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A Hybrid WKB-Galerkin Method and its Application

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An asymptotic approach presupposing the use of the hybrid WKB-Galerkin method is employed for the solution of some applied mathematics and mechanics problems. The extension of this methodology to some complicated mechanical problems with mixed boundary conditions and to nonlinear problems is possible.

1 Introduction

The analysis of complex mechanical models of nonhomogeneous structures necessitates the solution of systems of differential equations that contain variable coefficients and some small or large parameters. In these cases we cannot obtain, in general, the exact analytical solution. One of the possibilities is using approximate asymptotic methods. Geer and Anderson (1982, 1989, 1990, 1991) have discussed a two-step hybrid perturbation-Galerkin method for the solution of some types of differential equation and applied mechanical problems that involve a parameter.

The WKB(Wentzel-Kramers-Brillouin) method for numerous mechanical problems was discussed e.g. by Steele (1971, 1989). It was shown by Gristchak (1979, 1988), that the WKB-method can be used for some linear mechanical problems as well as for nonlinear buckling and vibration problems of nonhomogeneous structures as an inner expansion in the double asymptotic-perturbation procedure.

In the spirit of Geer and Anderson a hybrid WKB-Galerkin method for some applied mathematical problems that arise, the description of the behavior of mechanical structures under external loading is discussed. The results from the hybrid WKB-Galerkin method are compared with solutions obtained by purely numerical methods and with exact solutions where available.

A hybrid WKB-Galerkin technique is especially useful for approximate solutions of differential equations with the parameter near the higher order derivative. The main features of this approach include fully explored WKB-terms in final solutions as well as possibilities to take into account singularities. In this paper we will discuss some of them.

2 Description of the Method

The method we describe is a two-step hybrid analysis technique for the solution of linear differential equations. The technique of this approach includes two steps of solution: in the first step of the procedure the WKB-method determines the approximate solution of the initial equation; in the second step we use asymptotic coefficients as trial functions in the standard Bubnov-Galerkin method.

Suppose we are seeking an approximate solution $u(x,\varepsilon)$ to the boundary problem

$$L[u(x,\varepsilon),x,\varepsilon] = 0 \tag{1}$$

where L is some linear differential operator of n-th order (in the general case with variable coefficients), ε is a parameter near the highest derivative, x is located in some interval $[a,b]$, and $u(x,\varepsilon)$ is satisfied by the given boundary conditions.

In the first step we present the problem solution $u(x,\varepsilon)$ in correspondance with the WKB-procedure in the form

$$u(x,\varepsilon) = \exp\left(\int_a^x \sum_{i=0}^{\infty} u_i(x)\gamma_i(\varepsilon)\,dx\right) \tag{2}$$

where $\gamma_i(\varepsilon)$ is an appropriate asymptotic sequence $\left(\gamma_i(\varepsilon) = \varepsilon^{i-1}\right)$ and each $u_i(x)$ can be determined by a standard WKB-method. The approximate functions $u_i(x)$ are chosen as coordinate functions for the Bubnov-Galerkin technique and an approximation $\tilde{u}_i(x,\varepsilon)$ for $u_i(x,\varepsilon)$ is sought in the form

$$\tilde{u}(x,\varepsilon) = \exp\left(\int_a^x \sum_{i=0}^{N} u_i(x)\delta_i(\varepsilon)\,dx\right) \tag{3}$$

where the unknown parameters $\delta_i(\varepsilon)$ are complex functions of ε and all $u_i(x)$ are approximate coordinate functions that were found in the previous step. To determine the unknown coefficients $\delta_i\left(i = \overline{O,N}\right)$ we substitute equation (3) into equation (1). Thus we obtain a product of the right hand side of an expression (3) and some expression in which a leading derivative of functions $u_i(x)$ is one less than the leading derivative in equation (1).

$$L\left[\tilde{u}(x,\varepsilon),x,\varepsilon\right] = \exp\left(\int_a^x \sum_{i=0}^{N} \delta_i(\varepsilon)u_i(x)dx\right) R\left(\delta_0,\cdots\delta_N,\, u_0,\cdots u_N,\, u_0'\cdots u_N^{(n-1)},\, x,\varepsilon\right) \tag{4}$$

It is necessary to satisfy the right hand side of the governing equation (1), that is

$$R\left(\delta_0,\cdots\delta_N,\, u_0,\cdots u_N,\, u_0',\cdots u_N^{(n-1)},\, x,\varepsilon\right) \to 0 \tag{5}$$

therefore we demand that the residual R be orthogonal to the $N+1$ coordinate functions over the interval $[a,b]$, i.e.

