement $w^P(x,y)$ and the measured imperfection $w_0(x)$. In these solutions, only the effect of the initial imperfection on the buckling load is of interest. Hence, only a particular solution of equations (96) and (100) need to be considered. Second, the third equation of equilibrium (100) is solved approximately by substituting therein $f(x,y)$, $w^P(x,y)$, $b_x(x,y)$ and $w_0(x)$, and then applying Galerkin's procedure. This approach will yield a set of nonlinear algebraic equations in terms of the unknow amplitude $\xi$ (Arbocz, 1987).

An approximate solution can be obtained using an assumed mode of the form

$$w^P(x,y) = \xi\cos\frac{\pi x}{l_x}\cos\frac{\pi y}{l_y} \qquad (\xi \neq 0) \tag{101}$$

For greater generality, it would have been proper to have taken

$$w^p(x,y) = \xi \cos\frac{\pi x}{l_x} \cos\frac{k\pi y}{l_y} \tag{102}$$

and to compute buckling load curves for different values of $k$. The portions of those curves that correspond to minimum buckling loads would, of course, be the governing buckling criteria. There is little doubt that the minimizing value of $k$ would be greater than zero for sufficiently large $\lambda$. For the region of $\lambda$ in which $k$ is greater than about 3 or 4 it may even be sufficiently accurate, as in cylindrical shell buckling problems, to treat $k$ as continually variable and to formally minimize the general non-symmetrical solution with respect to $k$. Aside from the appreciable additional compution that would be required to calculate buckling curve for various values of $k$, it would be inconsistent to do so unless better approximations were made for the initial symmetrical state (Gjelsvik and Bonder, 1962).

Substituting the assumed mode into the compatibility and equilibrium equations, approximate particular solutions for $b_x(x,y)$ and $b_y(x,y)$ are obtained in terms of the coefficient $\xi$, i.e.

$$b_x(x,y) = \xi a \frac{\pi}{l_x} \sin\frac{\pi x}{l_x} \cos\frac{\pi y}{l_y} \qquad b_y(x,y) = \xi b \frac{\pi}{l_y} \cos\frac{\pi x}{l_x} \sin\frac{\pi y}{l_y} \tag{103}$$

where

$$\tau = l/\left(\sqrt{2} l_y\right) \qquad \delta = \varepsilon l \qquad a = \nabla_a/\nabla \qquad b = \nabla_b/\nabla \tag{104}$$

$$\nabla_a = 1 + \frac{1-\nu}{4}\chi_c\left(\rho^2+\tau^2\right) + \frac{1}{2}\frac{1-\nu}{2}\chi_c\frac{\delta^2}{2\pi^2} \qquad \nabla_b = 1 + \frac{1-\nu}{4}\chi_c\left(\rho^2+\tau^2\right) + \frac{1}{2}\chi_c\frac{\delta^2}{2\pi^2}$$

$$\nabla = \left[1 + \frac{1}{2}\chi_c\left(\frac{1-\nu}{2}\rho^2 + \tau^2 + \frac{1-\nu}{2}\frac{\delta^2}{2\pi^2}\right)\right]\left[1 + \frac{1}{2}\chi_c\left(\rho^2 + \frac{1-\nu}{2}\tau^2 + \frac{\delta^2}{2\pi^2}\right)\right] - \left(\frac{1+\nu}{4}\right)^2 \chi_c \rho^2 \tau^2$$

The compatibility equation will become approximately

$$\frac{R_*}{A}\left[\nabla^4 f + \varepsilon^2 f_{,xx}\right] \approx -\xi a_1 \cos\frac{\pi x}{l_x}\cos\frac{\pi y}{l_y} - \xi a_2 \sin\frac{3\pi x}{l_x}\cos\frac{\pi y}{l_y} + \xi^2 a_3\left[\cos\frac{2\pi x}{l_x} + \cos\frac{2\pi y}{l_y}\right] + \xi^2 a_4 \sin\frac{2\pi x}{l_x} \tag{105}$$

where

$$a_1 = \frac{2\rho^2\pi^2}{l^2}\left[1 + \frac{2\tau^2\gamma\mu\lambda_c}{\lambda_c - \lambda}\right] \qquad a_2 = \frac{3}{\sqrt{2}}\frac{\pi\rho^3\gamma\mu\delta\lambda_c}{\lambda_c-\lambda}\frac{1}{l^2}$$

$$a_3 = \frac{\pi^2\gamma\rho^2\tau^2}{2l^2} \qquad a_4 = \frac{\sqrt{2}\gamma\rho\delta}{2l^2}\left(\frac{\rho^2}{2}+\tau^2\right) \tag{106}$$

