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On the Stability of the Eye Shell under an Encircling Band

S. M. Bauer, P. E. Tovstik, A. B. Katchanov

The problem of local stability of the eye shell under an encircling band (circlage) is considered. An algorithm for construction of an asymptotic solution is proposed.

1 Introduction

The medical treatment of retinal detachment is one of the most important problems in ophthalmology. One of the methods for reattachment is the encircling band (circlage). The most commonly used variant of circlage is that located exactly along the equator (see Figure 1).

The material for circlage is one of either silk thread, stainless steel band or silicone (elastic) band (Friberg, 1990). Silk thread or a stainless steel band may be considered to be nonextensible. Besides the advantages of this ophthalmic surgical procedure there are a series of complications which must be considered. With large pressure, cutting of the eye shell may occur. Sometimes even with lower pressure, shrinking of the retina has been observed. The pressure disturbs the circulation of blood and leads to oedemata. Apparently, this is caused by local loss of stability in the neighbourhood of the circlage line. In this paper the problem of local stability of the eye shell under circlage is considered. We assume that in the neighbourhood of the equator the eye shell is a thin elastic spherical shell of radius R and constant thickness h. Circlage is considered to be a superposition of an edge effect along the equator and the membranous state

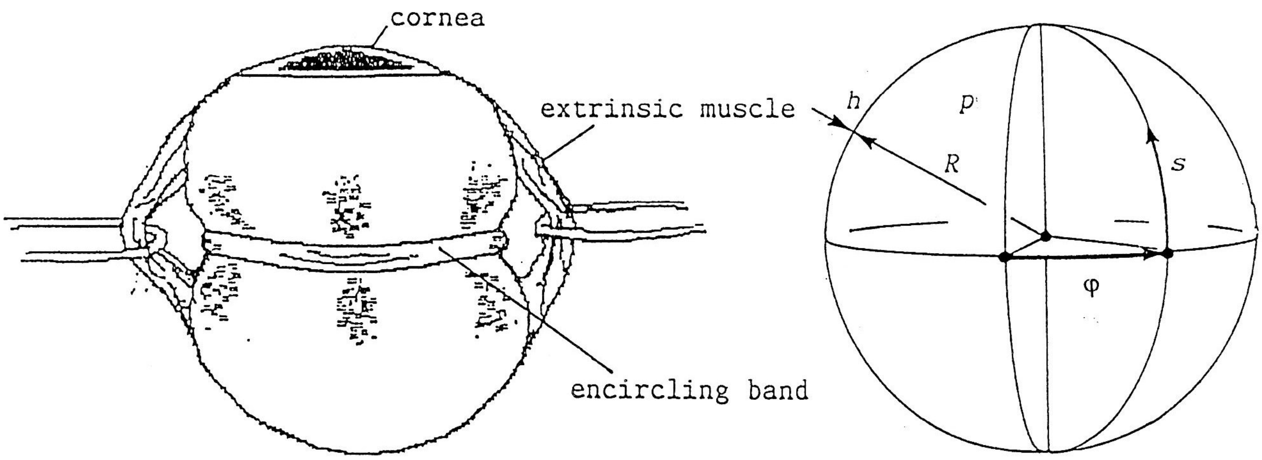

Figure 1. The eye shell under circlage

2 Axisymmetric Deformation under Circlage

Let the band pressure per unit length be q_0. Later the width of the band will be neglected. We introduce a coordinate system s,φ on the sphere, where s is the length of the meridian arc and φ is the angle in the circumferential direction. We assume that the band is located at the equator, i. e. at $s=0$ (see Figure 1). We will find the axisymmetric normal displacement of the eye shell $w_0(s)$ under pressure q_0. Subsequently we will study the buckling of the axisymmetric state. We limit ourselves to the determination of the approximate deflection in the neighbourhood of the equator. We use the equation for a simple edge effect

$$D\frac{d^4 w_0}{ds^4} + \frac{Eh}{R^2} w_0 = 0 \qquad \text{with} \qquad D = \frac{Eh^3}{12\left(1-\nu^2\right)} \tag{1}$$

where D is the bending stiffness, E is Young's modulus and ν is Poisson's ratio.

