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# On the Buckling of Structures

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*Using Schieck and Stumpf's superposition approach, the kinematics of buckling for continua has been investigated in the present paper. According to the properties of buckling phenomena the concept of "bifurcation configuration" has been introduced, and the total deformation gradient* $\mathbf{F}$ *can be expressed by pre-buckling deformation gradient* $\mathbf{F}_1$ *and post-buckling deformation gradient* $\mathbf{F}_2$, i.e. $\mathbf{F} = \mathbf{F}_2\mathbf{F}_1$. *As an extension, the elasto-plastic deformation has been investigated for the post-buckling stage using Lee and Liu's multiplicative decomposition* $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$.

## 1. Introduction

As we know, for thin-walled structures the membrane stiffness is generally several orders of magnitude greater than the bending stiffness. A thin-walled structure can absorb a great deal of membrane strain energy without deforming too much. It must deform much more in order to absorb an equivalent amount of bending strain energy. If the structure is loading in such a way that most of its strain energy is in the form of membrane compression, and if there is a way that this stored-up membrane energy can be converted into bending energy, the shell may fail rather dramatically in a process called "buckling", as it exchanges its membrane energy for bending energy. Very large bending deflections are generally required to convert a given amount of membrane energy into bending energy.

The way in which buckling occurs depends on how the structure is loaded and on its geometrical and material properties. The prebuckling process is often nonlinear if there is a reasonably large percentage of bending energy being stored in the structure throughout the loading history.

According to the percentage of bending energy, the two basic ways in which a conservative elastic system may lose its stability are: nonlinear collapse (snap-through, or over-the-hump) and bifurcation buckling (Figure 1.)

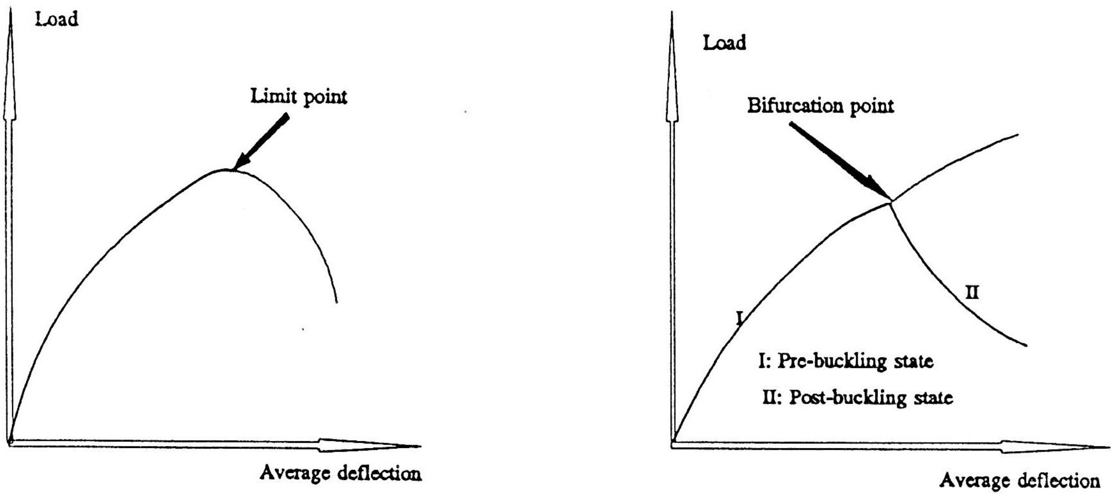


Figure 1. Load-deflection curves showing the two ways stability may be lost

Nonlinear collapse is predicated by means of a nonlinear analysis. The stiffness of the structure or the slope of the load-deflection curve, decreases with increasing load. At the collapse load the load-deflection curve has zero slope and, if the load is maintained as the structure deforms, failure of the structure is usually dramatic and almost instantaneous. This type of instability failure is often called "snap-through", a nomenclature derived from the many early tests and theoretical models of shallow arches, caps and cones. These very nonlinear systems initially deform slowly with increasing load. As the load approaches the maximum value, the rate of deformation increase until, reaching a status of neutral equilibrium in which the average curvature is almost zero, these shallow structures subsequently "snap-through" to a post-buckled state which resembles the original structure in an inverted form.

The term "bifurcation buckling" refers to a different kind of failure, the onset of which is predicted by means of an eigenvalue analysis. At the buckling load, or bifurcation point on the load-deflection path, the deformations begin to grow in an new pattern which is quite different from the prebuckling pattern.

In general, the shallower shells will snap-through, while the deeper shells will bifurcate. In this paper we shall restrict our discussion to those structures which lose their stability by bifurcation.

The general theory of buckling and post-buckling behavior of elastic structures enunciated by Koiter (1945) has spawned a considerable amount of research in this field. In additional to Koiter's original work, very many papers on the general theory have emerged, e. g. Sewell, 1968, and Thompson, 1969, almost exclusively in the language of finite-dimensional systems. Variations of the Koiter approach have be usually been based on continuum concepts, with a bias toward virtual work (Budiansky and Hutchinson, 1964; Budiansky, 1965, 1969; Fitch, 1968; Cohen, 1968; Masur, 1973; Arbocz, 1974, 1987; Budiansky, 1974; Stumpf, 1985; Pietraszkiewicz, 1993).