We have an approximate solution of the stress function

$$f(x,y) \approx -\xi\left[f_1\cos\frac{\pi x}{l_x}\cos\frac{\pi y}{l_y} + f_2\sin\frac{3\pi x}{l_x}\cos\frac{\pi y}{l_y}\right] + \xi^2\left[f_3\cos\frac{2\pi x}{l_x} + f_4\cos\frac{2\pi y}{l_y} + f_5\sin\frac{2\pi x}{l_x}\right] \tag{107}$$

$$f_1 = 2\alpha\rho^2\left[1+\frac{\tau^2\gamma\mu\lambda_c}{\lambda_c-\lambda}\right]\frac{1}{\left(\rho^2+\tau^2\right)^2-\frac{\delta^2}{2\pi^2}\rho^2} \qquad f_2 = \frac{\pi}{\sqrt{2}}\frac{\alpha\rho^3\gamma\mu\delta\lambda_c}{\lambda_c-\lambda}\frac{1}{\left(9\rho^2+\tau^2\right)^2-9\rho^2\frac{\delta^2}{2\pi^2}}$$

$$f_3 = \frac{\pi^2\alpha\gamma\tau^2}{8\pi^2\rho^2-\delta^2} \qquad f_4 = \frac{\pi^2\alpha\gamma\rho^2}{8\pi^2\tau^2-\delta^2} \qquad f_5 = \frac{\sqrt{2}\alpha\gamma\beta}{4\rho^2}\frac{\frac{\rho^2}{2}+\tau^2}{8\pi^2\rho^2-\delta^2} \tag{108}$$

We substitute equations (101), (103) and (107) into equation (100) and multiply by

$$\cos\frac{\pi x}{l_x}\cos\frac{\pi y}{l_y} \tag{109}$$

In the Galerkin procedure, the integral over the whole shell is formed and equated to zero

$$\int_0^{nl_x}\int_0^{2mR_*}[\cdots]\cos\frac{\pi x}{l_x}\cos\frac{\pi y}{l_y}dxdy = 0 \tag{110}$$

This yields an nonlinear algebraic equation for the coefficient $\xi$ in the following form:

$$\left[\lambda_c-\lambda+\alpha_0+\alpha_1\xi+\alpha_2\xi^2\right]\xi = 0 \tag{111}$$

where

$$\alpha_0 = \frac{\rho^2\tau^2\gamma\mu\lambda_c}{\lambda_c-\lambda}\left[\left(\rho^2+\tau^2\right)^2-\frac{\delta^2\rho^2}{2\pi^2}\right]^{-1} - \frac{27f_2}{8n\alpha} - \frac{\mu\gamma\tau^2}{\lambda_c-\lambda}\frac{2\rho^2\lambda f_1+\alpha\lambda}{4\alpha\rho^2}$$
$$-\frac{5f_2}{12n\alpha}\frac{\mu\gamma\tau^2\lambda_c}{\lambda_c-\lambda}+\frac{3\sqrt{2}}{16\pi^2}\frac{\mu\gamma\delta}{\lambda_c-\lambda}\frac{2\rho^2\lambda f_1+\alpha\lambda}{n\alpha\rho}-\frac{9\sqrt{2}f_2}{16\pi\alpha\rho}\frac{\mu\gamma\delta}{\lambda_c-\lambda}+\frac{5\sqrt{2}f_2}{8\pi n\alpha}\frac{\mu\gamma\delta\rho\lambda_c}{\lambda_c-\lambda} \tag{112}$$

$$\alpha_1 = -\gamma\tau^2\left(f_1-f_4\right)/(2\alpha) \qquad \alpha_2 = -\gamma\tau^2\left(nf_3+3f_5\right)/(2\pi\alpha) \tag{113}$$

and

$$\lambda_c = \bar{\lambda}_c + \alpha_0 \qquad \bar{\lambda}_c = \frac{k_0}{4\rho^2}+\frac{\rho^2}{\left(\rho^2+\tau^2\right)^2-\delta^2\rho^2/\left(2\pi^2\right)} \qquad k_0 = \left(\alpha\rho^2+b\tau^2\right)\left[\rho^2\tau^2-\delta^2/\left(2\pi^2\right)\right] \tag{114}$$

Since $\xi \neq 0$, equation (111) reduces to

$$\lambda_c - \lambda + \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 = 0 \tag{115}$$

This is a relation between $\lambda$ and amplitude $\xi$, and is called the pressure-deflection relation (Hutchinson and. Koiter, 1970).
If the nonlinear effect was not considered, the equation (115) becomes the following eigenvalue or bifurcation equation:

$$\lambda_c - \lambda + \alpha_0 = 0 \tag{116}$$

We find the value of $\lambda$ independent of the wave number $m$ in $y$ direction.