Displacement $w_0(s)$ is an even function in s, therefore it suffices to determine it only for $s > 0$. We seek the solution of equation (1), satisfying the boundary conditions

$$\frac{dw_0}{ds} = 0 \qquad \text{and} \qquad Q_0 = \frac{q_0}{2} = -D\frac{d^3 w_0}{ds^3} \qquad \text{for } s = 0$$

$$w_0 \to 0 \qquad \text{as} \qquad s \to \infty \tag{2}$$

The unknown solution has the form

$$w_0(s) = w^0 e^a (\cos a - \sin a) \qquad \text{with} \qquad a = \frac{s}{\sqrt{2}\mu R} \tag{3}$$

Here w_0 is the maximum shell displacement (under the band), $\mu > 0$ is a small parameter, and

$$w^0 = \frac{q_0 R}{2\sqrt{2}Eh} \qquad \text{and} \qquad \mu^4 = \frac{h^2}{12\left(1-\nu^2\right)R^2} \tag{4}$$

Solution (3) does not take into account the intraocular pressure p. If we take into account the intraocular pressure then the stretching meridianal stress $T_1 = pR/2$ is not equal to zero and we should replace equation (1) by

$$D\frac{d^4 w_0}{ds^4} - T_1\frac{d^2 w_0}{ds^2} + \frac{Eh}{R^2} w_0 = 0 \tag{5}$$

Boundary conditions (2) remain the same. The solution of equation (5) has the form

$$w_0(s) = w^0 e^{a_1}\left(\cos a_2 - \frac{a_1}{a_2}\sin a_2\right) \qquad \text{with} \qquad w^0 = \frac{q_0 R\sqrt{2+\delta}}{4Eh} \tag{6}$$

where

$$a_1 = \frac{s\sqrt{2+\delta}}{2\mu R} \qquad a_2 = \frac{s\sqrt{2-\delta}}{2\mu R} \qquad \delta = \frac{pR}{2Eh\mu^2} \tag{7}$$

In solution (6) the dimensionless parameter δ takes into account the influence of the intraocular pressure.

3 Equations of Axisymmetric State Bifurcation

We seek the adjacent nonaxisymmetric equilibrium mode with m waves in the circumferentional direction in the form

$$w(s,\varphi) = w(\theta)\cos(m\varphi) \tag{8}$$

In order to construct the adjacent equilibrium mode we use the Donnell system of equations for shallow shells (Donnell, 1976). After separation of variables (8) and introduction of dimensionless variables this system may be written as

$$\mu^2 \Delta\Delta w + \Delta_t w - \Delta_k \Phi = 0$$
$$\mu^2 \Delta\Delta \Phi + \Delta_k w = 0 \qquad (9)$$

where the differential operators Δ, Δ_k and Δ_t are given by

$$\Delta w = \frac{1}{b}\frac{d}{d\theta}\left(b\frac{dw}{d\theta}\right) - \frac{m^2}{b^2}w$$
$$\Delta_k w = \frac{1}{b}\frac{d}{d\theta}\left(b\frac{dw}{d\theta}\right) - \frac{m^2\left(1+\kappa_1^0\right)}{b^2}w \qquad (10)$$
$$\Delta_t w = -\frac{m^2 t_2^0}{b^2}w$$

In equations (9) $w(\theta)$ represents an additional displacement and $\Phi(\theta)$ is the stress function. Dimensionless variables in equations (9) and (10) are related to the corresponding dimensional variables by

$$b = \sin\theta \qquad \theta = \frac{s}{R} \qquad 0 \le \theta \le \pi \qquad (11)$$

In equation (10) the functions

$$t_2^0(\theta) = \frac{w_0}{\mu^2 R} \quad \text{and} \quad \kappa_1^0(\theta) = \frac{1}{R}\frac{d^2 w_0}{d\theta^2} \qquad (12)$$

describe the main prebuckling stresses and deformations. One can show (Donnell, 1976) that the effect of the other prebuckling stresses and deformations is small and is of relative order μ^2.

We now introduce the loading parameter λ and the wave number parameter ρ and the band pressure q_0.

$$w^0 = \mu^2\lambda R = \frac{\lambda h}{\sqrt{12\left(1-\nu^2\right)}} \qquad q_0 = \frac{\lambda E h^2}{R}\sqrt{\frac{2}{3\left(1-\nu^2\right)}} \qquad \rho = \mu m \qquad (13)$$

The problem would be solved if we find the smallest (by the parameter ρ) value of the loading parameter λ, such that there exists a nontrivial solution of system (9), decreasing as it moves away from the parallel $s = 0$.