## 2. Buckling and Change of Configuration

Now let us to discuss the process of buckling (Bushnell, 1985) and motion of configuration. To most laymen the word "buckling" evokes an image of failure of a structure which has been compressed in some way. Pictures and perhaps sounds come to mind of sudden , catastrophic collapse involving very large deformations. From a scientific and engineering point of view, the phenomenon can be described as follows: For the static analysis of perfect structures, the two phenomena loosely termed "buckling" are collapse at the maximum point in a load vs. deflection curve, and bifurcation buckling. These two types of instability failure are illustrated in Figure 2. and 3.

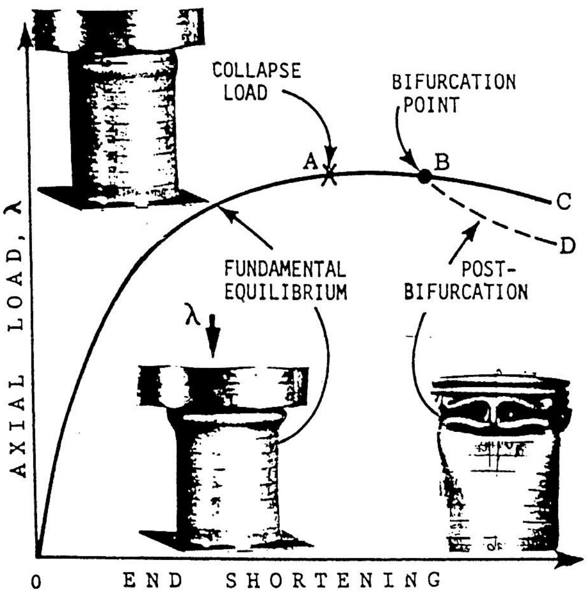


Figure 2. Load-end shortening curve with limit point A, bifurcation point B, and post-buckling equilibrium path BD (from Bushnell, 1985)

The axially compressed cylinder shown in Figure 2. deforms approximately axisymmetrically along the equilibrium path OA until a maximum or limit load $\lambda_L$ is reached at point A. If the axial load $\lambda$ is not sufficiently

relieved by the reduction in axial stiffness, the perfect cylinder will fail at this limit load, following either the path ABC along which it continues to deform axisymmetrically, or some other path ABD along which it first deforms axisymmetrically from A to B and then nonaxisymetrically from B to D. Limit point buckling, or "snap-through", occurs at point A and buckling at point B. The equilibrium path OABC, corresponding to the nonaxisymmetrical mode of deformation is called the fundamental or primary or prebuckling path; the post-buckling bifurcation equilibrium path BD, corresponding to the axisymmetrical mode of deformation is called the secondary or post-buckling path. Buckling of either collapse or bifurcation type may occur at loads for which some or all the structureal material has been stressed beyond its proportional limit. The example in Figure 2. is somewhat unusual in that the bifurcation point B is shown to occur after the collapse point has been reached. In this particular case, therefore, bifurcation buckling is of less engineering significance than axisymmetric collapse.
A commonly occuring situation is illustrated in Figure 3. The bifurcation point B is between O and A. If the fundamental path OAC corresponds to axisymmetrical deformation and BD to nonaxisymmetrical deformation, then initial failure of the structure would generally be characterized by rapidly growing nonaxisymmetric deformations. In this case the collapse load of the perfect structure $\lambda_L$ is of less engineering significance than the bifurcation point $\lambda_C$.

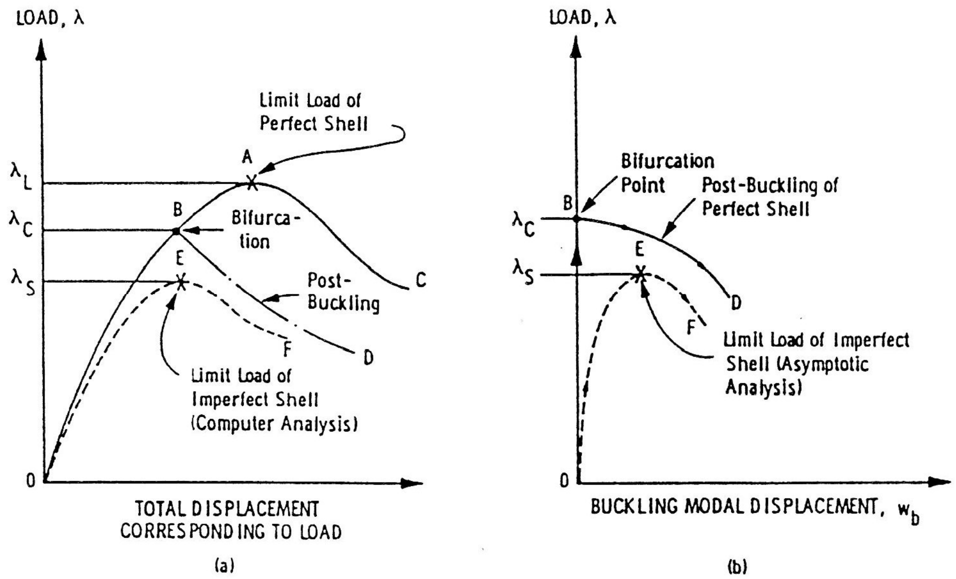


Figure 3. Load deflection curves showing limit and bifurcation points
(from Bushnell, 1985)
(a) general nonlinear analysis, (b) asymptotic analysis

