TECHNISCHE MECHANIK, Band 15, Heft 2, (1995), 91-98
Manuskripteingang: 03. Mai 1995

An Algorithm for the Construction of Influence Matrices for Shape Controlled Plates

S. M. Bauer, A. L. Smirnov, P. E. Tovstik, M. I. Ulitin

The algorithm for the construction of the influence matrix that allows the determination of the mirror surface points displacements and its mean square deviation from the given surface is proposed. From the minimum condition for the mean square deviation the optimal reactions in the supports are found. The proposed algorithm is applied for the providing of the optimal control of a circular mirror of constant thickness under external loading.

1 Determination of the Displacement of an Arbitrary Point

The nonuniform heating and the influences of the other external factors lead to the deformations and distortion of the geometric form of a plate. If the plate is the reflecting surface one has to control its form to get the desirable characteristics of the wave front (Tovstik and Ulitin, 1991). The algorithm for construction of the influence matrix proposed below helps to determine the displacements of the mirror surface points and the mean square deviation of the surface from the given surface. By choice of the support reactions the mean square surface deviation can be minimized. We consider a thin elastic plate. In n plate points M_j with the coordinates x_j, y_j the strings of stiffness c_j are attached. Let the low ends of the springs get the displacements z_j. These displacements cause the deflections (normal displacements) $w(M)$ of a plate, where M is an arbitrary point with the coordinates $\{x, y, M(x,y)\}$. We denote the deflections of the plate points M_j *as* $w_j = w(M_j)$. The set of forces F_j, with which the springs act on the plate are determined as

$$F_j = c_j(z_j - w_j) \qquad j = 1, \cdots, n \tag{1}$$

We suppose that there are no other forces acting on the plate. Then from the plate equilibrium equations we get

$$\sum_{j=1}^{n} F_j = 0 \qquad \sum_{j=1}^{n} x_j F_j = 0 \qquad \sum_{j=1}^{n} y_j F_j = 0 \tag{2}$$

We try to find the linear relations between the deflections w_j of the plate points M_j, the forces F_j and the displacements z_j such that

$$F_i = \sum_{j=1}^{n} F_{ij}^{w} w_j \qquad w_i = \sum_{j=1}^{n} W_{ij}^{z} z_j \qquad \text{or} \qquad z_i = \sum_{j=1}^{n} Z_{ij}^{w} w_j \qquad i = 1, \cdots, n \tag{3}$$

or in other words to find the matrices $\mathbf{F}^w$, $\mathbf{W}^z$, $\mathbf{Z}^w$

$$\mathbf{F}^w = \{F_{ij}^w\} \qquad i, j = 1, \cdots, n \qquad \mathbf{Z}^w = (\mathbf{W}^z)^{-1}$$

Firstly we find the plate deflections caused by the force $F_j = 1$, acting on the point M_j. Since there is only one acting force, the plate can not be in equilibrium. For that reason we apply at the plate point O $(x = y = 0)$ the compensating force $F = -1$ and the compensating moment M^0 (equal by absolute value to the distance $OM_j = \left(x_j^2 + y_j^2\right)^{1/2}$.

We denote as $w(M, M_j)$ the deflection of an arbitrary plate point M. This deflection is defined with accuracy to the term $b_1 + b_2 x_j + b_3 y_j$ (b_i are arbitrary constants), describing the displacements of a plate as a rigid body. We fix b_i and assume that the functions $w(M, M_j)$ are known. For the circular plate without shear, these functions are given in explicit form in Section 4.

We introduce matrices **X** of $n \times 3$ size and matrices **C** and **G** of $n \times n$ size

$$\mathbf{X} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \cdots & \cdots & \cdots \\ 1 & x_n & y_n \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c_n \end{pmatrix} \qquad \mathbf{G} = \{g_{ij}\}$$

where $g_{ij} = w(M_i, M_j)$ and the vectors

$$\mathbf{W} = (w_1, \cdots, w_n)^T \qquad \mathbf{Z} = (z_1, \cdots, z_n)^T \qquad \mathbf{F} = (F_1, \cdots, F_n)^T$$

where w_j are the deflections of the points M_j caused by the displacements z_j. Then

$$\mathbf{W} = \mathbf{G}\,\mathbf{F} + \mathbf{X}\,\mathbf{B} \qquad \mathbf{B} = (b_1, b_2, b_3)^T \tag{3a}$$

$$\mathbf{X}^T\,\mathbf{F} = \mathbf{0} \tag{3b}$$

We solve system (3a - 3b) with respect to vectors **F** and **W** and rewrite this system in the form

$$\mathbf{G}^*\mathbf{F}^* = \mathbf{W}^* \qquad \mathbf{G}^* = \begin{pmatrix} \mathbf{G} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{0} \end{pmatrix} \qquad \mathbf{F}^* = \begin{pmatrix} \mathbf{F} \\ \mathbf{B} \end{pmatrix} \qquad \mathbf{W}^* = \begin{pmatrix} \mathbf{W} \\ \mathbf{0} \end{pmatrix} \tag{4}$$

