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Approximated Shell Model based on Additive Decomposition of Deformation Gradient

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An analysis of approximated shell models is presented, based on an additive decomposition of the deformation gradient into mean stretching $(\mathbf{S})$ *and mean rotation* $(\mathbf{Q})$ *tensors. This analysis endeavours to follow Biot's original reasoning without sacrificing clarity and consistency of previous shell models. The simple but very instructive small* **S**-*stretching and moderate* **Q**-*rotation Kirchhoff-Love-type model of elastic shell is discussed in detail. Numerical examples based on the cubic-B-spline finite segment method are calculated and compared with other results.*

1 Introduction

In some recent research with regard to the Cauchy (1823) - Novozhilov (1948) mean rotations, the old question concerning the decomposition of the deformation gradient has been raised again. It is interesting, therefore, to investigate whether the introduction of the mean rotation leads to a useful simplification of the mathematical model as well as its computer implementations. The last works in this direction (Martins and Podio-Guidugli, 1992; Martins et al., 1987; Marzano, 1987; Zheng and Hwang, 1992) have provided a few attractive results and turned our attention from the polar rotation to the mean rotation.

According to Truesdell and Toupin (1960) there is no evidence of a suitable criterion to determine which rotation measure is better. However, a few circumstances advocate using the mean rotation and among those is a physically easy interpretation. It is mainly because the Cauchy-Novozhilov rotation is based on a skew-symmetric tensor $\widetilde{\mathbf{R}} = \widetilde{R}_{(K;M)}\mathbf{G}^K \otimes \mathbf{G}^M = \overline{U}_{[K;M]}\mathbf{G}^K \otimes \mathbf{G}^M$, being a simple function of the displacement gradient. This measure is calculated entirely within the material description and has been called the *mean* rotation tensor by Truesdell and Toupin (1960), owing to the direct relation with the Cauchy-Novozhilov rotation. Moreover, the important part played by the former has been recognized in geometrically non-linear, flexible, thin bodies by taking rotational terms into consideration. If one looks more carefully at recent consistent models of thin shells with small, moderate and large rotations (Pietraszkiewicz, 1981; Stumpf, 1986) then it is easy to find that such models are based on the mean rotation tensor $\widetilde{\mathbf{R}}$ rather than on the *polar* rotation measure $\mathbf{R} = R^i_M \mathbf{g}_i \otimes \mathbf{G}^M$, $\mathbf{F} = \mathbf{RU}$, in which $\mathbf{U}$ is the right Green-Cauchy stretch tensor. This approach to approximated shell models also has a long tradition coming from the works of Synge and Chien (1941), Chien (1943), Koiter (1966), Zern (1962) and others. It is natural to think about an extension of the mean rotation tensor $\widetilde{\mathbf{R}}$ into a proper rotation measure, say $\mathbf{Q}$, such that the fundamental properties of the rotation group SO (3) are fulfilled, i.e. $\mathbf{Q}^T = \mathbf{Q}^{-1}$, $\det \mathbf{Q} = 1$. As we expect, it would yield the essential features which are especially exhibited in the bending of thin shells and other thin-walled structures. Moreover, it would be an extension derived from the original Biot question as to how to distinguish which part of $\mathbf{F}$ is to be considered as a pure strain and which as a pure rotation.

A first solution is merely elementary and leads to an additive, say $\mathbf{S}$-$\mathbf{Q}$, decomposition of $\mathbf{F}$. It suggests itself as being the simplest of those that go beyond an $\widetilde{\mathbf{R}}$ approach. Note first that the skew-symmetric tensor $\widetilde{\mathbf{R}}$ possesses the form $\widetilde{\mathbf{R}} = \theta(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1)$ and has three eigenvalues $\widetilde{R}_{(1)} = i\theta$, $\widetilde{R}_{(2)} = i\theta$, $\widetilde{R}_{(3)} = 0$, where $\sin^2\theta = -tr\left(\widetilde{\mathbf{R}}^2\right)$ and the three vectors $\theta = \theta\mathbf{e}_3$, $\mathbf{e}_2$, $\mathbf{e}_1$ create an orthogonal base of eigenvectors. Introducing a reduced mean rotation tensor $\mathbf{L} = \left(\theta \sin^{-1}\theta\right)\widetilde{\mathbf{R}}$ and using a known formula of tensor algebra, one may define

$$\mathbf{Q} = \exp(\mathbf{L}) = \exp\left(\theta \sin^{-1}\theta \widetilde{\mathbf{R}}\right) = \mathbf{I} + \theta^{-1}\sin\theta\mathbf{L} + \theta^{-2}(1-\cos\theta)\mathbf{L}^2 \tag{1}$$

as a rotation tensor $\mathbf{Q} \in SO(3)$. Noting now that $\tilde{\mathbf{R}}^2$ as well as $\mathbf{L}^2$ are symmetric tensors, it is simple to involve the following transformations of the deformation gradient

$$\begin{aligned} \mathbf{F} &= \mathbf{I} + Grad\bar{\mathbf{U}} = \left(G_{MK} + \bar{U}_{(M;K)} + \bar{U}_{[M;K]}\right)\mathbf{G}^M \otimes \mathbf{G}^K = \left(G_{MK} + \tilde{E}_{MK} + \tilde{R}_{MK}\right)\mathbf{G}^M \otimes \mathbf{G}^K \\ &= \left[\tilde{\mathbf{E}} - \theta^{-2}(1-\cos\theta)\mathbf{L}^2\right] + \left[\mathbf{I} + \theta^{-1}\sin\theta\mathbf{L} + \theta^{-2}(1-\cos\theta)\mathbf{L}^2\right] \equiv \mathbf{S} + \mathbf{Q} \end{aligned} \tag{2}$$

Here the Love elongation tensor $\tilde{\mathbf{E}} = \tilde{E}_{MK}\mathbf{G}^M \otimes \mathbf{G}^K$ is replaced by a symmetric tensor **S** describing the mean stretching. The rotation tensor **Q** possesses three eigenvalues; $Q_{(1)} = \exp(i\theta)$, $Q_{(2)} = \exp(i\theta)$ and $Q_{(3)} = 1$, which enables us to express **Q** in terms of a so-called rotation vector $\theta = \theta\mathbf{e}_3$, $|\mathbf{e}_3| = 1$, as

$$\mathbf{Q} = \mathbf{I} + \theta^{-1}\sin\theta\, \theta \times \mathbf{I} + \theta^{-2}(1-\cos\theta)\, \theta \otimes \theta \tag{3}$$

Hereafter we shall assume for the sake of simplicity in further application, that the angle θ fulfils the condition $|\theta| < \pi$, retaining the validity of equation (3) parametrization. However, in general, due to the specific topological properties of the SO(3) group speaking about unlimited rotations, we need two types of parametrization. Using a more mathematical language one could say that since the first homotopy group of SO(3); $\Pi_1[SO(3)] = Z_2$, contains two elements then the atlas of SO(3) parametrization should be endowed with two essentially distinct maps.