In the case of real structures which contain unavoidable imperfections there is no such thing as true bifurcation buckling. The actual structure will follow a fundamental path OEF, with the failure corresponding to "snap-through" at point E at the collapse load $\lambda_s$. If point E in Figure 3. corresponds to bifurcation into a nonsymmetric buckling mode, the collapse at E will involve significant nonsymmetric displacement components. Although true bifurcation buckling is fictious, the bifurcation buckling analytical model is valid in that it is convenient and often leads to a good approximation of the actual failure load and mode.

From the above description, for the common situation illustrated in Figure 3., the motion of configuration corresponding to O, B and D can be described as follows. First we introduce a concept of "bifurcation configuration" which corresponds to bifurcation point B. Generally, the bifurcation configuration is a stressed and strained state, not the relaxed and stress-free state; and it is a real configuration and not a virtual one, which is different from the intermediate configuration (relaxed, stress-free) which was introduced by Lee (1967) in plasticity. Similarly, the initial or undeformed configuration and deformed or post-buckling one corresponding to the O and D states.

The configuration of the initially undeformed body is expressed by the particle coordinates labeled **X** in Figure 4. The body is subjected to deformation to the deformed or post-buckling configuration $B_2$ defined by the mapping

$$x = \phi(\mathbf{X}, t) \tag{1}$$

It is assumed that the mapping (1) is one-to-one and as many times differentiable as required. Hence the Jacobian of the transformation, $J = \det(\mathbf{F})$, is finite and positive, where det denotes determinant. The undeformed configuration is denoted by $B_0$, the bifurcation configuration is denoted by $B_1$ and the deformed configuration is denoted by $B_2$ corresponding to Figure 3.

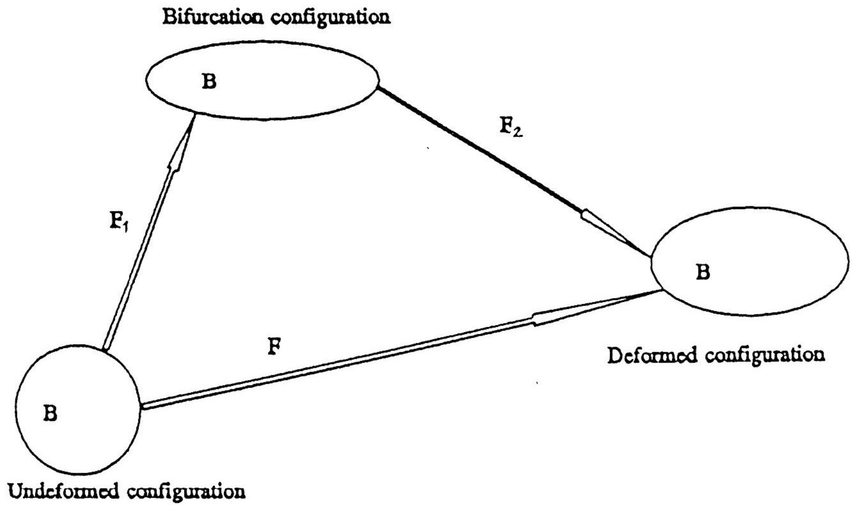


Figure 4. Change of configuration, undeformed, bifurcation and deformed configuration

The analysis of the general deformation is expressed in terms of the deformation gradient $\mathbf{F} = T\phi$, which is the tangent of the deformation (1), and it can be expressed as

$$\mathbf{F} = T\phi = \frac{\partial x}{\partial \mathbf{X}} \tag{2}$$

for the total deformation to the deformed configuration $B_2$. From the undeformed configuration $B_1$, the pre-buckling or primary deformation gradient $\mathbf{F}_1$ can be expressed as

$$\mathbf{F}_1 = \frac{\partial \bar{x}}{\partial \mathbf{X}} \tag{3}$$

and post-buckling configuration or secondary configuration gradient $\mathbf{F}_2$ can be expressed as

$$\mathbf{F}_2 = \frac{\partial x}{\partial \mathbf{X}} \tag{4}$$

Given the configuration $B_0$, $B_1$ and $B_2$, the total mapping $B_0 \rightarrow B_2$ can be expressed mathematically, by the sequence of mappings $B_0 \rightarrow B_1$ and $B_1 \rightarrow B_2$, and the chain rule then yields

$$\mathbf{F} = \mathbf{F}_2 \mathbf{F}_1 \tag{5}$$

which expresses the total deformation in terms of the pre-buckling and postbuckling components. It should be noted here that the meaning of equation (4) is different from the superposition formula of Schieck and Stumpf (1993) in plasticity, nevertheless the form of equation (4) is same as theirs. Their results are very general and of course we can derive some benefits from them.