$$\int_a^b R\left(\delta_0,\cdots\delta_N,\, u_0,\cdots u_N,\, u_0'.\cdots u_N^{(n-1)},\, x,\varepsilon\right) u_i(x)\,dx = 0 \tag{6}$$

with $\quad i = 0, \cdots, N$

Equation (6) represents a set of $N+1$ equations for the $N+1$ unknown coefficients $\delta_i(\varepsilon)$. If each δ_k represents a complex form as $\delta_k = \delta_{k1} + i\delta_{k2}$, we obtain a set of $2(N+1)$ equations (i.e. $N+1$ real and $N+1$ imaginary parts) for the $2(N+1)$ unknown coefficients. While equations (6) must generally be solved numerically, their solutions are simpler than a direct numerical solution of equation (1).

For example, we will illustrate the hybrid WKB-Galerkin method for the next two-point boundary problem

$$\varepsilon^2\ddot{u} + f(x)u = 0 \tag{7}$$

with $u(a) = u_a$ and $u(b) = u_b$, $f(x) > 0$ over $[a,b]$, which arises in some mechanical problems.

Step one. Following the WKB-procedure we represent the solution of equation (7) in the form

$$u(x,\varepsilon) = \exp\left[\int_a^x \left(\frac{1}{\varepsilon}u_0 + u_1 + \varepsilon u_2 + \cdots\right)dx\right] \tag{8}$$

For our example we shall take into account only the first term in the WKB-expansion (8) and after the substitution of equation (8) into equation (7) we obtain

$$u_0 = \pm i\sqrt{f(x)} \tag{9}$$

Step two. Now, following Geer and Andersen, we represent the approximation of $u(x,\varepsilon)$ in the form

$$\tilde{u}(x,\varepsilon) = \exp\left[\int_a^x \left(\delta_{01}(\varepsilon) + \delta_{02}(\varepsilon)\right) u_0(x) dx\right] \tag{10}$$

After the substitution of sequence (10) into equation (7) we obtain the residual, i.e.

$$R = \varepsilon^2\left[\left(\delta_{01} + i\delta_{02}\right)^2 u_0^2 + \left(\delta_{01} + i\delta_{02}\right) u_0'\right] + f(x) \tag{11}$$

Then we apply the Bubnov-Galerkin criterion (6) in order to obtain a set of equations

$$\begin{aligned} &\varepsilon^2\left[\left(\delta_{01}^2 - \delta_{02}^2\right)\int_a^b \left(\mp f^{3/2}(x)\right) dx - \delta_{02}\frac{1}{2}\int_a^b f'(x)\, dx\right] + \int_a^b \left(\pm f^{3/2}(x)\right) dx = 0 \\ &\varepsilon^2\left[\pm 2\delta_{01}\delta_{02}\int_a^b \left(f^{3/2}(x)\right) dx - \delta_{01}\frac{1}{2}\int_a^b f'(x)\, dx\right] = 0 \end{aligned} \tag{12}$$

In this case

$$\begin{aligned} \delta_{01} &= \sqrt{\left(\frac{1}{\varepsilon}\right)^2 - \left(\frac{f(b) - f(a)}{4\int_a^b f^{3/2}(x)\, dx}\right)^2} \\ \delta_{02} &= \pm\frac{f(b) - f(a)}{4\int_a^b f^{3/2}(x)\, dx} \end{aligned} \tag{13}$$

Now we obtain the hybrid WKB-Galerkin solution of the initial equation in the form

$$\tilde{u}(x,\varepsilon) = \exp\left(\frac{f(a) - f(b)}{4\int_a^b f^{3/2}(x)\, dx}\int_a^x f^{1/2}(x)\, dx\right)\left(c_1 \cos\left(\Delta\int_a^x f^{1/2}(x) dx\right) + c_2 \sin\left(\Delta\int_a^x f^{1/2}(x) dx\right)\right) \tag{14}$$

where $\Delta = \sqrt{\left(\frac{1}{\varepsilon}\right)^2 - \left(\frac{f(b) - f(a)}{4\int_a^b f^{3/2}(x) dx}\right)}$, and c_1, c_2 are determined from the given boundary conditions.

If the function $f(x) < 0$ within the interval $[a,b]$, our hybrid solution becomes

$$\tilde{u} = (x,\varepsilon) = \exp\left(\frac{f(a) - f(b)}{4\int_a^b f^{3/2}(x)\, dx}\int_a^x f^{1/2}(x)\, dx\right)\left(c_1 \cos\left(\Delta\int_a^x f^{1/2}(x) dx\right) + c_2 \sin\left(\Delta\int_a^x f^{1/2}(x) dx\right)\right) \tag{15}$$

at $$\Delta = \sqrt{\left(\frac{1}{\varepsilon}\right)^2 + \left(\frac{f(a) - f(b)}{4\int_a^b f^{3/2}(x)dx}\right)}$$

We note here that for the turning point problem of the initial differential equation when $f(x) < 0$ at $-a \leq x < 0$, $f(x) > 0$ at $0 < x \leq b$, and $f(0) = 0$, we can use a special presentation of the WKB approximation through Airy functions.