The additive decompostion has recently been introduced by Chen (1979), with quite similar arguments. Other complementary problems such as the stored elastic energy and the requirements of material frame indifference were also discussed comprehensively. Examples of applications of **S-Q** decomposition were successfully considered as well by Chen (1987), Qin and Chen (1988), Sun (1989a), Sun (1989b), Li and Chen (1994), mainly from the point of view of engineering applications. However, from the standpoint of the foundations of the Cauchy simple continuum, the **S-Q** decomposition should be treated as a first step beyond the notions of the mean rotation tensor $\tilde{\mathbf{R}}$ and the Love elongation tensor $\tilde{\mathbf{E}}$. Therefore from the conceptual point of view there is no serious difference between the polar $(\mathbf{F} = \mathbf{RU})$ and the additive $(\mathbf{F} = \mathbf{S} + \mathbf{Q})$ rotation measures. It is not the proper place here to present a comprehensive discussion of the polar decomposition, (see Pietraszkiewicz and Badur, 1983; Nolte, 1983). Let us only remark that many implementational complex formulae related with **R** are naturally superfluous in the **S-Q** approach. Passing on to thin shell theory, judging from what we know of the early problems appearing in the course of mathematical modelling of shells (see the pioneering discussion by Kirchhoff (1985) and Love (1988)) we may expect that the **S-Q** decomposition will throw new light on details of shell modelling. For instance, at present we do not yet know whether Love's model of first approximation is based on the polar rotation measure, or whether it involves **Q**-type rotations. We have no grounds for assuming that non-linear shell models, especially those up to circa 1927, have only been constructed with the polar rotation. It is wonderful to see how, within the traditional shell models of three or six parameters, the kinematical equations acquire a clearer meaning with the appearance of the **Q**-tensor. It is instructive to compare, using simple examples, what kind of obstacles may come up during the modelling of approximate theories.

The problem is far from being merely academic. Above all, there is no clarity among the finite element models based on the polar rotation tensor (Gruttmann et al., 1989; Sansour and Bufler, 1992; Chroscielewski et al., 1992; Buechter and Ramm, 1992). Yet a more complex situation deals with another numerical model - the model with a deformable director. Additionally, there is no difference whether the director undergoes elongation (Hughes and Liu, 1981) or whether the length of the director remains constant (Simo and Fox, 1989). As we have mentioned above, the modern discussion about an additive decomposition of **F** has been started by Biot (1939). Some extensions of Biot's line of reasoning, especially those toward the so-called Biot stress measure, are set out in the literature, mainly in the context of an appropriate formulation of stability of thin shell-like bodies. It is commonly known that elastic flexible bodies can undergo very large rotations simultaneously with a small deformation within a shell domain. Therefore, a properly formulated problem of shell stability admits only the case of small rotations superimposed on finite rotations. Being closer to Biot's original line of thought we would now ask: how are finite rotations defined $(\mathbf{R}, \tilde{\mathbf{R}} \text{ or } \mathbf{Q})$ and how is the superimposed rotation defined?

Noteworthy papers concerned with these issues are those by Von Karman and Tsien (1939), and Sun and Badur (1993). In this paper we shall sketch briefly the main outline of the **S-Q** decomposition. A mathematical model of the thin shell will be an example of a physical application of the **S-Q** approach and one that will be of good use later. It will give us an opportunity to formulate the problem of the approximated rotation a little more clearly. To make the paper self-contained, it is important to repeat part of the results obtained previously. In section 2 we introduce the additive decomposition of the deformation gradient. In section 3 we perform the **S-Q** decomposition for shell kinematics undergoing the Timoshenko hypothesis. We get an appropriate formula for series expansions of both the mean stretching **S** as well as the mean rotation **Q** tensor. Further, in section 4 we analyze approximate models undergoing either moderate or large **Q**-rotation but having small **S**-stretching. In section 5 a model with Kirchhoff-Love type assumptions and moderate **Q**-rotation is analyzed numerically and compared with previous results.

2 Decomposition of Deformation Gradient F

According to the traditional shell theory (Danker, 1943; Green and Zerna, 1950) a mechanical deformation of a thin, shell-like, solid body is analyzed by using the Lagrangian description and a cenvective system of curvilinear coordinates, usually denoted by ξ^i $(i = 1, 2, 3)$. If we denote the position vectors of an arbitrary particle $B \in \Im$ of a body under consideration by $\mathbf{P}(\xi^i)$, $\mathbf{p}(\xi^i)$ then $\mathbf{G}_i \equiv \mathbf{P}(\xi^j)_{,i}$, $\mathbf{g} \equiv \mathbf{p}(\xi^j)_{,i}$, are the holonomic bases in the undeformed and deformed configuration (Libai and Simmonds, 1983) respectively. Additionally, when the convective system (ξ^i) is initially identified with any material coordinate system X^M (Truesdell and Toupin, 1960) then $\mathbf{G}_i \equiv \delta_i^M \mathbf{g}_M$, $x_M^i = \delta_M^i$, $\mathbf{g}_m \neq \delta_m^K \mathbf{G}_K$. Since within the convective parametrization all information about a deformation of a small Cauchy continuum is contained in $\mathbf{G}_i$ the deformation gradient becomes

$$\mathbf{F} = x_{,M}^{m}\, \mathbf{g}_m \otimes \mathbf{G}^M \equiv Grad\, \mathbf{p} = \delta_M^i \mathbf{g}_i \otimes \mathbf{G}^M = \mathbf{g}_i \otimes \mathbf{G}^i \tag{4}$$

together with $\mathbf{F}^T = \mathbf{G}^i \otimes \mathbf{g}_i$, $\mathbf{F}^{-1} = \mathbf{G}_i \otimes \mathbf{g}^i$, etc. any two-point tensor also has a one-point representation (Truesdell and Toupin, 1960). Therefore, taking Hamilton's nabla operator in the convected form $\nabla = \partial_i \mathbf{G}^i$ as well as introducing the displacement vector $\overline{\mathbf{U}}(\xi^i) = \mathbf{p} - \mathbf{P} = \overline{U}_i \mathbf{G}^i$ we get

$$\mathbf{F} = \mathbf{I} + \overline{\mathbf{U}} \otimes \nabla = \left(G_{ij} + \overline{U}_{i|j}\right)\mathbf{G}^i \otimes \mathbf{G}^j = F_j^i \mathbf{G}_i \otimes \mathbf{G}^j \tag{5}$$

where $\mathbf{I} = \mathbf{P} \otimes \nabla = \mathbf{G}_i \otimes \mathbf{G}^i = G_{ij} \mathbf{G}^i \otimes \mathbf{G}^j$ denotes the Gibbs idemfactor and the symbol $(|)$ denotes, as usual, the derivative based on the Γ_{ij}^k coefficients of the holonomic connection.

S-Q *Decomposition* Scheme (Chen, 1979) suggested another decomposition rule for **F** which, owing to its additivity, was simply denoted as **S-Q** decomposition. (Here we use **S-Q** instead of Chen's **S-R** to distinguish between mean and polar rotation tensor.)