Indeed, equation (5) represents the superposition of two finite deformations. For the analysis of buckling problems, firstly we find the solution of pre-buckling, then the buckling solution and finally the solution of post-buckling.
In the following section we will follow Schieck and Stumpf (1993), Stumpf and Schieck (1993), to formulate the kinematics of buckling for continua.

## 3 Kinematics for Pre-Buckling, Bifurcation and Post-Buckling

As we know, the whole process of buckling analysis includes three steps. The first one is prebuckling analysis, the second is bifurcation analysis, and the third is post-buckling analysis. The corresponding kinematics will be listed in this section.

### 3.1 Pre-Buckling Analysis

In this case, without secondary path, i.e. $\mathbf{F}_2 = 1$. The pre-buckling process is a normal process of nonlinear deformation. Generally nonlinear analysis should be used for pre-buckling investigations. For this kind of problem, there are numerous papers about this topic.

### 3.2 Bifurcation Analysis

The kinematics of bifurcation analysis can be considered as small deformation superposed on a large one. This problem has been investigated in detail.

### 3.3 Postbuckling Analysis

The problem for kinematics of post-buckling is how to get total strain tensor **E** after the pre-buckling is updated. This is the central task of this paper.

The transformation (1) rotates and stretches material line elements, and therefore, a material neighbourhood is, in general, rotated as well as distorted. To uncouple the rigid rotation form the pure distorsion, the polar decomposition of **F** is used

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \tag{6}$$

where the symmetric positive-definite tensors **U** and **V** are called the right and left stretch tensors, and where **R** is a proper orthogonal matrix, and $\mathbf{R}^{-1} = \mathbf{R}^T$, $\det \mathbf{R} = +1$. From equation (6) it follows that

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 = \mathbf{R}^T\mathbf{V}^2\mathbf{R} = \mathbf{R}^T\mathbf{B}\mathbf{R} \tag{7}$$

where

$$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T = \mathbf{R}\mathbf{C}\mathbf{R}^T \tag{8}$$

The quantity **B** is called the left Cauchy-Green tensor, and we recall that **C** is the right Cauchy-Green tensor. It is clear that the principle values of **C** and **B** are the same, but their principal directions are not. Furthermore, equation (7) states that **U** and **C** are coaxial in the sense that they have the same principal directions, but the principal values of **C** are the squares of those of **U**. Similar comments apply to **V** and **B**.

Applying the polar decomposition theorem to the pre-buckling deformation gradient $\mathbf{F}_1$ and to the post-buckling deformation gradient $\mathbf{F}_2$, we have

$$\mathbf{F}_1 = \mathbf{R}_1\mathbf{U}_1 \qquad \mathbf{F}_2 = \mathbf{R}_2\mathbf{U}_2 \tag{9}$$

where

$$\mathbf{U}_1^2 = \mathbf{F}_1^T\mathbf{F}_1 \qquad \mathbf{U}_2^2 = \mathbf{F}_2^T\mathbf{F}_2$$

are positive and symmetric pre- and post-buckling stretches. A schematic sketch of equations (5), (6) and (9) is shown in Figure 5.

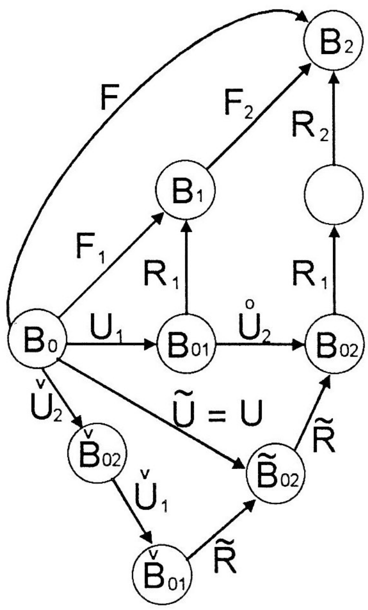


Figure 5. A schematic sketch of decomposition

Here we use the approach of Schieck and Stumpf (1993), then one can get the following decomposition of deformation gradient $\mathbf{F}$

$$\mathbf{F} = \mathbf{R}_2\mathbf{R}_1\mathring{\mathbf{U}}_2\mathbf{U}_1 \tag{10}$$

where

$$\mathring{\mathbf{U}}_2 = \mathbf{R}_1^T\mathbf{U}_2\mathbf{R}_1 \tag{11}$$

is the pull-back of $\mathbf{U}_2$ with rotation $\mathbf{R}_1$. Decomposition (10) leads the following decomposition of total stretch tensor into pre- and post-buckling parts