Now we consider the general case where the coefficients of the initial equation are complex functions. Suppose an approximate solution in the form

$$u(x,\varepsilon) = \operatorname{Re} u(x,\varepsilon) + i \operatorname{Im} u(x,\varepsilon) \tag{16}$$

to the boundary problem

$$\varepsilon^2 u^* - f(x)u = 0 \tag{17}$$

where $f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x)$, with $u(a) = u_a$ and $u(b) = u_b$.

Step one. Following the WKB procedure the solution of equation (16) can be represented in the form

$$u_0 = \pm\sqrt{f(x)} = \pm \operatorname{Re} u_0 \pm i \operatorname{Im} u_0 \tag{18}$$

Step two. We represent the function $u(x,\varepsilon)$ in an approximate form, such as

$$\tilde{u}(x,\varepsilon) = \exp\left[\int_a^x \delta_0(\varepsilon) u_0(x) dx\right] \tag{19}$$

After the substitution of equation (19) into equation (17) we obtain the residual

$$R = \varepsilon^2\left(\delta_0^2 u_0^2 + \delta_0 u_0'\right) - f(x) \tag{20}$$

or, from equation (18),

$$R = \varepsilon^2\left(\delta_0^2 f(x) + \delta_0 \frac{f'(x)}{\pm 2\sqrt{f(x)}}\right) - f(x) \tag{21}$$

We apply the Bubnov-Galerkin criterion (6) and after that we obtain a quadratic equation for δ_0.

$$\delta_0^2 \int_a^b \varepsilon 2\left(\pm f^{3/2}(x)\right) dx + \delta_0 \int_a^b \varepsilon^2\left(\frac{f'(x)}{2}\right) dx - \int_a^b \left(\pm f^{3/2}(x)\right) dx = 0 \tag{22}$$

The solutions of this equation are

$$\delta_0^{1,2} = \mp \frac{f(b) - f(a)}{4\int_a^b \left(f^{3/2}(x)\right) dx} + \sqrt{\frac{1}{\varepsilon^2} + \left(\frac{f(b) - f(a)}{4\int_a^b \left(f^{3/2}(x)\right) dx}\right)^2} \tag{23}$$

Thus, finally we obtain the hybrid WKB-Galerkin solution of equation (17) as

$$\tilde{u}(x,\varepsilon) = \exp\left[\left(\int_a^x \frac{f(b)-f(a)}{4\int_a^b \left(f^{3/2}(x)\right)dx} + \sqrt{\frac{1}{\varepsilon^2} + \left(\frac{f(b)-f(a)}{4\int_a^b \left(f^{3/2}(x)\right)dx}\right)^2}\right)\sqrt{f(x)dx}\right] \tag{24}$$

The solutions for equations (14) and (15) can be obtained by substitution of corresponding functions $f(x)$ into equation (23).

3 A Hybrid Approximate Solution of the Euler Equation

For the case of Euler's equation, when the function $f(x) = \frac{1}{x^2}$, the initial equation (7) has an exact solution for parameter $\varepsilon^2 < 4$.

$$u(x,\varepsilon) = \sqrt{x}\cos(\omega \ln x) + \left[\frac{1}{\sqrt{e}\sin\omega} - \cot\omega\right]\sqrt{x}\sin(\omega \ln x) \tag{25}$$

where $\omega = \sqrt{\frac{1}{\varepsilon^2} - \frac{1}{4}}$

We take into account the boundary conditions as

$$u(1) = u(e) = 1 \tag{26}$$

Function (25) has singularities at

$$\varepsilon = \left(\frac{1}{\pi^2} + \frac{1}{4}\right)^{1/2}, \left(\frac{1}{4^2\pi^2} + \frac{1}{4}\right)^{1/2}, \ldots, \left(\frac{1}{n^2\pi^2} + \frac{1}{4}\right)^{1/2}, \ldots \tag{27}$$

After the substitution of equation (8) into equation (7) we obtain

$$\exp\left[\int_a^x \left(\frac{1}{\varepsilon}u_0 + u_1\right)dx\right]\left(u_0^2 + 2\varepsilon u_0 u_1 + \varepsilon^2 u_1^2 + \varepsilon u_0' + \varepsilon^2 u_1' + \frac{1}{x^2}\right) = 0 \tag{28}$$