S-Q *Proposition*: Suppose that in a three-dimensional physical space, a physically admissible displacement function $\overline{\mathbf{U}}(\xi^i)$ is given for a deformable continuous point set. Let $\overline{\mathbf{U}}(\xi^i)$ be a single valued continuous differentiable function; the transformation induced by the motion of the point set can then be decomposed into a symmetrical transformation and an orthogonal transformation. According to the proposition, the deformation gradient **F** would be decomposed as follows

$$\mathbf{F} = \mathbf{S} + \mathbf{Q} = \left(S_j^i + Q_j^i\right)\mathbf{G}_i \otimes \mathbf{G}_j \qquad \mathbf{S} = \mathbf{S}^T \qquad \mathbf{Q}\mathbf{Q}^T = \mathbf{I} \qquad \det \mathbf{Q} = 1 \tag{6}$$

and due to equations (1), (2), (6), we have the following explicit formulae for the components of **S** and **Q**

$$S_{ij} = \frac{1}{2}\left(\overline{U}_i|_j + \overline{U}_j|_i\right) - \theta^{-2}(1-\cos\theta)L_{ik}L_j^k \qquad Q_{ij} = G_{ij} + \theta^{-1}\sin\theta L_{ij} + \theta^{-2}(1-\cos\theta)L_{ik}L_j^k \tag{7}$$

Here $L=-L^T$ is the reduced tensor of the mean rotation and $L_{ij}=\theta\sin^{-1}\widetilde{R}_{ij}$, $\widetilde{R}_{ij}=\frac{1}{2}\left(\overline{U}_i\big|_j-\overline{U}_j\big|_i\right)$, and θ is the mean rotation angle $\theta = \pm\arcsin\left(-\widetilde{R}_{ij}R^{ji}\right)^{\frac{1}{2}}$.

The „mathematical" proof on **S-Q** decomposition was given by Chen (1979). We must now prove that the present explanation of **S-Q** leads accurately to the proper expressions of a symmetric **S** and a proper rotation tensor **Q**. This will furnish the proof which was originally presented by Sun and Badur (1993). To do this we cannot use the method of direct calculation as above in the introduction, but must apply the following rather elegant considerations. From $\mathbf{F}=\mathbf{S}+\mathbf{Q}$, we can get $\mathbf{F}^T=\mathbf{S}+\mathbf{Q}^T$. If we denote the skew-symmetric part of **F** by **A**,

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{F} - \mathbf{F}^T\right) \qquad \mathbf{A}^T = -\mathbf{A} \tag{8}$$

then we have

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{Q} - \mathbf{Q}^T\right) \tag{9}$$

which means that **A** is the antisymmetric part of **Q**, and then **Q-A** must be a symmetric tensor. Post- and pre-multiplying the above equation (8) by **Q**, we have

$$\mathbf{Q}^2 - 2\mathbf{A}\mathbf{Q} - \mathbf{I} = \mathbf{0} \qquad \mathbf{A}\mathbf{Q} = \mathbf{Q}\mathbf{A} \tag{10}$$

the second of equations (10) can be considered as a restriction condition. Using this condition, the first of equations (10) is rewritten as

$$(\mathbf{Q}-\mathbf{A})^2 = \mathbf{I} + \mathbf{A}^2 \tag{11}$$

where $(\mathbf{Q}-\mathbf{A})^2$ is a positive and symmetric tensor, so that $\mathbf{I}+\mathbf{A}^2$ is also positive, which is the necessary and sufficient condition for the existence of **Q**. Then we have

$$tr\mathbf{A}^2 \geq -2 \tag{12}$$

Hence, the rotation tensor **Q** from equation (11) can also be obtained as

$$\mathbf{Q} = \mathbf{A} \pm \sqrt{\mathbf{I}+\mathbf{A}^2} \tag{13}$$

By the Cayley-Hamilton theorem, for **A**

$$\mathbf{A}^3 - \frac{1}{2}\left(tr\mathbf{A}^2\right)\mathbf{A} = \mathbf{0} \tag{14}$$

we have

$$\sqrt{\mathbf{I}+\mathbf{A}^2} = \mathbf{I} + \kappa^{-2}\left[1\pm\sqrt{1-\kappa^2}\right]\mathbf{A}^2 \tag{15}$$

where

$$\kappa^2 = -\frac{1}{2}tr\mathbf{A}^2 \geq 0 \tag{16}$$

Equation (16) is the existence condition of **S-Q** decomposition. Notice that for rigid body motion $\mathbf{F}=\mathbf{I}$, and therefore $\kappa=0$ and $\mathbf{Q}=\mathbf{I}$. Then the abstract form of the **S-Q** decomposition is

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{F}+\mathbf{F}^T\right) - \mathbf{I} = \frac{1-\sqrt{1-\kappa^2}}{\kappa^2}\mathbf{A}^2 \qquad \mathbf{Q} = \mathbf{I} + \frac{1}{2}\left(\mathbf{F}-\mathbf{F}^T\right) + \frac{1-\sqrt{1-\kappa^2}}{\kappa^2}\mathbf{A}^2 \tag{17}$$

The above form can simply be transformed into Chen's results.

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{F}+\mathbf{F}^T\right) - \mathbf{I} - \theta^{-2}(1-\cos\theta)\mathbf{L}^2 \qquad \mathbf{Q} = \mathbf{I} + \theta^{-1}\sin\theta\mathbf{L} + \theta^{-2}(1-\cos\theta)\mathbf{L}^2 \tag{18}$$

where $\kappa = \sin\theta$, and $\widetilde{\mathbf{R}} = \mathbf{A}$, $\mathbf{L} = \theta\sin^{-1}\theta\widetilde{\mathbf{R}}$. From equations (18) we find that the **S-Q** decomposition can be considered as an natural generalization of the decomposition of small deformation, since it assumes a simultaneous occurrence of a rotation and a stretching deformation. This is an attractive feature of the **S-Q** decomposition; however, it should satisfy the condition given by equation (16). Under this condition, the additive decomposition equations (18) is also available. The incremental variational equation of using the additive decomposition can be found by Li and Chen (1994).

3 Deformation Geometry of Shells

Here we consider a shell as a three-dimensional deformable space continuum which extends through its small thickness. The midsurface of an undeformed shell is regarded as a smooth surface. In this section, we shall describe the geometry of shell deformation coming from the **S-Q** decomposition of the deformation gradient.