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \mathbf{U}_1\mathring{\mathbf{U}}_2^2\mathbf{U}_1 \tag{12}$$

According to Schieck and Stumpf the back-rotated post-buckling stretch tensor is for the general case of non-coaxial deformation. The composition of two stretch tensors in equation (10) is non-symmetric and non-commutative and therefore connected with an additional rotation

$$\tilde{\mathbf{R}} = \mathbf{U}_2\mathring{\mathbf{U}}_1\mathbf{U}^{-1} \tag{13}$$

Then we have another decomposition of $\mathbf{F}$ as follows

$$\mathbf{F} = \mathbf{R}_2\mathbf{R}_1\tilde{\mathbf{R}}\mathbf{U} = \mathbf{R}\mathbf{U} \qquad \mathbf{R} = \mathbf{R}_2\mathbf{R}_1\tilde{\mathbf{R}} \tag{14}$$

where $\mathbf{R}$ is total rotation of deformation. Using the additional rotation (13) another decomposition of deformation gradient $\mathbf{F}$ can be written as

$$\mathbf{F} = \mathbf{R}_2\mathbf{R}_1\tilde{\mathbf{R}}^2\breve{\mathbf{U}}_1\breve{\mathbf{U}}_2 \tag{15}$$

where

$$\breve{\mathbf{U}}_1 = \tilde{\mathbf{R}}^T \mathbf{U}_1 \tilde{\mathbf{R}} \qquad \breve{\mathbf{U}}_2 = \tilde{\mathbf{R}}^T \mathring{\mathbf{U}}_2 \tilde{\mathbf{R}} \tag{16}$$

are pull-back of $\mathbf{U}_1$ and $\mathbf{U}_2$ with the additional rotation (13). The decomposition (15) leads to another decomposition of total stretch into pre- and post-buckling parts

$$\mathbf{U}^2 = \breve{\mathbf{U}}_2 \breve{\mathbf{U}}_1 \breve{\mathbf{U}}_2 \tag{17}$$

Recall the following relation between **U** and the Lagrange-Green strain tensor **E**

$$\mathbf{U}^2 = 1 + 2\mathbf{E} \tag{18}$$

then from equations (12) and (17) the following results can be obtained:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{U}_1 \mathring{\mathbf{U}}_2^2 \mathbf{U}_1 - \mathbf{I}\right) \tag{19}$$

and

$$\mathbf{E} = \frac{1}{2}\left(\breve{\mathbf{U}}_2 \breve{\mathbf{U}}_1^2 \breve{\mathbf{U}}_2 - \mathbf{I}\right) \tag{20}$$

where **I** is the identity tensor, det $\mathbf{I} = 1$. If one introduces the definitions for the Lagrange-Green strain tensor as follows:

$$\begin{aligned}
\mathbf{E}_1 &= \frac{1}{2}\left(\mathbf{F}_1^T \mathbf{F}_1 - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{U}_1^2 - \mathbf{I}\right) & \mathbf{E}_2 &= \frac{1}{2}\left(\mathbf{F}_2^T \mathbf{F}_2 - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{U}_2^2 - \mathbf{I}\right) \\
\breve{\mathbf{E}}_1 &= \frac{1}{2}\left(\breve{\mathbf{U}}_1^2 - \mathbf{I}\right) & \breve{\mathbf{E}}_2 &= \frac{1}{2}\left(\breve{\mathbf{U}}_2^2 - \mathbf{I}\right) \\
\mathring{\mathbf{E}}_1 &= \frac{1}{2}\left(\mathring{\mathbf{U}}_1^2 - \mathbf{I}\right) & \mathring{\mathbf{E}}_2 &= \frac{1}{2}\left(\mathring{\mathbf{U}}_2^2 - \mathbf{I}\right)
\end{aligned} \tag{21}$$

then we have the following relations:

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_1 + \mathbf{F}_1^T \mathbf{E}_2 \mathbf{F}_1 \\
\mathbf{E} &= \mathbf{E}_1 + \mathbf{U}_1 \mathring{\mathbf{E}}_2 \mathbf{U}_1 \\
\mathbf{E} &= \breve{\mathbf{E}}_2 + \breve{\mathbf{U}}_2 \breve{\mathbf{E}}_1 \breve{\mathbf{U}}_2
\end{aligned} \tag{22}$$

which means the total Lagrange-Green strain tensor **E** can be additively decomposed into pre- and post-buckling parts by means of pull-back and push-forward operation.