Equating the coeffcients in equation (28) of the same power of ε we find that

$$u_0 = \pm\frac{i}{x} \quad \text{and} \quad u_1 = \frac{1}{2x} \tag{29}$$

For this type of equation the two term WKB approximation becomes

$$\tilde{u}(x,\varepsilon) = \sqrt{x}\cos\left(\frac{1}{\varepsilon}\ln x\right) + \left[\frac{1}{\sqrt{e}}\sin\frac{1}{\varepsilon} - \cot\frac{1}{\varepsilon}\right]\sqrt{x}\sin\left(\frac{1}{\varepsilon}\ln x\right) \tag{30}$$

Following the approach that we have discussed above, the approximation to function $u(x,\varepsilon)$ can be written in the form (10). Substituting equation (10) into Euler's equation we obtain the residual R.

$$R = \varepsilon^2\left[\left(\delta_{01} + i\delta_{02}\right)^2 u_0^2 + \left(\delta_{01} + i\delta_{02}\right)\dot{u}_0\right] + \frac{1}{x^2} \tag{31}$$

The set of equations (12) for the coefficients δ_{01} and δ_{02} becomes

$$
\begin{aligned}
&\varepsilon^2\left(\mp\delta_{01}^2 \pm\delta_{02}^2 + \delta_{02}\right) + (\pm 1) = 0 \\
&\pm 2\delta_{02} + 1 = 0
\end{aligned}
\tag{32}
$$

From equations (32) it follows that

$$
\begin{aligned}
\delta_{01} &= \sqrt{\frac{1}{\varepsilon^2} - \frac{1}{4}} \\
\delta_{02} &= \mp\frac{1}{2}
\end{aligned}
\tag{33}
$$

After the substitution of equation (33) into equation (10) and taking into account the boundary conditions (26) we obtain the hybrid WKB-Galerkin first approximation solution of Euler's equation.

$$
\tilde{u}(x,\varepsilon) = \sqrt{x}\cos(\omega\ln x) + \left[\frac{1}{\sqrt{e}\sin\omega} - \cot\omega\right]\sqrt{x}\sin(\omega\ln x) \tag{34}
$$

where $\omega = \delta_{01}$.

We can obtain the solution (34) at once if we substitute the function $f(x) = \frac{1}{x^2}$ into equation (14). It is easily seen, that the function $\tilde{u}(x,\varepsilon)$ coincides with the exact solution (16) using the one-term hybrid WKB-Galerkin method.

The two-term WKB approximation in equation (30) is not exact but it is accurate up to some distance from the singularity points. In Figure 1 we compare the calculations for the two-term WKB approximation with the exact (or with the one-term hybrid WKB-Galerkin method) solution for parameter $\frac{1}{\varepsilon} \to \sqrt{4\pi^2 + \frac{1}{4}}$ when $\frac{1}{\varepsilon} = 6.28$.