Splitting matrix $\left(\mathbf{G}^*\right)^{-1}$ into the same blocks as matrix $\mathbf{G}^*$ in equations (4), we find

$$\left(\mathbf{G}^*\right)^{-1} = \begin{pmatrix} \mathbf{F}^w & \mathbf{B}^w \\ \mathbf{B}^{wT} & \mathbf{B}^q \end{pmatrix} \qquad \mathbf{F} = \mathbf{F}^w\mathbf{W} \qquad \mathbf{B} = \mathbf{B}^w\mathbf{W}$$

Now submitting these expressions into equation (1) we get Z, from which it follows that

$$\mathbf{Z}^w = \mathbf{E} + \mathbf{C}^{-1}\mathbf{F}^w$$

Remark

We assumed above that none of the points M_j coincides with the point O. In connection with this, the additional forces and moments applied to the point O are mutually annihilated according to equations (2). Now let the point M_k coincide with the point O $(M_k = 0)$. In this case k-column of the matrix **G** is to be simply changed into a column of zeros, and the action of the force, applied at the zero-point, would be taken automatically into account according to equations (2). the deflection of an arbitrary point $M(x, y)$ may be calculated by the formula

$$w(M) = \mathbf{g}^T(M)\mathbf{F} + \mathbf{Y}^T(M)\mathbf{B} \tag{5}$$

where

$$\mathbf{g}(M) = \{w(M,M_1), \cdots, w(M,M_n)\}^T \qquad \mathbf{Y}(M) = \{1,x,y\}^T \tag{6}$$

and **F** and **B** are the vectors found earlier.

2 The Determination of the Mean Square Deviation and its Minimization

Now let us have the function $f(M) = f(x,y)$ and n points M_j. We are required to set the displacements w_j at these points such that the mean square deviation of the surface $w(x,y)$ from $f(x,y)$ is minimal

$$\sigma = S^{-1}\left[\iint_S \left(w(M) - f(M)\right)^2 dS\right]^{1/2} \tag{7}$$

where S is the area of the plate surface. Evaluating equation (7) after submitting $w(M)$ from equation (5) we get

$$\sigma^2 = \mathbf{F}^T\mathbf{K}_1\mathbf{F} + 2\mathbf{F}^T\mathbf{K}_2\mathbf{B} + \mathbf{B}^T\mathbf{K}_3\mathbf{B} - 2\mathbf{F}^T\mathbf{K}_4 - 2\mathbf{B}^T\mathbf{K}_5 + \mathbf{K}_6 \tag{8}$$

where

$$\begin{array}{llll} \mathbf{K}_1 = S^{-1}\iint_S \mathbf{g}\mathbf{g}^T dS & (n\times n) & \mathbf{K}_2 = S^{-1}\iint_S \mathbf{g}\mathbf{Y}^T dS & (n\times 3) \\ \mathbf{K}_3 = S^{-1}\iint_S \mathbf{Y}\mathbf{Y}^T dS & (3\times 3) & \mathbf{K}_4 = S^{-1}\iint_S \mathbf{g} f dS & (n\times 1) \\ \mathbf{K}_5 = S^{-1}\iint_S \mathbf{Y} f dS & (3\times 1) & \mathbf{K}_6 = S^{-1}\iint_S f^2 dS & (1\times 1) \end{array} \tag{9}$$

The sizes of the corresponding matrices are shown in parentheses. The evaluations of integrals (9) for the circular plate are given in Section 4 of the paper. The right side of equation (8) can also be written in the form

$$\sigma^2 = \mathbf{F}^{*T}\mathbf{K}\mathbf{F}^* - 2\mathbf{F}^{*T}\mathbf{K}^0 + \mathbf{K}_6 \tag{10}$$

where the vector $\mathbf{F}^*$ is the same as in formula (4) and

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^T & \mathbf{K}_3 \end{pmatrix} \qquad \mathbf{K}^0 = \begin{pmatrix} \mathbf{K}_4 \\ \mathbf{K}_5 \end{pmatrix}$$

Using equations (8) or (10) one can solve some problems of the plate deflection regulation.