Now we consider the decomposition of deformation rate **D**. Let **l** denote the compatible velocity gradient with respects to the deformed configuration, then

$$\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{F}}_2 \mathbf{F}_2^{-1} + \mathbf{F}_2\left(\dot{\mathbf{F}}_1 \mathbf{F}_1^{-1}\right)\mathbf{F}_2^{-1} = \mathbf{l}_2 + \mathbf{F}_2 \mathbf{l}_1 \mathbf{F}_2^{-1} \tag{23}$$

where

$$\mathbf{l}_1 = \dot{\mathbf{F}}_1 \mathbf{F}_1^{-1} \qquad \mathbf{l}_2 = \dot{\mathbf{F}}_2 \mathbf{F}_2^{-1} \tag{24}$$

Using l, one can define the deformation rate **D** as follows

$$\mathbf{D} = \dot{\mathbf{E}} = \dot{\mathbf{E}}_1 + \left(\mathbf{F}_1^T\mathbf{E}_2\mathbf{F}_2\right)^{\cdot} = \dot{\mathbf{E}}_1 + \mathbf{F}_1^T\mathbf{E}_2^{\circ}\mathbf{F}_1 \tag{25}$$

where

$$\mathbf{E}_2^{\circ} = \dot{\mathbf{E}}_2 + \mathbf{l}_1^T\mathbf{E}_2 + \mathbf{E}_2\mathbf{l}_1 \tag{26}$$

is the objective rate of $\mathbf{E}_2$, which can also be defined as the Lee derivative of $\mathbf{E}_2$. And equation (21) can be written as

$$\dot{\mathbf{E}}_1 = \frac{1}{2}\left(\dot{\mathbf{F}}_1^T\mathbf{F}_1 + \mathbf{F}_1^T\dot{\mathbf{F}}_1\right) = \mathbf{F}_1^T\mathbf{d}_1\mathbf{F}_1 \qquad \mathbf{d}_1 = \frac{1}{2}\left(\mathbf{l}_1 + \mathbf{l}_1^T\right) \tag{27}$$

then equation (25) becomes

$$\mathbf{D} = \mathbf{F}_1^T\left(\mathbf{d}_1 + \mathbf{E}_2^{\circ}\right)\mathbf{F}_1 \tag{28}$$

3.4 Consequence

For some problems, the pre-buckling deformation is very small, that is $\mathbf{F}_1 \approx 1$. In this case, there is no difference between bifurcation configuration and undeformed configuration. Then we have the following simplified results:

$$\begin{aligned}
&\dot{\mathbf{U}}_2 \approx \mathbf{U}_2 \qquad \mathbf{U}^2 \approx \mathbf{U}_1\mathbf{U}_2^2\mathbf{U}_1 \\
&\widetilde{\mathbf{R}} = \mathbf{U}_2\mathbf{U}_1\mathbf{U}^{-1} \qquad \widetilde{\mathbf{U}}_i = \widetilde{\mathbf{R}}^T\mathbf{U}_i\widetilde{\mathbf{R}} \qquad i = 1,2 \\
&\mathbf{E} \approx \mathbf{E}_1 + \mathbf{E}_2 = \overset{\circ}{\mathbf{E}}_1 + \mathbf{E}_2 \\
&\mathbf{D} \approx \dot{\mathbf{E}}_1 + \mathbf{E}_2^{\circ} = \dot{\mathbf{E}}_1 + \mathbf{E}_2
\end{aligned} \tag{29}$$

From the above we can see that the total strain tensor **E** and total strain rate **D** can be additively decomposed into pre- and post-buckling parts.

## 4 Introduction of Elasto-Plastic Effect

During the deformation elasto-plastic phenomena often exist. Sometimes during the pre-buckling process there is plastic deformation, and sometimes only during the post-buckling process. For the general case, we can refer to Schieck and Stumpf (1993), only changing the meaning of updated configuration into bifurcation one. In this section we are only interested in the following special problem: pre-buckling is elastic and post-buckling is elasto-plastic.

In order to introduce the plastic effect, we use Lee's (1967, 1969) multiplicative decomposition of deformation gradient. According to his suggestion, deformation gradient **F** can be multiplicativily decomposed into elastic and plastic parts, $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, based on the concept of local, current and relaxed intermediate configuration. However, how to define plastic strain has remained a controversial issue for over 20 years.

In this case, we have decomposition $\mathbf{F}_1 = \mathbf{F}_1^e$ (for convenience we denote $\mathbf{F}_1$ with "$e$"), and $\mathbf{F}_2 = \mathbf{F}_2^e\mathbf{F}_2^p$. Then we have total deformation gradient **F**

$$\mathbf{F} = \mathbf{F}_2\mathbf{F}_1 = \mathbf{F}_2^e\mathbf{F}_2^p\mathbf{F}_1^e \tag{30}$$

Using the polar decomposition theorem

$$\mathbf{F}_1^e = \mathbf{R}_1^e\mathbf{U}_1^e \qquad \mathbf{F}_2^e = \mathbf{R}_2^e\mathbf{U}_2^e \qquad \mathbf{F}_2^p = \mathbf{R}_2^p\mathbf{U}_2^p \tag{31}$$

then the total **F** can be written as

$$\mathbf{F} = \mathbf{R}_2^e\mathbf{U}_2^e\mathbf{R}_2^p\mathbf{U}_2^p\mathbf{R}_1^e\mathbf{U}_1^e \tag{32}$$

A schematic sketch of the polar decompositions (31) and (32) is shown in Figure 6.