3.1 Displacement of an Arbitrary Point $\mathbf{B} \in \Im$ in a Shell

Let $\mathbf{r}\left(\xi^1,\xi^2\right)$ be the position vector of a midsurfarce M in the undeformed configuration, then an arbitrary point $\mathrm{B} \in \Im$ can be represented by coordinates $\left(\xi^1,\xi^2,\xi^3\right)$, where $\xi^3 = \xi$ is the normal distance from this point to the midsurface M. The covariant base vectors in a convective system are then

$$\mathbf{A}_\alpha = \mathbf{r}_{,\alpha} \qquad \mathbf{A}_3 = \frac{1}{2}\in^{\alpha\beta}\mathbf{A}_\alpha \times \mathbf{A}_\beta \tag{19}$$

The metric and curvature tensors are defined to be (Deuker, 1943)

$$\mathbf{A}_{\alpha\beta} = \mathbf{A}_\alpha\mathbf{A}_\beta \qquad b_{\alpha\beta} = -\frac{1}{2}\left(\mathbf{A}_\alpha\mathbf{A}_{3,\beta} + \mathbf{A}_\beta\mathbf{A}_{3,\alpha}\right) = b_{\beta\alpha} \tag{20}$$

If the convective coordinate line ξ is to be defined as a straight line then the position vector of arbitrary point $\mathrm{B} \in \Im$ in the undeformed shell configuration is expressed as follows:

$$\mathbf{P} = \mathbf{r} + \xi\mathbf{A}_3 \tag{21}$$

Using equation (21) and equation (20) the covariant base vectors of the point B are

$$\mathbf{G}_\alpha = \mathbf{P}_{,\alpha} = \mathbf{A}_\alpha - \xi b_\alpha^\beta\mathbf{A}_\beta = \left(\delta_\alpha^\beta - \xi b_\alpha^\beta\right)\mathbf{A}_\beta \equiv \mu_\alpha^\beta\mathbf{A}_\beta \qquad \mathbf{G}_3 = \mathbf{A}_3 \tag{22}$$

where the shifter tensor $\mu = \mu_{\alpha\beta}\mathbf{A}^\alpha \otimes \mathbf{A}^\beta = \mathbf{G}_\alpha \otimes \mathbf{A}^\alpha$, $\mu = \mu^T$, is used to perform a lift of the tensor fields from the midsurface to the arbitrary point under discussion (Green and Zerna, 1950).

The position vector and its base vectors of the point $\mathrm{B} \in \Im$ in a deformed configuration may be expressed by

$$\mathbf{p} = \mathbf{P} + \overline{\mathbf{u}}\left(\xi^i\right) \qquad \mathbf{g}_i = \mathbf{p}_{,i} = \mathbf{G}_i + \overline{\mathbf{u}}_{,i} = \mathbf{F}\mathbf{G}_i \tag{23}$$

Then the displacement vector $\overline{\mathbf{u}}(\xi^{\alpha},\xi)$ of an arbitrary point $B \in \Im$ can be expressed by the displacement vector $\mathbf{u}$ of points on the midsurface and the directors difference vector β to the surface

$$\overline{\mathbf{u}} = \mathbf{u} + \xi\beta \qquad \beta = \mathbf{a}_3 - \mathbf{A}_3 \qquad \mathbf{a}_3 = \mathbf{g}_3(\xi^1,\xi^2,0) \tag{24}$$

It is worth pointing out that the vector $\mathbf{a}_3$ is generally neither a unit vector nor orthogonal to the deformed midsurface M. From equations (24), setting $\xi = 0$, we obtaine the base vectors on the deformed midsurface

$$\mathbf{a}_i = \widetilde{F}_i^{\,j}\mathbf{A}_j = \widetilde{F}_{i.}^{\alpha}\mathbf{A}_\alpha + \widetilde{F}_{i.}^{3}\mathbf{A}_3 \tag{25}$$

where $\widetilde{F}_{i.}^{j} = F_i^j$ if $\xi = 0$, is a three-dimensional deformation gradient of the midsurface particles. If we expand $\mathbf{u} = u^\alpha\mathbf{A}_\alpha + u^3\mathbf{A}_3$ and $\beta = \beta^\alpha\mathbf{A}_\alpha + \beta^3\mathbf{A}_3$ within the initial base $(\mathbf{A}_\alpha,\mathbf{A}_3)$, then we have

$$\overline{\mathbf{u}} = \overline{u}^\alpha\mathbf{A}_\alpha + u^3\mathbf{A}_3 \qquad \overline{u}^\alpha = u^\alpha + \xi\beta^\alpha \qquad \overline{u}^3 = u^3 + \xi\beta^3 \tag{26}$$

here are six fundamental unknown functions in equations (26). Let us take note of the connection between the definition of β and equation (25). Since β can always be written as

$$\beta = \mathbf{a}_3 - \mathbf{A}_3 = \left(\widetilde{F}_3^{\,j} - \delta_3^j\right)\mathbf{A}_j = \widetilde{F}_3^{\alpha}\mathbf{A}_\alpha + \left(\widetilde{F}_3^{3} - 1\right)\mathbf{A}_3 \tag{27}$$

we have the following representation for β:

$$\beta^\alpha = \widetilde{F}_{3.}^{\alpha} \qquad \beta^3 = \widetilde{F}_{3.}^{3} - 1 \tag{28}$$

such that the displacement vector of an arbitrary point can be rewritten as

$$\overline{\mathbf{u}} = \mathbf{u} + \xi\left(\widetilde{F}_3^{\,i} - \delta_3^i\right)\mathbf{A}_i \tag{29}$$

which is important in numerical analyses, especially those which have been used in the finite element method.

3.2 S-Stretching and Q-Rotation

In order to construct the stretching and rotation distributions by the **S-Q** approach, the deformation gradient **F** should be obtained first. Let us suppose that a shell is indeed thin, and therefore, its shifter tensor is almost a unit tensor, $\mu \approx \mathbf{I}$, then from equation (24), we get

$$\overline{\mathbf{u}}_{,\alpha} = \mathbf{u}_{,\alpha} + \xi\beta_{,\alpha} = \left(\phi_\alpha^\lambda + \xi\psi_\alpha^\lambda\right)\mathbf{A}_\lambda + \left(\phi_\alpha + \xi\psi_\alpha\right)\mathbf{A}_3 \qquad \overline{\mathbf{u}}_3 = \beta = \beta^\alpha\mathbf{A}_\alpha + \beta^3\mathbf{A}_3 \tag{30}$$

Using the definition of **F**, equation (4), we have the following representation within the $(\mathbf{A}_\alpha,\mathbf{A}_3)$ base:

$$\mathbf{F} = F_{j.}^{i}\mathbf{A}_i \otimes \mathbf{A}^i \qquad \left[F_{j.}^{i}\right] = \begin{pmatrix} 1+\phi_1^1+\xi\psi_1^1 & \phi_1^2+\xi\psi_1^2 & \phi_1+\xi\psi_1 \\ \phi_2^1+\xi\psi_2^1 & 1+\phi_2^2+\xi\psi_2^2 & \phi_2+\xi\psi_2 \\ \beta^1 & \beta^2 & 1+\beta^3 \end{pmatrix} \tag{31}$$

where we have introduced the following notations:

$$\begin{aligned} \phi_\alpha^\lambda &= u^\lambda\big|_\alpha = u^\lambda\big\|_\alpha - b_\alpha^\lambda u^3 \qquad & \psi_\alpha^\lambda &= \beta^\lambda\big|_\alpha = \beta^\lambda\big\|_\alpha - b_\alpha^\lambda\beta^3 \\ \phi_\alpha &= u^3\big|_\alpha = u_{,\alpha}^3 + b_{\lambda\alpha}u^\lambda \qquad & \psi_\alpha &= \beta^3\big|_\alpha = \beta_{,\alpha}^3 + b_{\lambda\alpha}\beta^\lambda \end{aligned} \tag{32}$$

using, additionally, the symbol $(\|)$ for the covariant or the contravariant surface derivatives (Weatherburm, 1927; Koiter, 1966). Starting with the deformation gradient represented by equation (31) and using **S-Q** decomposition, the **S**-stretching and **Q**-rotation components take the forms