Problem 1

Let the displacements w_j of the points M_j be given, i. e. the vector **W** is determined. For example, one can require the deviation of the deflection $w(M)$ from the given function $f(M)$ be equal to zero at the points M_j, i. e.

$$w_j = f(M_j) \tag{11}$$

To solve this problem we firstly get the vector $\mathbf{F}^*$ from equation (4), and then the value $\sigma = \sigma_1$ is obtained from equation (10).

Problem 2

Let the forces F_j satisfying equations (2) of equilibrium be given. Now it is necessary to find the value of σ in supposition that the displacements of a plate as a rigid body (vector **B**) are selected according to minimum conditions for σ. From minimum conditions on the left side of equation (8) we get

$$\mathbf{B} = \mathbf{K}_3^{-1}\left(\mathbf{K}_5 - \mathbf{K}_2^T\mathbf{F}\right)$$

and then we find the value $\sigma = \sigma_2$ from equation (10).

Problem 3

We are required to determine the deflections ω_j or F_j to minimize σ. We search for the minimum of σ in (10) with respect to $\mathbf{F}^*$ under conditions (2) and obtain the equation

$$\mathbf{K}^{**}\mathbf{F}^{**} = \mathbf{K}^{0*} \tag{12}$$

where

$$\mathbf{K}^{**} = \begin{pmatrix} \mathbf{K}_1 & \mathbf{K}_2 & \mathbf{X} \\ \mathbf{K}_2^T & \mathbf{K}_3 & \mathbf{0} \\ \mathbf{X}^T & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad \mathbf{F}^{**} = \begin{pmatrix} \mathbf{F} \\ \mathbf{B} \\ \Lambda \end{pmatrix} \qquad \mathbf{K}^{0*} = \begin{pmatrix} \mathbf{K}_4 \\ \mathbf{K}_5 \\ \mathbf{0} \end{pmatrix}$$

Here Λ is a Lagrange multiplier, appearing under consideration of the expression $\sigma^2 - \Lambda^T\mathbf{X}^T\mathbf{F}$, where $\Lambda = \left(\Lambda_1, \Lambda_2, \Lambda_3\right)^T$. Solving equation (12) we determine the value $\sigma = \sigma_3$ from equation (10) and the corresponding deflections w_j from equation (3).

3 The Control of the Deflection of a Loaded Plate

Let a plate be under the external load of intensity $q(x, y)$ and under a temperature gradient along the plate thickness. Let $w_q^0(x, y)$ be a deflection of a free plate under self balanced load $q_0(x, y)$. This load can be obtained from q by subtracing the force F_{q1}, the moment with projections F_{q2}, F_{q3} on the axes x, y applying to the point O and the temperature gradient. Here

$$\mathbf{F}_q = \left\{F_{q1}, F_{q2}, F_{q3}\right\}^T = \iint_S \mathbf{Y} q dS$$

where vector **Y** is the same as in equation (6). In this case the solution of Problems 1, 2 and 3 is obtained by the same scheme as above with the following changes:

(i) Equation (2) is changed into

$$\mathbf{X}^T\mathbf{F} + \mathbf{F}_q = 0$$

In particular, the forces determined in Problem 2 have to satisfy this equation.

(ii) The right side of equation (4) transforms into

$$\mathbf{W}^* = \begin{pmatrix} \mathbf{W} - \mathbf{W}_q^0 \\ -\mathbf{F} \end{pmatrix} \qquad \mathbf{W}_q^0 = \left\{ w_q^0(M_1), \cdots, w_q^0(M_n) \right\}^T$$

(iii) The vectors $\mathbf{K}_4$, $\mathbf{K}_5$ and a value of $\mathbf{K}_6$ are replaced by $\mathbf{K}_4^q$, $\mathbf{K}_5^q$, $\mathbf{K}_6^q$, which are obtained from $\mathbf{K}_4$, $\mathbf{K}_5$, $\mathbf{K}_6$ replacing f in equation (9) by $f^q = f - w_q^0$.