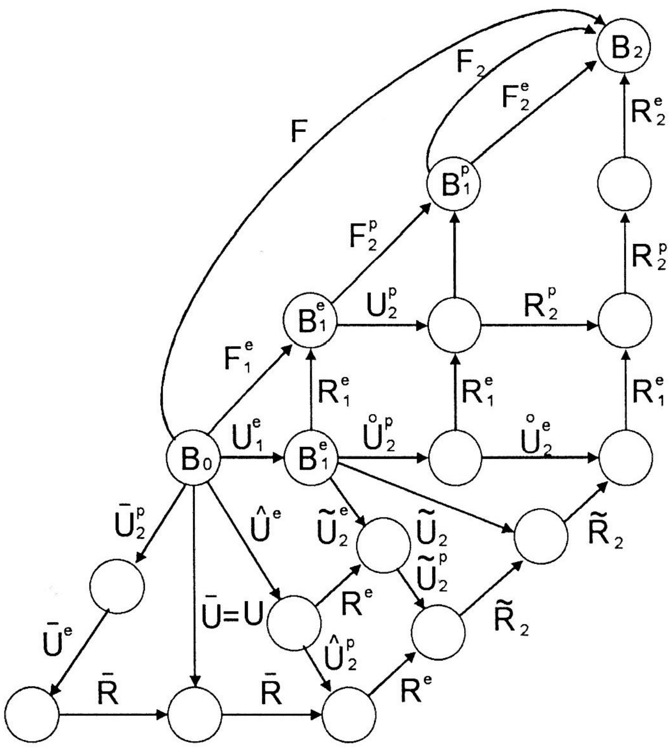


Figure 6. A schematic sketch of decomposition

The central problem now is how to get the total elastic and plastic stretch. First we define the two back-rotated stretch tensors

$$\mathring{\mathbf{U}}_2^p = \mathbf{R}_1^{eT}\mathbf{U}_2^p\mathbf{R}_1^e \qquad \mathring{\mathbf{U}}_2^e = \mathbf{R}_2^{pT}\mathbf{R}_1^{eT}\mathbf{U}_2^e\mathbf{R}_1^e\mathbf{R}_2^p \tag{33}$$

According to Schieck and Stumpf the back-rotated post-buckling stretch tensor is for the general case of non-coaxial deformations. The composition of two stretch tensors in equation (10) is non-symmetric and non-commutative and therefore connected with an additional rotation

$$\tilde{\mathbf{R}}_2 = \mathring{\mathbf{U}}_2^e\,\mathring{\mathbf{U}}_2^p\tilde{\mathbf{U}}_2^{-1} \tag{34}$$

With the rotation (34) we are able to define the following two back-rotated stretch tensors

$$\tilde{\mathbf{U}}_2^e = \tilde{\mathbf{R}}_2^T\,\mathring{\mathbf{U}}_2^e\tilde{\mathbf{R}}_2 \qquad \tilde{\mathbf{U}}_2^p = \tilde{\mathbf{R}}_2^T\,\mathring{\mathbf{U}}_2^p\tilde{\mathbf{R}}_2 \tag{35}$$

From the above analysis the total deformation gradient **F** can be expressed as

$$\mathbf{F} = \mathbf{R}_2^e\mathbf{R}_2^p\mathbf{R}_1^e\tilde{\mathbf{R}}_2^2\tilde{\mathbf{U}}_2^p\mathbf{U}_1^e \tag{36}$$

yielding the total stretch decomposition

$$\mathbf{U} = \mathbf{F}^T\mathbf{F} = \widetilde{\mathbf{U}}_2^p\widetilde{\mathbf{U}}_2^e\mathbf{U}_1^{e2}\widetilde{\mathbf{U}}_2^e\widetilde{\mathbf{U}}_2^p \tag{37}$$

As it can be seen from equation (36) elastic and plastic stretch appear seperately for the pre- and post-buckling deformation. Our intention is now to determine the total elastic stretch. Let us apply the polar decomposition theorem to pre- and post-buckling elastic composition

$$\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e = \widetilde{\mathbf{U}}_2^e\mathbf{U}_1^e \tag{38}$$

yielding the total elastic stretch

$$\mathbf{U}^{e2} = \mathbf{U}_1^e\widetilde{\mathbf{U}}_2^{e2}\mathbf{U}_1^e \tag{39}$$

It can be seen that total elastic stretch has been scperatly decomposed into pre- and post-buckling parts. With the rotation $\mathbf{R}^e$ as defined in equation (38) we can obtain a back-rotated plastic stretch tensor

$$\hat{\mathbf{U}}_2^p = \widetilde{\mathbf{R}}^{eT}\mathbf{U}_2^p\widetilde{\mathbf{R}}^p \tag{40}$$

Then the total deformation gradient $\mathbf{F}$ can also be expressed as

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{R}_2^e\mathbf{R}_2^p\mathbf{R}_1^e\widetilde{\mathbf{R}}_2^2\mathbf{R}^e\,\hat{\mathbf{U}}_2^p\mathbf{U}^e \tag{41}$$

yielding the total stretch tensor $\mathbf{U}$

$$\mathbf{U}^2 = \mathbf{U}^e\,\hat{\mathbf{U}}_2^{p2}\mathbf{U}^e \tag{42}$$

This equation means that the total stretch tensor has been decomposed into total elastic stretch and total plastic stretch parts.