$$
\begin{aligned}
S_1^1 &= \phi_1^1 + \xi\psi_1^1 + (1-\cos\theta)\left[\left(L_2^1\right)^2+\left(L_3^1\right)^2\right] & S_2^2 &= \phi_2^2 + \xi\psi_2^2 + (1-\cos\theta)\left[\left(L_2^1\right)^2+\left(L_3^2\right)^2\right] \\
S_3^3 &= \beta^3 + (1-\cos\theta)\left[\left(L_1^3\right)^2+\left(L_3^2\right)^2\right] & S_2^1 &= \frac{1}{2}\left[\phi_1^2+\phi_2^1+\xi\left(\psi_1^2+\psi_2^1\right)\right] - (1-\cos\theta)L_1^3L_3^2 \\
S_3^2 &= \frac{1}{2}\left(\phi_2+\beta^2\xi\psi_2\right) - (1-\cos\theta)L_2^1L_1^3 & S_1^3 &= \frac{1}{2}\left(\phi_1+\beta^1+\xi\psi_1\right) - (1-\cos\theta)L_3^2L_2^1
\end{aligned} \tag{33}
$$

and

$$
\begin{aligned}
Q_1^1 &= 1 - (1-\cos\theta)\left[\left(L_2^1\right)^2+\left(L_1^3\right)^2\right] & Q_2^2 &= 1 - (1-\cos\theta)\left[\left(L_2^1\right)^2+\left(L_3^2\right)^2\right] \\
Q_3^3 &= 1 - (1-\cos\theta)\left[\left(L_1^3\right)^2+\left(L_3^2\right)^2\right] & Q_2^1 &= \frac{1}{2}\left[\phi_1^2+\phi_2^1+\xi\left(\psi_1^2-\psi_2^1\right)\right] + (1-\cos\theta)L_1^3L_3^2 \\
Q_3^2 &= \frac{1}{2}\left(\beta^2-\phi_2-\xi\psi_2\right) + (1-\cos\theta)L_2^1L_1^3 & Q_1^3 &= \frac{1}{2}\left(\phi_1-\beta^1+\xi\psi_1\right) + (1-\cos\theta)L_3^2L_2^1
\end{aligned} \tag{34}
$$

where

$$
\begin{aligned}
\sin\theta &= \frac{1}{2}\left\{\left[\phi_1^2-\phi_2^1+\xi\left(\psi_1^2-\psi_2^1\right)\right]^2 + \left(\beta^1-\phi_1-\xi\psi_1\right)^2 + \left(\phi_2-\beta^2+\xi\psi_2\right)^2\right\}^{1/2} \\
L_2^1 &= \left[\phi_1^2-\phi_2^1+\xi\left(\psi_1^2-\psi_2^1\right)\right]/(2\sin\theta) \\
L_3^2 &= \left(\beta^2-\phi_2-\xi\psi_2\right)/(2\sin\theta) \qquad L_1^3 = \left(\phi_1-\beta^1+\xi\psi_1\right)/(2\sin\theta)
\end{aligned} \tag{35}
$$

The above results are the **S**-stretching and **Q**-rotation components for a shell under the kinematical assumption that a straight fibre remains straight after deformation. Note that, in contradistinction to the Green measure, the **S**-stretching as well as **Q**-rotation are nonlinear functions of the coordinate ξ.

4 Kirchhoff-Love Type Model under a Moderate Q-Rotation

Some consequences can now be distinguished. Depending on the estimate of admissible values of the mean roation angle (Qin and Chen, 1988; Sun, 1989a; Sun, 1989b), they are a) small **Q**-rotation model, if $\theta = O(\eta_0)$, b) moderate **Q**-rotation model, if $\theta = O\left(\eta_0^{1/2}\right)$, c) large **Q**-rotation model, if $\theta = \left(\eta_0^{1/4}\right)$, and d) finite **Q**-rotation model, if θ unrestricted, where $\eta_0 = \|\tilde{\mathbf{E}}\| > 0$, and $\tilde{\mathbf{E}}$ is the infinitesimal strain tensor of shells defined earlier by equation (25). Within the large **Q**-rotation model, we have $\sin\theta \approx \theta - (1/3)\theta^3$, $\cos\theta \approx 1 - (1/2)\theta^2$. In this work we will not discuss any further the possible simplifications of geometrical relations and basic equations for the large **Q**-rotation model, which are studied comprehensively in Sun, (1989a). Within the moderate rotation model, we have $\sin\theta = \theta + O\left(\theta^3\right)$, $\cos\theta \approx 1 + O\left(\theta^2\right)$, thus corresponding simplifications of geometrical relations will be presented below. Within the small rotation model, we have $\sin\theta \cong \theta$, $\cos\theta \cong 1$, where all geometrical and physical relations reduce to the linear forms from the known classical linear theory of shells. Let us discuss now the simplest case of the Kirchhoff-Love model in more detail, expressing also the resulting formulae within the physical components. Recalling briefly that if $\overset{\circ}{A}_1, \overset{\circ}{A}_2$ denote Lamé's coefficients of the orthogonal Gauss parametrization ξ^α, such that $\mathbf{A}_\alpha = \overset{\circ}{A}_\alpha \mathbf{e}_\alpha$ $(\alpha = 1, 2)$, then the differentiation of a nonholonomic (physical) base $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{A}_3$ is expressed (Mushtari and Galimov, 1957) by

$$\mathbf{e}_{1,1} = \left(-\mathring{A}_2^{-1} \mathring{A}_{1,2} \right) \mathbf{e}_2 - \mathring{R}_1 \mathring{A}_1 \mathbf{e}_3 \qquad \mathbf{e}_{2,1} = \left(\mathring{A}_2 \mathring{A}_{1,2} \right) \mathbf{e}_1 \qquad \mathbf{e}_{3,1} = \mathring{R}_1^{-1} \mathring{A}_1 \mathbf{e}_1 \tag{36}$$

Using now the physical components of a two-dimensional Hamilton nabla operator $\nabla_2 = \mathring{A}_1^{-1} \mathbf{e}_1 \partial_1 + \mathring{A}_2^{-1} \mathbf{e}_2 \partial_2$, of the displacement vector $\mathbf{u} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3$ and taking into account equation (24), one obtains the surface deformation gradient

$$\begin{aligned} \Gamma &= (\mathbf{r}+\mathbf{u}) \otimes \nabla_2 = \mathbf{g}_\alpha \otimes \mathbf{G}^\alpha = (1+e_{11}) \mathbf{e}_1 \otimes \mathbf{e}_1 + \omega_{21} \mathbf{e}_2 \otimes \mathbf{e}_1 \\ &+ \omega_{12} \mathbf{e}_1 \otimes \mathbf{e}_2 + (1+e_{22}) \mathbf{e}_2 \otimes \mathbf{e}_2 + \phi_1 \mathbf{e}_3 \otimes \mathbf{e}_2 \end{aligned} \tag{37}$$

where, in this case, the physical components of the two point tensor Γ are denoted

$$\begin{aligned} e_{11} &= \mathring{A}_1^{-1} u_{,1} + \left(\mathring{A}_1 \mathring{A}_2 \right) \mathring{A}_{1,2} v + \mathring{R}_1^{-1} w \qquad \omega_{12} = \mathring{A}_2^{-1} u_{,2} - \left(\mathring{A}_1 \mathring{A}_2 \right)^{-1} \mathring{A}_{2,1} v \qquad (1\square 2) \\ \phi_1 &= \mathring{A}_1 w_{,1} - \mathring{R}_1^{-1} u \end{aligned} \tag{38}$$