4 Asymptotic Integration of System (9)

System (9) contains the small parameter μ at the derivatives, therefore we can use the asymptotic method. We rescale the independent variable

$$\theta = \frac{\pi}{2} + \mu\xi \qquad (14)$$

and use the expansion

$$\sin\theta = 1 - \frac{\mu^2\xi^2}{2} + \cdots \qquad (15)$$

Then the solution of this system can be represented formally by asymptotic series

$$w(\theta,\mu) = \sum_{n=0}^{\infty}\mu^n w_n(\xi) \qquad \text{and} \qquad \Phi(s,\mu) = \sum_{n=0}^{\infty}\mu^n \Phi_n(\xi) \tag{16}$$

We construct only the zeroth-order aproximations $w_0(\xi)$ and $\Phi_0(\xi)$ which satisfy the system of equations

$$\begin{aligned} \Delta_0\Delta_0 w_0 + \Delta_{t0}w_0 - \Delta_{k0}\Phi_0 &= 0 \\ \Delta_0\Delta_0\Phi_0 + \Delta_{k0}w_0 &= 0 \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Delta_0 w_0 &= \frac{d^2 w_0}{d\xi^2} - \rho^2 w_0 \\ \Delta_{k0} w_0 &= \frac{d^2 w_0}{d\xi^2} - \rho^2\left(1+\lambda\kappa_{10}\right)w_0 \\ \Delta_{t_0} w_0 &= \lambda t_{20}^k w_0 \end{aligned} \tag{18}$$

and

$$\begin{aligned} t_2^{00}(\xi) &= e^{a_0}\left(\cos a_0 - \sin a_0\right) \\ \kappa_1^{00}(\xi) &= -e^{a_0}\left(\cos a_0 + \sin a_0\right) \end{aligned} \qquad a_0 = \frac{\xi}{\sqrt{2}} \tag{19}$$

5 Asymptotic Solution of System (17)

System (17) has variable coefficients $t_2^{00}(\xi)$ and $\kappa_1^{00}(\xi)$. We rewrite them in the form

$$t_2^{00} = \frac{1+i}{2}e^{\alpha_1\xi} + \frac{1-i}{2}e^{\alpha_2\xi} \qquad \text{and} \qquad \kappa_2^{00} = -\frac{1-i}{2}e^{\alpha_1\xi} - \frac{1+i}{2}e^{\alpha_2\xi} \tag{20}$$

where

$$\alpha_1 = \frac{1+i}{\sqrt{2}} \qquad \text{and} \qquad \alpha_2 = \frac{1-i}{\sqrt{2}} \tag{21}$$

The linearly independent solutions of system (17) satisfying the conditions of damping

$$w(\xi),\ \Phi(\xi) \to 0 \qquad \text{as} \qquad \xi \to -\infty$$

we seek in the form

$$w^{(j)}(\xi) = \sum_{n=0}^{\infty} \sum_{n_1+n_2=n} w_{n_1,n_2}^{(j)} e^{p_{n_1,n_2}^{(j)} \xi} \qquad \text{with} \qquad p_{n_1,n_2}^{(j)} = p^{(j)} + n_1\alpha_1 + n_2\alpha_2$$

$$\Phi^{(j)}(\xi) = \sum_{n=0}^{\infty} \sum_{n_1+n_2=n} \Phi_{n_1,n_2}^{(j)} e^{p_{n_1,n_2}^{(j)} \xi} \tag{22}$$

where $p^{(j)}$ are the roots of the characteristic equation of system (17) for $\kappa_1^{00} = t_2^{00} = 0$, i. e.

$$\left(p^2 - \rho^2\right)^4 + \left(p^2 - \rho^2\right)^2 = 0 \tag{23}$$

and

$$\Re\left(p^{(j)}\right) > 0 \tag{24}$$

Series (22) are convergent. The algorithm for the evaluation of the coefficients $w_{n_1,n_2}^{(j)}$ and $\Phi_{n_1,n_2}^{(j)}$ is described in Tovstik (1995).