Similar to the procedure outlined in the above we can decompose the deformation gradient $\mathbf{F}$ in the following alternative form

$$\begin{aligned} &\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{R}_2^e\mathbf{R}_2^p\mathbf{R}_1^e\widetilde{\mathbf{R}}_2^2\mathbf{R}^e\overline{\mathbf{R}}^2\overline{\mathbf{U}}^e\overline{\mathbf{U}}_2^p \\ &\overline{\mathbf{U}}^e = \overline{\mathbf{R}}^T\mathbf{U}^e\overline{\mathbf{R}} \qquad\qquad \overline{\mathbf{U}}_2^p = \overline{\mathbf{R}}^T\,\hat{\mathbf{U}}_2^p\overline{\mathbf{R}} \end{aligned} \tag{43}$$

yielding the total stretch

$$\mathbf{U}^2 = \overline{\mathbf{U}}_2^p\overline{\mathbf{U}}^{e2}\overline{\mathbf{U}}_2^p \tag{44}$$

Based on the above decompositions, the Langrange-Green strain tensor can be represented as

$$\mathbf{E} = \mathbf{E}_1^e + \mathbf{U}_1^e\widetilde{\mathbf{E}}_2\mathbf{U}_1^e \qquad\qquad \widetilde{\mathbf{E}}_2 = \widetilde{\mathbf{E}}_2^e + \widetilde{\mathbf{U}}_2^e\widetilde{\mathbf{E}}_2^p\widetilde{\mathbf{U}}_2^e \tag{45}$$

and

$$\mathbf{E} = \mathbf{E}^e + \mathbf{U}^e\,\hat{\mathbf{E}}^p\mathbf{U}^e \tag{46}$$

and

$$\mathbf{E} = \overline{\mathbf{E}}_2^p + \overline{\mathbf{U}}_2^p\overline{\mathbf{E}}^e\overline{\mathbf{U}}_2^p \tag{47}$$

in which

$$\widetilde{\mathbf{E}}^e = \frac{1}{2}\left(\mathbf{U}^{e2}-1\right) \qquad \hat{\mathbf{E}}^p = \frac{1}{2}\left(\hat{\mathbf{U}}^{p2}-1\right)$$

$$\widetilde{\mathbf{E}}_1^e = \frac{1}{2}\left(\widetilde{\mathbf{U}}^{e2}-1\right) \qquad \widetilde{\mathbf{E}}_2^e = \frac{1}{2}\left(\widetilde{\mathbf{U}}_2^{e2}-1\right) \qquad \widetilde{\mathbf{E}}_2^p = \frac{1}{2}\left(\widetilde{\mathbf{U}}_2^{p2}-1\right) \tag{48}$$

$$\overline{\mathbf{E}}^e = \frac{1}{2}\left(\overline{\mathbf{U}}^{e2}-1\right) \qquad \overline{\mathbf{E}}_2^p = \frac{1}{2}\left(\overline{\mathbf{U}}^{p2}-1\right)$$

From equations (45),(46) and (47), some consequences can be obtained as follows:

1. pre-buckling deformation is small, i.e., $\mathbf{F}_1^e \approx 1$, then

$$\mathbf{E} \approx \mathbf{E}_1^e + \widetilde{\mathbf{E}}_2^e + \widetilde{\mathbf{U}}_2^e\widetilde{\mathbf{U}}_2^p\widetilde{\mathbf{U}}_2^e \approx \mathbf{E}^e + \mathring{\mathbf{E}}_2^e + \widetilde{\mathbf{U}}_2^e\widetilde{\mathbf{U}}_2^p\widetilde{\mathbf{U}}_2^e \tag{49}$$

2. post-buckling plastic deformation is small, i.e. $\mathbf{F}_2^p \approx 1$, then

$$\mathbf{E} \approx \widetilde{\mathbf{E}}_2^p + \widetilde{\mathbf{E}}^e \tag{50}$$

3. post-buckling elastic deformation is small, i.e. $\mathbf{F}_2^e \approx 1$, then

$$\mathbf{E} \approx \mathbf{E}_1^e + \mathbf{U}_1^e\left(\widetilde{\mathbf{E}}_2^e + \widetilde{\mathbf{E}}_2^p\right)\mathbf{U}_1^e \tag{51}$$

4. post-buckling deformation (elastic and plastic) is small, i.e. $\mathbf{F}_2 \approx 1$, then

$$\mathbf{E} \approx \mathbf{E}_1^e + \widetilde{\mathbf{E}}_2^e + \widetilde{\mathbf{E}}_2^p \approx \mathbf{E}^e + \widetilde{\mathbf{E}}_2^p \tag{52}$$

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