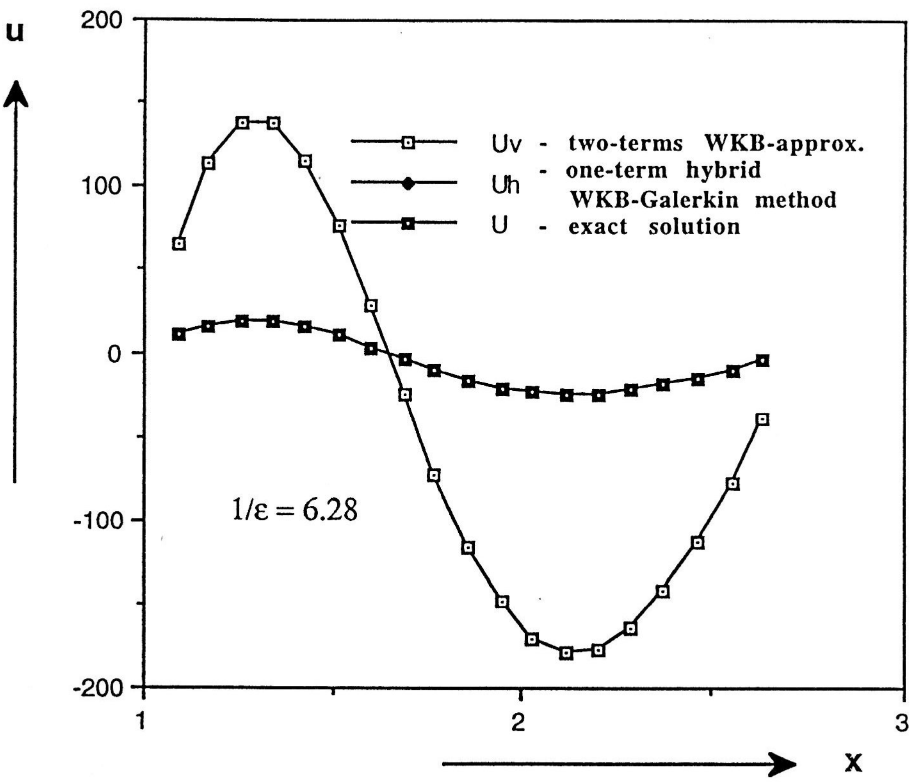

Figure 1. Solution of Euler's Equation by Hybrid WKB-Galerkin Method

Some of the reasons why the hybrid WKB-Galerkin method provides a good enough approximation for problem (16) we will discuss in the next section.

For comparison, in Figure 2 are represented the results of calculations according to the perturbation-Galerkin method in the spirit of Geer and Andersen for this problem, with parameter $\frac{1}{\varepsilon} = 0.6$.

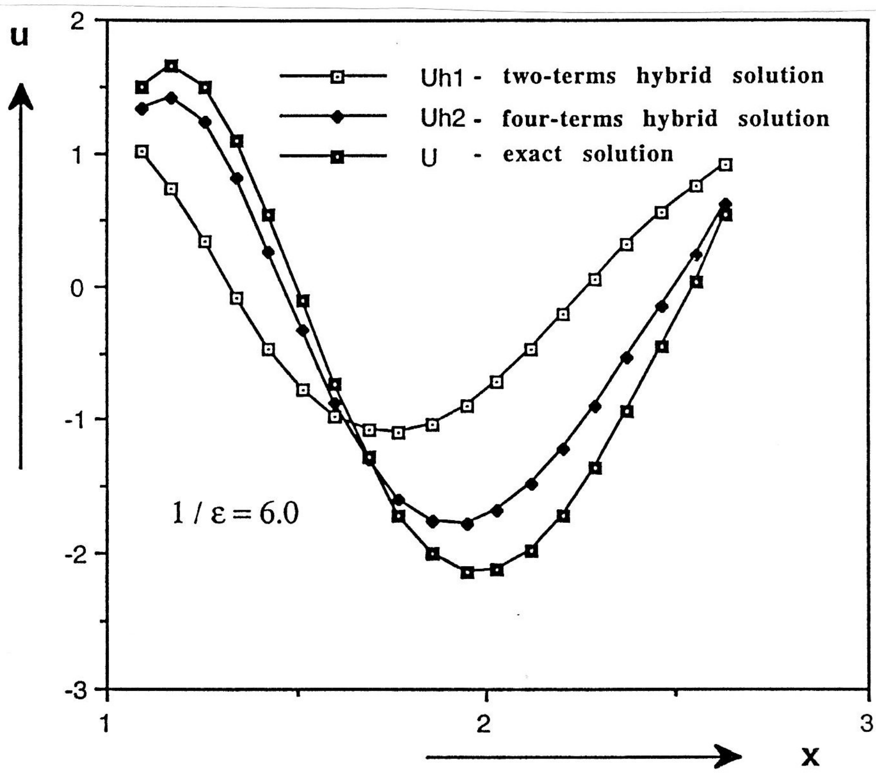

Figure 2. Hybrid Regular Perturbation-Galerkin Solution

4 Application of the Hybrid WKB-Galerkin Technique for the Stress Strain State of an Orthotropic Cone under an Axisymmetric Loading

Let an orthotropic shell of revolution be given an axis of rotation z. Suppose that the shell is loaded axisymmetrically with respect to this axis.

$$X = X(s) \qquad\qquad Z = Z(s) \qquad\qquad Y = 0$$

The boundary conditions shall also be axisymmetric. The shell is an orthotropic body with an anisotropy of rotation (Figure 3).

We consider the stress-strain state of the orthotropic conical shell given by a differential equation in complex form (Ambartsumian, 1974).

$$\ddot{\sigma} - \frac{1}{s'-s}\dot{\sigma} - \left(\lambda\frac{1}{(s'-s)^2} - \frac{ik^2 ctg\alpha}{h(s'-s)}\right)\sigma = \Phi(s) \tag{35}$$