To satisfy the boundary conditions at $\xi = 0$ we need four linearly independent solutions. Unfortunately, the root $p = \rho$ of equation (23) is multiple and the corresponding solution has a form which differs from equation (22). In order not to change the algorithm, we may modify system (17), assuming that

$$\Delta_{k0} w_0 = \frac{d^2 w_0}{d\xi^2} - \rho^2\left(k_1 + \lambda\kappa_{10}\right) w_0 \qquad \text{with} \qquad k_1 = 1 + \varepsilon \qquad \varepsilon << 1 \tag{25}$$

This system describes an ellipsoid close to a sphere. Now

$$\left(p^2 - \rho^2\right)^4 + \left(p^2 - k_1\rho^2\right)^2 = 0 \tag{26}$$

becomes simple.

6 Boundary Conditions

In the general case eight conditions for the generalized displacements and forces should be fulfilled along the line $\xi = 0$. But we consider the buckling mode with deflection $w(\xi)$, where $w(\xi)$ is an even function in ξ. There may be two formulations of the problem. In the first we assume that the pressure q_0 does not change during buckling and in the second case we assume that the pressure q_0 changes due to the shell deformation during buckling.

In the first case for an even buckling mode we have the following boundary conditions

$$u = S = \Delta Q_1 = \gamma_1 = 0 \qquad \text{for} \qquad \xi = 0 \tag{27}$$

where u is the projection of the displacement in the meridional direction, S is the shear stress and ΔQ_1 is the additional shear stress, γ_1 is the angle of rotation of the tangent to the meridian. Expressing the variables in equation (27) through the fundamental variables w and Φ, we can rewrite it in the form

$$\frac{dw}{d\xi} = \frac{d^3 w}{d\xi^3} = \frac{d\Phi}{d\xi} = \frac{d^3\Phi}{d\xi^3} = 0 \qquad \text{for} \qquad \xi = 0 \tag{28}$$

from which it follows that the buckling mode is even. Now let the change of the pressure be taken into account. The pressure $q(\varphi)$ is found from

$$q(\varphi) = T_0\left(\frac{1}{R}+\kappa_2\right) \qquad \text{with} \qquad \kappa_2 = \frac{1}{R^2}\frac{\partial^2 w}{\partial\varphi^2} \tag{29}$$

where $T_0 = q_0 R$ is the band tension, which is assumed to be constant under buckling, κ_2 is the change of curvature for the line $s = 0$ during buckling. The shear stress Q_1 is discontinuous at $s = 0$, and yields

$$Q_1 = \frac{q}{2} \qquad \text{and} \qquad \Delta Q_1 = \frac{q_0 R}{2}\kappa_2 \tag{30}$$

Now the boundary conditions have the form

$$\frac{dw}{d\xi} = \frac{d^3 w}{d\xi^3} - \sqrt{2}\lambda\rho^2 w = \frac{d\Phi}{d\xi} = \frac{d^3\Phi}{d\xi^3} = 0 \qquad \text{for} \qquad \xi = 0 \tag{31}$$

7 Results

In Table 1 the critical (minimum) values of the loading parameter λ and the corresponding value of the wave number parameter ρ for the following loading are given for different values of intraocular pressure parameter δ. These values correspond to the symmetric buckling mode. The antisymmetric mode gives higher critical loading.

δ	0	0.1	0.2	0.3
λ	2.157	2.276	2.414	2.584
ρ	0.69	0.67	0.64	0.60

Table 1. The values of the parameters λ and ρ vs the values of the parameter δ

$\delta = 0.2$			$\delta = 0.3$		
μ	m	λ	μ	m	λ
0.12	6	2.4193	0.12	5	2.5840
0.13	5	2.4146	0.13	5	2.5916
0.14	5	2.4234	0.14	4	2.5910
0.15	4	2.4193	0.15	4	2.5840
0.16	4	2.4144	0.16	4	2.5888

Table 2. The critical loading λ for different values of the parameters μ and δ

According to many authors (Friberg, 1990; Kobayashi et al., 1973) for the eye shell Poisson's ratio $\nu = 0.45$, $R = 11\text{-}12$ mm and Young's modulus $E = 10\text{-}14.3$ MPa. Hence the dimensionless intraocular pressure parameter $\delta = 0.3$ corresponds to 34-36 mm Hg. (If the intraocular pressure is larger than 34-36 mm Hg the blood inflow into the eye shell is interrupted.)