As far as Γ is concerned, a simple two dimensional analogue of the Love elongation tensor $\tilde{\mathbf{E}}$, equation (2), and the mean rotation tensor $\tilde{\mathbf{R}}$ are similarly defined as follows: defining first $\mathbf{a} = \mathbf{I} - \mathbf{A}_3 \otimes \mathbf{A}^3$ to be a surface indemfactor one gets a decomposition of equation (37) into $\mathbf{a}\Gamma = \mathbf{a} + \tilde{\mathbf{E}}_M + \tilde{\mathbf{R}}_M$, in which

$$\begin{aligned} \tilde{\mathbf{E}}_M &= \frac{1}{2} \mathbf{a} \left(\Gamma + \Gamma^T - \mathbf{a} \right) \mathbf{a} = e_{11} \mathbf{e}_1 \otimes \mathbf{e}_1 + e_{12} (\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) + e_{22} \mathbf{e}_2 \otimes \mathbf{e}_2 \\ \tilde{\mathbf{R}}_M &= \frac{1}{2} \mathbf{a} \left(\Gamma - \Gamma^T \right) \mathbf{a} = (1/2)(\omega_{12} - \omega_{21}) \mathbf{e}_1 \otimes \mathbf{e}_2 + (1/2)(\omega_{12} - \omega_{21}) \mathbf{e}_2 \otimes \mathbf{e}_1 = \phi_3 (\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) \end{aligned} \tag{39}$$

here $e_{12} \equiv e_{21} = (1/2)(\omega_{12} + \omega_{21})$, $\phi_3 = (1/2)(\omega_{12} - \omega_{21}) = \left(\mathring{A}_1 \mathring{A}_2 \right)^{-1} \left[\left(\mathring{A}_1 u_2 \right)_{,2} - \left(\mathring{A}_2 v \right)_{,1} \right]$. Now, the three-dimensional gradient $\tilde{\mathbf{F}} \equiv \mathbf{F}|_{\xi=0}$, equation (25), has the following convenient physical representation (still $\mu \approx \mathbf{a}$):

$$\tilde{\mathbf{F}} = \mathbf{a} + \tilde{\mathbf{E}}_M + \tilde{\mathbf{R}}_M + \phi_1 \mathbf{e}_3 \otimes \mathbf{e}_1 + \phi_2 \mathbf{e}_3 \otimes \mathbf{e}_2 + (\beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \beta_3 \mathbf{e}_3) \otimes \mathbf{e}_3 \tag{40}$$

Since we are interested here only in the case of moderate rotations [the large rotation case leads to cumbersome calculations but it involves the same procedure (Sun, 1989a; Sun, 1989b)], without further discussion of Kirchhoff's hypothesis we take

$$\begin{aligned} \beta_1 &= [\phi_2 \omega_{21} - \phi_1 (1+e_{22})] \kappa \approx -\phi_1 \qquad & \beta_2 &= [\phi_1 \omega_{12} - \phi_2 (1+e_{11})] \kappa \approx -\phi_2 \\ \beta_3 &= [(1+e_{11})(1+e_{22}) - \omega_{12} \omega_{21}] \kappa \approx 1 \qquad & \kappa &\approx 1 \end{aligned} \tag{41}$$

Hence, from equations (40) and (41) within the moderate rotation limit we have a surface **S-Q** decomposition

$$\tilde{\mathbf{F}} = \mathbf{a} + \tilde{\mathbf{E}}_M + \tilde{\mathbf{R}}_M + \phi_1 (\mathbf{e}_3 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_3) + \phi_2 (\mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3) + \mathbf{e}_3 \otimes \mathbf{e}_3 \equiv \tilde{\mathbf{S}} + \tilde{\mathbf{Q}} \tag{42}$$

where the mean stretching $\tilde{\mathbf{S}}$ is defined by

$$\tilde{\mathbf{S}} \equiv \tilde{\mathbf{E}}_M - \left[\tilde{\mathbf{R}}_M + \phi_1(\mathbf{e}_3 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_3) + \phi_2(\mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3)\right]^2$$

$$\begin{aligned} &= \left(e_{11} + \frac{1}{2}\phi_1^2 + \frac{1}{2}\phi_3^2\right)\mathbf{e}_1 \otimes \mathbf{e}_1 + \left(e_{12} + \frac{1}{2}\phi_1\phi_2\right)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \\ &+ \left(e_{22} + \frac{1}{2}\phi_2^2 + \frac{1}{2}\phi_3^2\right)\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{1}{2}\phi_1\phi_3(-\mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3) \\ &+ \frac{1}{2}\phi_2\phi_3(\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3) + \frac{1}{2}\left(\phi_1^2 + \phi_2^2\right)\mathbf{e}_3 \otimes \mathbf{e}_3 \end{aligned} \tag{43}$$

together with the mean rotation tensor $\tilde{\mathbf{Q}}$

$$\begin{aligned} \tilde{\mathbf{Q}} \equiv &\left(1 - \frac{1}{2}\phi_1^2 - \frac{1}{2}\phi_3^2\right)\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{2}(\phi_3 + \phi_1\phi_2)\mathbf{e}_1 \otimes \mathbf{e}_2 \\ &+ \left(-\phi_3 - \frac{1}{2}\phi_1\phi_2\right)\mathbf{e}_2 \otimes \mathbf{e}_1 + \left(1 - \frac{1}{2}\phi_2^2 - \frac{1}{2}\phi_3^2\right)\mathbf{e}_2 \otimes \mathbf{e}_2 \\ &+ \left(-\phi_1 + \frac{1}{2}\phi_2\phi_3\right)\mathbf{e}_1 \otimes \mathbf{e}_3 + \left(\phi_1 + \frac{1}{2}\phi_2\phi_3\right)\mathbf{e}_3 \otimes \mathbf{e}_1 + \left(-\phi_2 - \frac{1}{2}\phi_1\phi_3\right)\mathbf{e}_2 \otimes \mathbf{e}_3 \\ &+ \left(\phi_2 - \frac{1}{2}\phi_1\phi_3\right)\mathbf{e}_3 \otimes \mathbf{e}_2 + \left(1 - \frac{1}{2}\phi_1^2 - \frac{1}{2}\phi_2^2\right)\mathbf{e}_3 \otimes \mathbf{e}_3 \end{aligned} \tag{44}$$

The corresponding surface mean stretching is now only part of equation (43).

$$\in = \in^T = \mathbf{a}\tilde{\mathbf{S}}\mathbf{a} = \in_{11} \mathbf{e}_1 \otimes \mathbf{e}_1 + \in_{12} (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \in_{22} \mathbf{e}_2 \otimes \mathbf{e}_2 \tag{45}$$