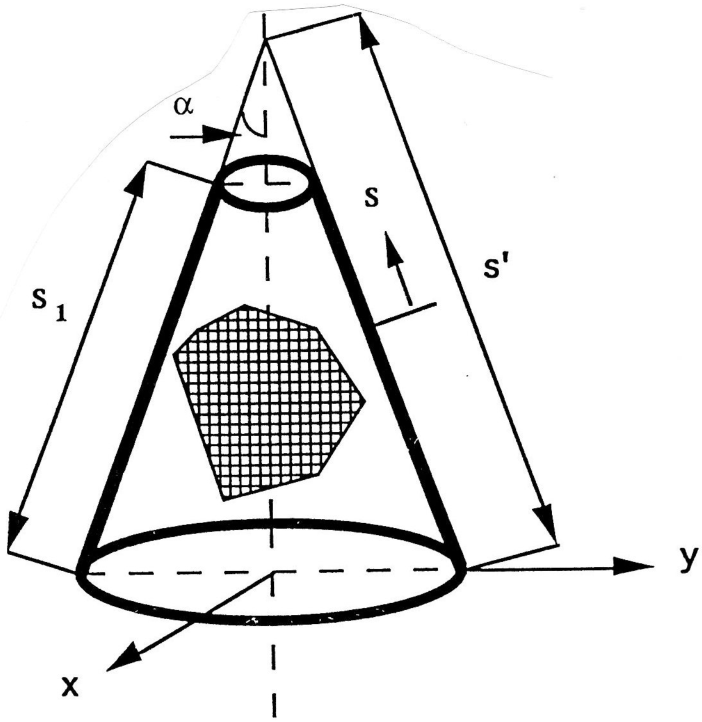

Figure 3. Geometry of the Orthotropic Cone

The boundary conditions for this problem are

$$\sigma(s_0) = \sigma(s_1) = 0 \tag{36}$$

where

$$\sigma(s) = W(s) - i\frac{c_{11}k^2}{\Omega_0}V(s) \tag{37}$$

and $W(s)$ and $V(s)$ are the functions to be determined, $\Phi(s)$ is the surface load function, $\lambda = \frac{c_{22}}{c_{11}} = \frac{\Delta_{22}}{\Delta_{11}}$, h is the shell thickness, $k^2 = \sqrt{\frac{\Omega_0}{c_{11}\Delta_{11}}}$, $\Omega_0 = c_{11}c_{22} - c_{12}^2$, and $c_{11}, c_{22}, c_{12}, \Delta_{11}, \Delta_{22}$ are corresponding parameters of rigidity of the cone. One obtains

$$\Phi(s) = -\frac{F_2}{(s'-s)\sin\alpha\Delta_{11}} - \frac{ik^2}{(s'-s)\sin\alpha\Omega_0}\left[c_{12}\frac{dF_1}{ds} - c_{22}\frac{F_1}{s'-s}\right] \tag{38}$$

where $F_i(s)$ are the functions of surface loading.

$$\begin{aligned} F_1 &= \sin\alpha\int_{s_0}^{s}(s'-s)\sin\alpha\, E_r ds + \cos\alpha\left(\frac{P_z^0}{2\pi} - \int_{s_0}^{s}(s'-s)\sin\alpha\, E_z ds\right) \\ F_2 &= -\cos\alpha\int_{s_0}^{s}(s'-s)\sin\alpha\, E_r ds + \sin\alpha\left(\frac{P_z^0}{2\pi} - \int_{s_0}^{s}(s'-s)\sin\alpha\, E_z ds\right) \end{aligned} \tag{39}$$

The quantities E_r and E_z are the components of the surface loading with respect to the r and z directions respectively, P_z^0 is the value of the main vector of an external force applied to the circle $s = s_0$ with the radius r_0, as shown in Figure 4.

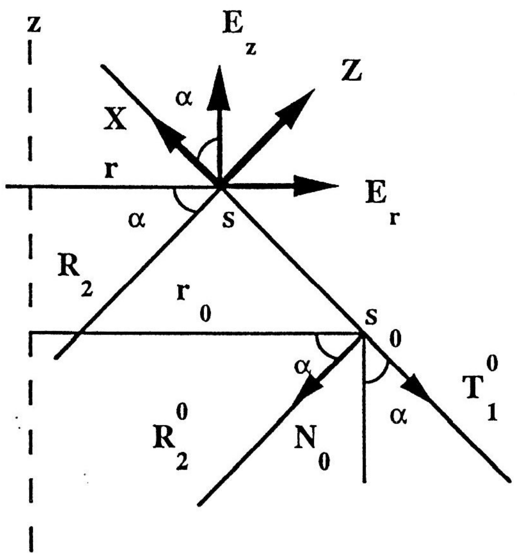

Figure 4. Relations between the Components of Surface Loading

From Figure 4 it follows that

$$E_r \;=\; Z\cos\alpha \;-\; X\sin\alpha$$

$$E_r \;=\; Z\sin\alpha \;+\; X\cos\alpha \tag{40}$$

$$P_z^0 \;=\; \left(T_1^0\cos\alpha \;-\; N^0\sin\alpha\right)2\pi r_0$$

Considering this problem for large values of the parameter k^2 the solution obtained by Ambartsumian (1974) has the form