(iv) Relations (11) in problem 1 are changed into

$$w_j = f(M_j) - w_q^0(M_j)$$

(v) The right part $\mathbf{K}^{0*}$ of equation (12) is changed into

$$\mathbf{K}_q^{0*} = \left\{ \mathbf{K}_4^{qT}, \mathbf{K}_5^{qT}, -\mathbf{F}_q^T \right\}^T$$

4 Circular Plate

Now we try to construct the functions $w\,(M, M_j)$ and to evaluate integrals (9), containing these functions. We consider a circular plate of radius R, cylindrical stiffness D and Poisson's ratio ν. The equation of the plate bending has the form (Donnell, 1976)

$$D\Delta\Delta\hat{w} = q \tag{13}$$

We nondimensionalize equation (13) in such a way to make plate radius equal to unity. We consider the deflection $\hat{w}\,(r, \phi)$ at the point $M(r, \phi)$, whose position is described by the polar coordinate $r, \phi\ (0 \le r \le 1,\ 0 \le \phi \le 2\pi)$, under a self-balanced system of forces, connected with the point $M_j(r_j, \phi_j)$. This system consists of the force F_j applied to the point M_j, the force - F_j and the moment M_0, applied to the plate center O. We represent the deflection $\hat{w}\,(r, \phi)$ in the form

$$\hat{w}_j(r, \phi) = F_j R^2 D^{-1} w_j$$

where the dimensionless deflection w_j is equal to

$$w_j = w(M, M_j) = w(r, \phi, r_j, \phi_j) = \sum_{k=0}^{\infty} u_k(r, r_j) \cos k(\phi - \phi_j) \tag{14}$$

Functions u_k are given by different expressions for $r < r_j$ and for $r > r_j$. Denoting these functions by u_k^- and u_k^+ respectively we get

$$u_0^- = (8\pi)^{-1} r^2 \left[v_1 r_j^2 + 1 - \ln\left(r / r_j\right) \right]$$
$$u_0^+ = (8\pi)^{-1} r_j^2 \left[v_1 r^2 + 1 + \ln\left(r / r_j\right) \right]$$
$$u_1^- = (4\pi)^{-1} r r_j \ln\left(r / r_j\right) + \left(16\pi r_j\right)^{-1} r^3 \left(v_2 r_j^4 - 1\right)$$
$$u_1^+ = (16\pi r)^{-1} r_j^3 \left(v_2 r^4 - 1\right)$$
$$u_k^- = C_1^- r^k + C_{3k}^- r^{k+2} \qquad \text{for } k \geq 2$$
$$u_k^+ = C_1^+ r^k + C_{2k}^+ r^{-k} + C_{3k}^+ r^{k+2} + C_{4k}^+ r^{2-k}$$
$$C_{1k}^- = C_{1k}^+ + k_1 r_j^{2-k} \qquad C_{1k}^+ = r_j^k k_1 \left[\left(r_j^2 (1-k) + k v_2 + v_3 / k \right] \right.$$
$$C_{2k}^+ = -k_2 r_j^{k+2} \qquad C_{4k}^+ = k_1 r_j^k$$
$$C_{3k}^- = C_{3k}^+ - k_2 r_j^{-k} \qquad C_{3k}^+ = -k_2 v_2 \left(k + 1 - k r_j^2\right) r_j^k$$

where

$$v_1 = \frac{1-v}{2(1+v)} \qquad v_2 = \frac{1-v}{3+v} \qquad v_3 = \frac{8(1+v)}{(1-v)(3+v)}$$

and

$$k_1 = \left(8\pi k(k-1)\right)^{-1} \qquad k_2 = \left(8\pi k(k+1)\right)^{-1}$$

In integrals (9) the components of vector $\mathbf{g}\ (M)$ have the form of equation (14), and

$$\mathbf{Y} = \left(1,\ r\cos\phi,\ r\sin\phi\right)^T$$

We introduce the expansion of the function f into a Fourier series

$$f = \sum_{k=0}^{\infty} \left[f_k^c \cos k\phi + f_k^s \sin k\phi \right]$$

Then it is possible to make the integration over ϕ in integrals (9). We denote as $K_{1,ij}$, $K_{2,ij}$, $K_{3,ij}$, $K_{4,i}$, $K_{5,i}$ the elements of the corresponding matrices. For their evaluation we have