The physical components of the terms in equation (45), are easily anticipated from equation (43). It is interesting to compare this result with the one following from the surface Green tensor $\gamma \equiv (1/2)\left(\Gamma^T\Gamma - \mathbf{a}\right)$. This is done quite easily by means of equations (37) and (38) which lead to

$$\begin{aligned} \gamma \equiv &(1/2)\left(2e_{11} + \underline{e_{11}^2} + \overline{\omega_{21}^2} + \phi_1^2\right)\mathbf{e}_1 \otimes \mathbf{e}_1 + (1/2)\left[2e_{12} + \underline{e_{12} + (e_{11} + e_{22})} + 2\phi_1\phi_2\right](\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \\ &+ (1/2)\left(2e_{22} + \underline{e_{22}^2} + \overline{\omega_{12}^2} + \phi_2^2\right)\mathbf{e}_2 \otimes \mathbf{e}_2 \end{aligned} \tag{46}$$

The terms underlined are dropped in the small deformation approximation, however, the difference between the mean stretching tensor $\in$, equation (45), and the Green strain tensor γ, equation (46), appear in the overlined terms.

For completeness, let us recall the form of the curvature term $\tilde{\mathbf{S}}_1$ being an important part of the thickness expansion $\mathbf{S}(\xi^\alpha, \xi) = \tilde{\mathbf{S}}(\xi^\alpha) + \xi\mathbf{S}_1(\xi^\alpha) + \cdots$. Its surface part $\chi = \mathbf{a}\tilde{\mathbf{S}}_1\mathbf{a}$ for Kirchhoff's hypothesis undergoing the moderate **Q**-rotation aproximation, as has been shown in Sun (19989a, 1989b), is equal to

$$\begin{aligned} \chi = &\left[-\mathring{A}_1\phi_{1,1} - \left(\mathring{A}_1\mathring{A}_2\right)^{-1}\mathring{A}_{1,2}\phi_2\right]\mathbf{e}_1 \otimes \mathbf{e}_1 + \left[-\mathring{A}_2^{-1}\phi_{2,2} - \left(\mathring{A}_1\mathring{A}_2\right)^{-1}\mathring{A}_{2,1}\phi_1\right]\mathbf{e}_2 \otimes \mathbf{e}_2 \\ &+ (1/2)\left[-\mathring{A}_1^{-1}\mathring{A}_2\left(\mathring{A}_2^{-1}\phi_2\right)_{,1} - \mathring{A}_2^{-1}\mathring{A}_1\left(\mathring{A}_1^{-1}\phi_1\right)_{,2}\right](\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \end{aligned} \tag{47}$$

Note that the same expression is to be obtained if the nonlinear bending measure $\kappa = -\left(\Gamma^T\mathbf{b}\Gamma - \mathbf{b}\right)$ undergoes the moderate **R**-rotation approximation. Passing now into a consideration of the equilibrium equations for the above model we note firstly that on account of the estimations of the first approximation to the elastic stored energy $\mathbf{W}^M$ (Koiter, 1966), we are permitted, in our case, to repeat Koiter's procedure making no separate analysis. Taking the best known example of the harmonic or John's material (Libai and Simmonds, 1983) one expresses the Lagrangian-type constitutive relations as

$$\mathbf{N} = \mathbf{W}^M_{,\in} = \frac{\partial \mathbf{W}^M}{\partial \in} \qquad \mathbf{M} = \mathbf{W}^M_{,\chi} = \frac{\partial \mathbf{W}^M}{\partial \chi} \tag{48}$$

the Bio-type resultant stress and couples. Next, continuing along Kirchhoff's (1850, §3) line of reasoning making use of equation (48), and the expressions for variations of $\delta \in, \delta \chi$ from the principle of virtual work one gets the following equilibrium equation in vector form

$$\Delta iv\left[\left(\mathbf{a}+2\tilde{\mathbf{R}}_M\right)\mathbf{N} + 2(\mathbf{I}-\mathbf{a})\Gamma\mathbf{N} - \mathbf{bM} + \mathbf{A}_3 \otimes \mathbf{a}\Delta iv\mathbf{M}\right] + \mathbf{p} = 0 \tag{49}$$

together with the appropriate force-type boundary conditions. Using equations (37), (39) and (48) and

$$\mathbf{b} = b_{\alpha\beta}\mathbf{A}^\alpha \otimes \mathbf{A}^\beta = \left(\overset{\circ}{R}_1\right)^{-1} \mathbf{e}_1 \otimes \mathbf{e}_2 + \left(\overset{\circ}{R}_2\right)^{-1} \mathbf{e}_2 \otimes \mathbf{e}_2 \qquad \mathbf{e}_3 = \mathbf{A}_3 \qquad \mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3$$

and Weatherburn's (1927) definition of the divergence operator

$$\Delta iv\mathbf{M} \equiv \left(\overset{\circ}{A}_1 \overset{\circ}{A}_2\right)^{-1}\left[\left(\overset{\circ}{A}_2 \mathbf{M}\right)_{,1} \mathbf{e}_1 + \left(\overset{\circ}{A}_1 \mathbf{M}\right)_{,2} \mathbf{e}_2\right] \qquad \mathbf{M} = M_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + M_{12}\left(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1\right) + M_{22}\mathbf{e}_2 \otimes \mathbf{e}_2$$

we obtain the physical components of the equilibrium equations (49). For example, the equation along the normal direction $\mathbf{e}_3$ can be read

$$\begin{aligned}
&\left(-\overset{\circ}{A}_1 \overset{\circ}{A}_2\right)\left(\overset{\circ}{R}_1^{-1} N_{11} + \overset{\circ}{R}_2^{-1} N_{22}\right) + \overset{\circ}{A}_1 \overset{\circ}{A}_2 \phi_3 N_{12}\left(\overset{\circ}{R}_2^{-1} - \overset{\circ}{R}_1^{-1}\right) + \left[\overset{\circ}{A}_2\left(\phi_1 N_{11} + \phi_2 N_{12}\right)\right]_{,1} \\
&+ \left[\overset{\circ}{A}_1\left(\phi_1 N_{12} + \phi_2 N_{22}\right)\right]_{,2} + \overset{\circ}{A}_{2,1}\left(\phi_1 N_{11} + \phi_2 N_{12}\right) + \overset{\circ}{A}_{1,2}\left(\phi_1 N_{12} + \phi_2 N_{22}\right) \\
&- \overset{\circ}{A}_1 \overset{\circ}{A}_2\left(\overset{\circ}{R}_2^{-1} M_{22}\right) + \left(\overset{\circ}{A}_2 Q_1\right)_{,1} + \left(\overset{\circ}{A}_2 Q_2\right)_{,2} + \overset{\circ}{A}_{1,2} Q_2 + \overset{\circ}{A}_{2,1} Q_1 + \overset{\circ}{A}_1 \overset{\circ}{A}_2 p_3 = 0
\end{aligned} \tag{50}$$

where $\mathbf{a}\Delta iv\mathbf{M} = Q_1\mathbf{e}_1 + Q_2\mathbf{e}_2$ is represented via the so-called transverse resulting force

$$Q_1 = \left(\overset{\circ}{A}_1 \overset{\circ}{A}_2\right)^{-1}\left[\left(\overset{\circ}{A}_2 M_{11}\right)_{,1} + \left(\overset{\circ}{A}_1 M_{12}\right)_{,2} + 2\overset{\circ}{A}_{1,2} M_{12} + \overset{\circ}{A}_{2,1}\left(M_{11} - M_{22}\right)\right] \qquad (1\square2) \tag{51}$$

5 Numerical Analysis for Large Axisymmetric Deformation of Shells of Revolution

Based on the above discussions, the numerical implementation for large elastic axisymmetric deformation of shells of revolution will be carried out in detail. According to the geometrical feature of the shells, the cubic B-spline function shall be used to construct of the midsurface, its displacement, the resultant stress and the couples. The numerical results have shown the B-spline element to be very efficient for this kind of shell, and our formulation of stretches and rotations are competitive compared with some analytical results. Detailed numerical formulation can be found in Sun (1989a, 1989b), Sun and Badur (1993).