Taking into account that the buckling wave number m should be an integer, we can list more precise critical values for some values of the parameter μ as has been done in Table 2.

In order to create critical (buckling) conditions an encircling band of a length *shorter* than $2\pi R$ must be installed. If the circlage is made of silk thread or a steel band, both of which may be considered nonextensible, then due to equations (9), we get the critical length reduction of thread or steel band, (i.e. the amount by which it must be shorter than the length of the equator, i. e. $2\pi R$ without band). For these materials the critical length reduction is 3.4-4 mm.

Next we consider the circlage made of a silicone (elastic) band. Let S be the cross-sectional area of the silicone band, and E_s Young's modulus of silicone. Then for a band reduction of Δl, the relative shortening of the band

$$\varepsilon = \frac{q_0 R}{E_s S} = \frac{\Delta l - 2\pi w^0}{2\pi R - \Delta l}$$

where the critical value q_0 is to be found from equations (9). Hence

$$\Delta l = \frac{2\pi\left(R\varepsilon + w^0\right)}{1+\varepsilon} \qquad \text{and} \qquad \varepsilon = \frac{Eh^2}{E_s S}\sqrt{\frac{2}{3\left(1-\nu^2\right)}}$$

The plot of the critical length reduction of a silicone band vs the cross section area of the band for various intraocular pressure parameters δ is shown in Figure 2. Young's modulus of silicone is $E_s = 1.93$ MPa.

Thus these results show that the usage of the silicone bands for circlage is preferable.

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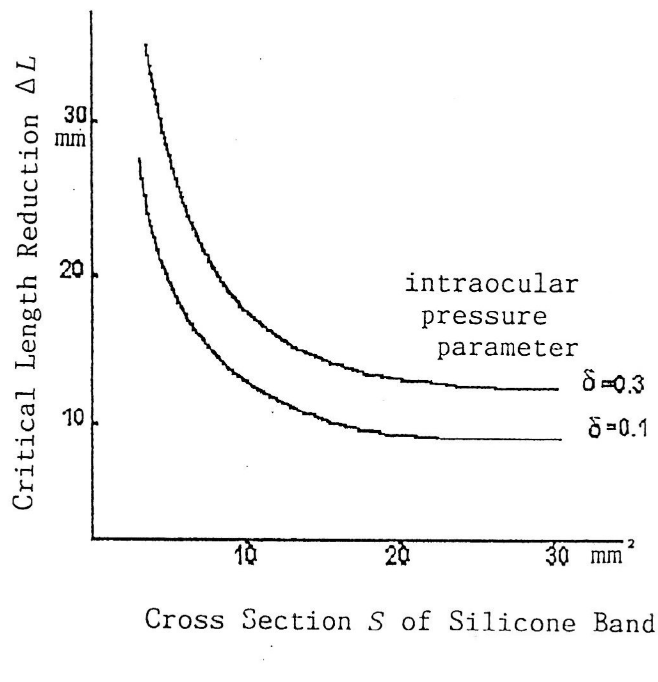

Figure 2. The critical length reduction of a silicone band vs its cross sectional area

Literature

1. Donnell, L. H.: Beams, Plates and Shells, McGraw-Hill, New York, (1976), 774p.

2. Friberg, T. R.: Scleral Buckling and Ocular Rigity, Arch. Ophthalmol., 108, (1990), 1622-1627.

3. Gol'denveizer, A. L.: Theory of Elastic Thin Shells, Nauka, Moscow, (1976), 658p., (in Russian).

4. Kobayachi, A. S.; Staberg, L. G.; Schlegel, W. A.: Viscoelastic Properties of Human Cornea, Experimental Mechanics, 12, (1973), 497-503

5. Tovstik, P. E.: The bifurcation of the axisymmetric state of shells with due regard for the moment stresses, Vestnik St. Petersburgskogo Universiteta, (1995), (to appear), (in Russian).

Addresses: Professor Svetlana M. Bauer and Professor Peter E. Tovstik, Faculty of Mathematics and Mechanics, St. Petersburg State University, 2 Bibliotechnaya Square, Stary Peterhof, RUS-198904 St. Petersburg; Andrew B. Katchanov, St. Petersburg City Eye Hospital