$$K_{1,ij} = \sum_{k=0} \delta_k l_{ij}^k \cos k\left(\phi_1 - \phi_j\right) \qquad \delta_0 = 2 \qquad \delta_k = 1 \quad \text{for } k > 0$$
$$l_{ij}^k = \int_0^1 u_k(r, r_i) u_k(r, r_j) r dr$$
$$K_{2,i1} = 2\int_0^1 u_0(r, r_i) r dr \qquad K_{2,i2} = l_i \cos\left(\phi_i\right) \qquad K_{2,i3} = l_i \sin\phi_i$$
$$l_i = \int_0^1 u_1(r, r_i) r^2 dr$$
$$K_3 = \mathrm{diag}(1,\ 1/4,\ 1/4) \qquad K_{4,i} = \sum_{k=0}^{\infty} \delta_k \left[l_i^{kc} \cos k\phi_i + l_i^{ks} \sin k\phi_i \right]$$
$$l_i^{kc} = \int_0^1 u_k(r, r_i) f_k^c(r) r dr \qquad l_i^{ks} = \int_0^1 u_k(r, r_i) f_k^s(r) r dr$$
$$K_{5,1} = 2\int_0^1 f_0^c(r) r dr \qquad K_{5,2} = \int_0^1 f_1^c(r) r^2 dr \qquad K_{5,3} = \int_0^1 f_1^s(r) r^2 dr$$

$$K_6 = 2\int_0^1 \left(f_0^c(r)\right)^2 rdr + \sum_{k=1}^{\infty}\int_0^1\left[\left(f_k^c(r)\right)^2 + \left(f_k^s(r)\right)^2\right] rdr$$

5 The Compensation of Deflections, caused by Temperature Deformations

The deflection w_q^0 of the circular plate, caused by temperature deformation is equal to

$$w_q^0 = -\xi r^2 + C \qquad \xi = \alpha \Delta T R^2 h^{-1} / 2$$

where α is the coefficient of temperature expansion, R-radius of a plate, h-plate thickness, ΔT-temperature difference between face surfaces.

Taking $w_q^0 = r^2$ in equations (10) and (3) we find dimensional values of σ for arbitrary vector **W** and for vector $\mathbf{W}^*$ respectively , which gives the minimum for σ. To obtain the dimensional value of σ its dimensionless value has to be multiplied by ξ.

To estimate the rate of decreasing of the deflection with moving supports, we note that without the restriction $\sigma = 0.288$.

We consider two cases. In the first, the plate is supported at $n = 2n_0 + 1$ points (one point in the center and n_0 points on each of the circles of radii r_1 and r_2). In the second case the plate is supported at $2n_0$ points (without support in the center). One can see the results of calculation in Tables 1 and 2.

n_0	n	r_1	r_2	σ_1	σ_3
3	7	0.5	0.8	0.152	0.071
4	9	-	-	0.103	0.060
5	11	-	-	0.070	0.038
6	13	-	-	0.054	0.025
3	7	0.45	-	0.160	0.074
4	9	-	-	0.100	0.059
5	11	-	-	0.067	0.036
6	13	-	-	0.053	0.037

Table 1. Meansquare deviation of the plate points with $2n_0 + 1$ supports

n_0	n	r_1	r_2	σ_1	σ_3
3	6	0.45	0.8	0.200	0.157
4	8	-	-	0.105	0.062
5	10	-	-	0.067	0.039
6	12	-	-	0.053	0.035

Table 2. Meansquare deviation of the plate points with $2n_0$ supports

Here σ_1 corresponds to the zero deflection at the points of support and σ_3 corresponds to the minimal deflection. Making a regular triangle mesh with the distances between the points equal to 0.45, we obtain for a plate supported in $n = 19$ points that $\sigma_1 = 0.030$ and $\sigma_3 = 0.020$.

Making a regular rectangular mesh with the distances r_0 between the points, we obtain for a plate supported at $n = 21$ points that

$$w(r) = \xi w_q^0(r) + C \qquad \xi = PR^2 / q \qquad P = \pi R^2 q$$

$$w_q^0(r) = \frac{(3+v)}{r} 32\pi(1+v) - \frac{r^2 \log(r)}{8\pi} + \frac{r^4}{64\pi}$$

where P is the plate weight and D is the cylindrical stiffness.

6 Acknowledgment

The research described in this publication was made possible in part by Grant N°NW5000 from the International Science Foundation

Literature

1. Donnell, L. H.: Beams, Plates and Shells, McGraw-Hill, New York, (1976), 774p.

2. Tovstik, P. E.; Ulitin, M.I.: Control of the Bending of Elastic Plates, Leningrad University Mechanics Bulletin No. 2, (1991), 40-45.

Addresses: Svetlana M. Bauer, Professor Andrei L. Smirnov, Professor Peter E. Torstik, Mikhail I. Ulitin, Faculty of Mathematics and Mechanics, St. Petersburg State University, 2 Bibliotechnaya, Stary Peterhof, RUS-198904 St. Petersburg