6 Numerical Examples

Based on the above formulation, a few typical numerical examples have been carried out (Sun, 1989a; 1989b) to demonstrate the accuracy, and to compare the findings with some well-known results. The basic equations have been solved by the Newton-Raphson method and the control parameter in the calculation process has been chosen to be a load and a displacement, respectively.

a) *Large Deflection of a Circular Clamped Plate under Uniform Pressure (Figure 1)*

This one-dimensional example was solved analytically by Chien et al. (1951) using the perturbation method. For this problem the pressure was chosen as the control parameter, since there is no snap-through phenomenon. We use a 4-segment model with 2 Gauss integration points. From Figure 1 one can find that our resulting deflection is consistent with Chien's perturbation solution (Chien et al., 1951). The convergence of this solution has been examined for various finite segment meshes (Sun, 1989a and 1989b).

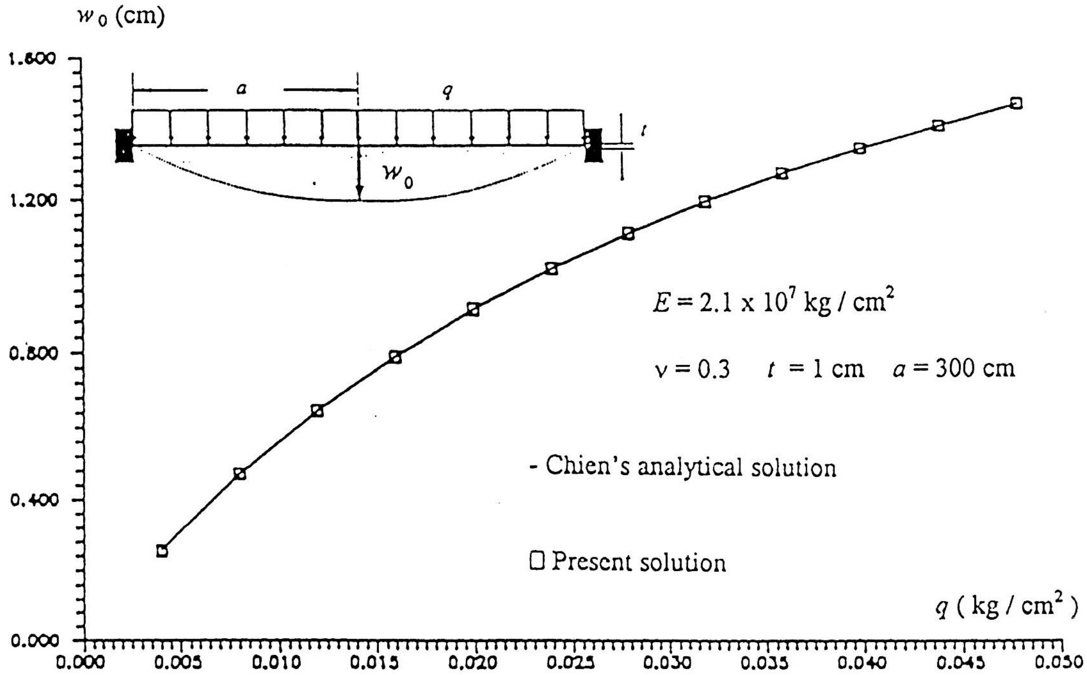

Figure 1. Central deflection of clamped plate by uniform lateral pressure

b) *Belleville Spring (Figure 2)*

Another example is the study of the load-deflection characteristics of a Belleville spring. This problem has been calculated by Surana (1982) in detail. Since there is the possibility of a snap-through, we chose a deflection parameter δ to be the control parameter. We use a 4-segment model with 2 Gauss points. It is shown in Figure 2 that there are some differences between our results and Surana's. The reason is that our strain formulation is different from his.

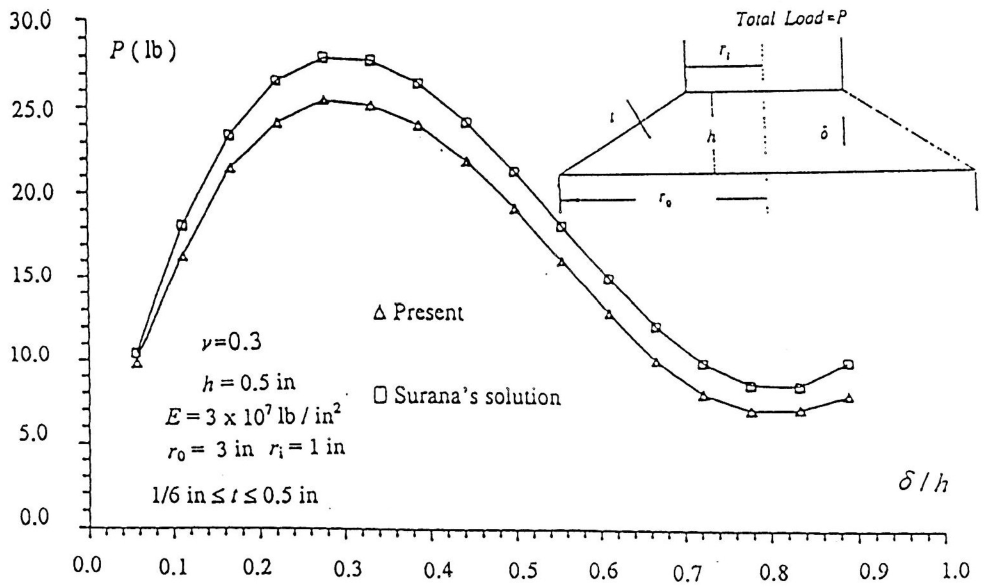

Figure 2 Load-deflection of a Belleville spring

From both numerical examples it follows that our results are consistent with those of the classical Kirchhoff-Love plate problem, but that there are some differences for shells. The reason is that we use the same strain formula for plates but a different strain formula for shells compared with those in former papers.

Our considerations concerning the shell model based on the **S-Q** decomposition are far from complete. Further tasks will involve, for instance, the consideration of large and finite **Q**-rotations and the development of an elasto-plastic shell model.

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