$$\overline{W}_A(s) \;=\; \left(E_1\cos\beta \;-F_1\sin\beta\right)e^{-\beta} + \left(E_2\cos\beta \;+F_2\sin\beta\right)e^{\beta}$$

$$\overline{V}_A(s) \;=\; -\frac{\Omega_0}{c_{11}k^2}\left[\left(E_1\sin\beta \;+F_1\cos\beta\right)e^{-\beta} - \left(E_2\sin\beta \;-F_2\cos\beta\right)e^{\beta}\right] \;-\; \frac{F_2(s)}{\cos\alpha} \tag{41}$$

where

$$\beta \;=\; \sqrt{\frac{k^2}{2}}\int_{s_0}^{s}\frac{\cot^{1/2}\alpha}{\sqrt{s'-s}}\,ds \tag{42}$$

In order to construct the hybrid WKB-Galerkin solution we consider the initial equation (35) as

$$\varepsilon^2 \;=\; \ddot{\varphi} \;-\; \left\{\frac{4\lambda-1}{4k^2\left(s'-s\right)^2} \;-\; i\,\frac{\cot(\alpha)}{h\left(s'-s\right)}\right\}\varphi \;=\; \Phi_1(s) \tag{43}$$

where

$$\Phi_1(s) = \Phi(s)\exp\left(-\frac{1}{2}\int_{s_0}^{s}\frac{ds}{s'-s}\right) \tag{44}$$

with the boundary conditions

$$\varphi(s_0) = \varphi(s_1) = 0 \tag{45}$$

where

$$\begin{aligned} \varphi(s) &= \sigma(s)\exp\left(-\frac{1}{2}\int_{s_0}^{s}\frac{ds}{s'-s}\right) \\ \varepsilon^2 &= \frac{1}{k^2} \end{aligned} \tag{46}$$

The partial solution of differential equation (43) is

$$\varphi^* = \sigma^*\exp\left(-\frac{1}{2}\int_{s_0}^{s}\frac{ds}{s'-s}\right) \qquad \text{with} \qquad \sigma^* = i\frac{c_{11}k^2}{\Omega_0\cos(\alpha)} \tag{47}$$

where F_2 is an external loading function. For example, we may assume the function F_2 in the form

$$F_2 = \sin\alpha\left(\frac{z_0}{2}(s-s_0)\,(s+s_0-2s') + \frac{p_0}{2\pi}\right) \tag{48}$$

where z_0 and p_0 are parameters.

The WKB approximation of the homogeneous equation that corresponds to initial equation (43) is

$$\begin{aligned} W_{WKB}(s) &= (E_1\cos\beta_1 - F_1\sin\beta_1)e^{(\alpha_1+\gamma)} + (E_2\cos\beta_1 + F_2\sin\beta_1)e^{(-\alpha_1+\gamma)} \\ V_{WKB}(s) &= -\frac{\Omega_0}{c_{11}k^2}\left[(E_1\sin\beta_1 + F_1\cos\beta_1)e^{(\alpha_1+\gamma)} - (E_2\sin\beta_1 - F_2\cos\beta_1)e^{(-\alpha_1+\gamma)}\right] - \frac{F_2(s)}{\cos\alpha} \end{aligned} \tag{49}$$

with

$$\begin{aligned} \beta_1 &= \frac{1}{\varepsilon}\int_{s_0}^{s} R_2(s)ds \qquad \alpha_1 = \frac{1}{\varepsilon}\int_{s_0}^{s} R_1(s)ds \qquad \gamma = \frac{1}{2}\int_{s_0}^{s}\frac{1}{s'-s} \\ R_1 &= \mathrm{Re}\sqrt{f} \qquad R_2 = \mathrm{Im}\sqrt{f} \end{aligned} \tag{50}$$

Now we consider an approximation of $\varphi(s)$ as

$$\varphi_H(s) = \exp\int_{s_0}^{s}\delta_0\varphi_0(s)ds \tag{51}$$

Following the Bubnov-Galerkin method we obtain a quadratic equation in the form (21). The solutions of this equation for our problem are

$$\delta_0^{1,2} = \mp \frac{f(s_1)-f(s_0)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds} + \sqrt{\frac{1}{\varepsilon^2} + \left(\frac{f(s_1)-f(s_0)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds}\right)^2} \tag{52}$$

where

$$f(s) = \frac{4\lambda-1}{4k^2(s'-s)^2} - i\frac{\cot\alpha}{h(s'-s)} \tag{53}$$