Perfect Cones

The buckling load for a geometrically perfect sandwich cone can be easily obtained by setting $\mu = 0$ in equation (115), i.e.

$$\lambda_c - \lambda + \beta_1\xi + \beta_2\xi^2 = 0 \tag{117}$$

where

$$\beta_1 = \frac{1}{2}\gamma\rho^2\tau^2\left[2\left(\rho^2\tau^2 - \frac{\delta^2\rho^2}{2\pi^2}\right)^{-1} - \frac{\pi^2\gamma}{8\pi^2\tau^2-\delta^2}\right]$$
$$\beta_2 = -\frac{1}{2n}\,\frac{\gamma^2\tau^2}{8\pi^2\rho^2-\delta^2}\left[n\pi^2\tau^2 + \frac{3\sqrt{2}}{4}\,\frac{\delta}{\rho^2}\left(\frac{1}{2}\rho^2+\tau^2\right)\right] \tag{118}$$

If the nonlinear term is omitted,

$$\lambda = \frac{k_0}{4\rho^2} + \frac{\rho^2}{\left(\rho^2+\tau^2\right)^2 - \delta^2\rho^2/\left(2\pi^2\right)} \tag{119}$$

For the sandwich cylinder, that is $\delta = 0$, equation (119) reduces to

$$\lambda = \frac{1}{1+\frac{1}{2}\chi_c\left(\rho^2+\tau^2\right)}\,\frac{\left(\rho^2+\tau^2\right)^2}{4\rho^2} + \frac{\rho^2}{\left(\rho^2+\tau^2\right)^2} \tag{120}$$

In the limiting case of a non-shear deformable core equation (120) reduces to:

$$\lambda = \frac{1}{2}\left(\Lambda+\frac{1}{\Lambda}\right) \qquad \Lambda = \frac{\left(\rho^2+\tau^2\right)^2}{2\rho^2} \tag{121}$$

From

$$\frac{d\lambda}{d\Lambda} = \frac{1}{2}\left(1-\frac{1}{\Lambda^2}\right) = 0 \tag{122}$$

we have that any combination of $\rho$ and $\tau$ that satisfies

$$\rho^2 + \tau^2 - \sqrt{2}\rho = 0 \tag{123}$$

will yield a minimum at $\lambda = 1$.

Equation (123) is the well known Koiter circle for sandwich cylinders with non-shear deformable core (Tennyson and Chan, 1990), which is the locus of a family of modes belonging to the lowest eigenvalue $\lambda_c = 1.0$ (Arbocz, 1987).

For a sandwich cylinder, we get the generalized Koiter circle

$$\left(\rho^2+\tau^2\right)^2 = 2\rho^2\sqrt{1+\frac{1}{2}\chi_c\left(\rho^2+\tau^2\right)} \tag{124}$$

Imperfect Cone

The pressure-deflection relation (115) yields the limit buckling coefficient $\lambda_s$ for an axisymmetric imperfect sandwich cone by minimizing $\lambda$ with respect to the circumferential wave number $\tau$ for a given imperfection wave number $\rho$. Because of the complexity of the equation, the minimization has to be done numerically in a forthcoming paper and the smallest root of eigenvalue equation is selected as $\lambda_s$.

Since there is no asymmetric imperfection, in this case, the limit buckling coefficient will be taken to have the following form:

$$\lambda_s = \lambda_c - \frac{\alpha_1^2}{4\alpha_2} \tag{125}$$

## 9 Conclusions

1. The imperfections have a pronounced effect on the buckling (Koiter, 1945).
2. Sandwich shells can be considered as a material imperfect, or damaged shell $(\chi_c)$ compared with ist corresponding perfect shell $(\chi_c = 0)$.
3. The small parameter $\varepsilon_0$ or $\delta$ have an effect on the stationary value of the buckling coefficient. When the parameter is increasing the buckling coefficient is decreasing.
4. The bifurcation value of buckling load and the stress for the perfect cone are

$$P_{cr} = 2\pi R_1 t\sigma_{cr} = \frac{2\pi E t^2}{\sqrt{3(1-\nu^2)}} \frac{R_1}{R_c}\lambda_a \cos^2\alpha \tag{126}$$

$$\sigma_{cr} = \frac{Et}{\sqrt{3(1-\nu^2)}} \frac{1}{R_c}\lambda_a \sin^2\varphi = \frac{Et}{\sqrt{3(1-\nu^2)}} \frac{1}{R_c}\lambda_a \cos^2\alpha \tag{127}$$

5. The stationary value of buckling load and stress for imperfect cones are

$$P_s = 2\pi R_1 t\sigma_s = \frac{2\pi E t^2}{\sqrt{3(1-\nu^2)}} \frac{R_1}{R_c}\lambda_s \cos^2\alpha \tag{128}$$

$$\sigma_s = \frac{Et}{\sqrt{3(1-\nu^2)}} \frac{1}{R_c}\lambda_s \sin^2\varphi = \frac{Et}{\sqrt{3(1-\nu^2)}} \frac{1}{R_c}\lambda_s \cos^2\alpha \tag{129}$$

It should be noted that the above relations between the bifurcation coefficient and stationary coefficient for buckling are very important in the practical design of sandwich cones.

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- For Literature and Addresses see Band 14, Heft 3/4, (1994), S. 248 -