Thus we have obtained two solutions for the function $\varphi_0(s)$

$$\begin{aligned} \varphi_0(s) &= \sqrt{f(s)} = R_1(s) + \operatorname{Im} R(s) \\ \varphi_0(s) &= -\sqrt{f(s)} = -R_1(s) - \operatorname{Im} R(s) \end{aligned} \tag{54}$$

Finally the hybrid WKB-Galerkin solution of the initial equation (43) can be written as

$$\begin{aligned} W_H(s) &= (E_1\cos\beta_{21} - F_1\sin\beta_{21})e^{(\alpha_{21}+\gamma)} + (E_2\cos\beta_{22} + F_2\sin\beta_{22})e^{(-\alpha_{22}+\gamma)} \\ V_H(s) &= -\frac{\Omega_0}{c_{11}k^2}\left[(E_1\sin\beta_{21} + F_1\cos\beta_{21})e^{(\alpha_{21}+\gamma)} + (E_2\sin\beta_{22} + F_2\cos\beta_{22})e^{(-\alpha_{22}+\gamma)}\right] - \frac{F_2(s)}{\cos\alpha} \end{aligned} \tag{55}$$

where

$$\begin{aligned} \beta_{21} &= \delta_{01}^1\int_{s_0}^{s} R_2(s)ds + \delta_{02}^1\int_{s_0}^{s} R_1(s)ds \\ \alpha_{21} &= -\delta_{01}^1\int_{s_0}^{s} R_1(s)ds - \delta_{02}^1\int_{s_0}^{s} R_2(s)ds \\ \beta_{22} &= -\delta_{01}^2\int_{s_0}^{s} R_2(s)ds - \delta_{02}^2\int_{s_0}^{s} R_1(s)ds \\ \alpha_{22} &= -\delta_{01}^2\int_{s_0}^{s} R_1(s)ds + \delta_{02}^2\int_{s_0}^{s} R_2(s)ds \\ \delta_{01}^1 &= \operatorname{Re}\left[\frac{f(s_0)-f(s_1)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds} + \sqrt{\frac{1}{\varepsilon^2} + \left(\frac{f(s_1)-f(s_0)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds}\right)^2}\right] \\ \delta_{02}^1 &= \operatorname{Im}\left[\frac{f(s_0)-f(s_1)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds} - \sqrt{\frac{1}{\varepsilon^2} + \left(\frac{f(s_1)-f(s_0)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds}\right)^2}\right] \end{aligned} \tag{56}$$

$$\delta_{01}^2 = \mathrm{Re}\left[\frac{f(s_0)-f(s_1)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds} - \sqrt{\frac{1}{\varepsilon^2} + \left(\frac{f(s_1)-f(s_0)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds}\right)^2}\right]$$

$$\delta_{02}^2 = \mathrm{Im}\left[\frac{f(s_0)-f(s_1)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds} - \sqrt{\frac{1}{\varepsilon^2} + \left(\frac{f(s_1)-f(s_0)}{4\int_{s_0}^{s_1} f^{\frac{3}{2}}(s)ds}\right)^2}\right]$$

The coefficients E_1, E_2, F_1, F_2 in the solution (55) can be determined from the boundary conditions. The exact solution (Ambartsumian, 1974) of the problem is

$$\sigma = C_1 I_{\sqrt{4\lambda}}\left(2\sqrt{-\frac{i\cot\alpha}{\varepsilon^2 h}x}\right) + C_2 I_{-\sqrt{4\lambda}}\left(2\sqrt{-\frac{i\cot\alpha}{\varepsilon^2 h}x}\right) + i\frac{c_{11}k^2}{\Omega_0\cos\alpha}F_2 \qquad (57)$$

where $I_p(z), I_{-p}(z)$ are Bessel functions.

To illustrate the accuracy of the hybrid method we consider the case when $p=\frac{5}{2}$. In this case the Bessel functions can be expressed as elementary functions. We compare our hybrid WKB-Galerkin approximation (55) with the exact solution (57) as well as with the one-term WKB approximation, with Ambartsumian's solution (41) and with the purely numerical solutions for small parameters ε. In the Figures 5 to 7 are presented the results of calculations for the same parameters ε and for $\lambda=\frac{25}{16}$, $s=10$, $\alpha=\frac{\pi}{3}$, $h=0.5$, $c_{11}=1.1$, $\Omega_0=0.9$, $x_1=8$, $p_z^0=7$, $X=0$, $Z=4$.

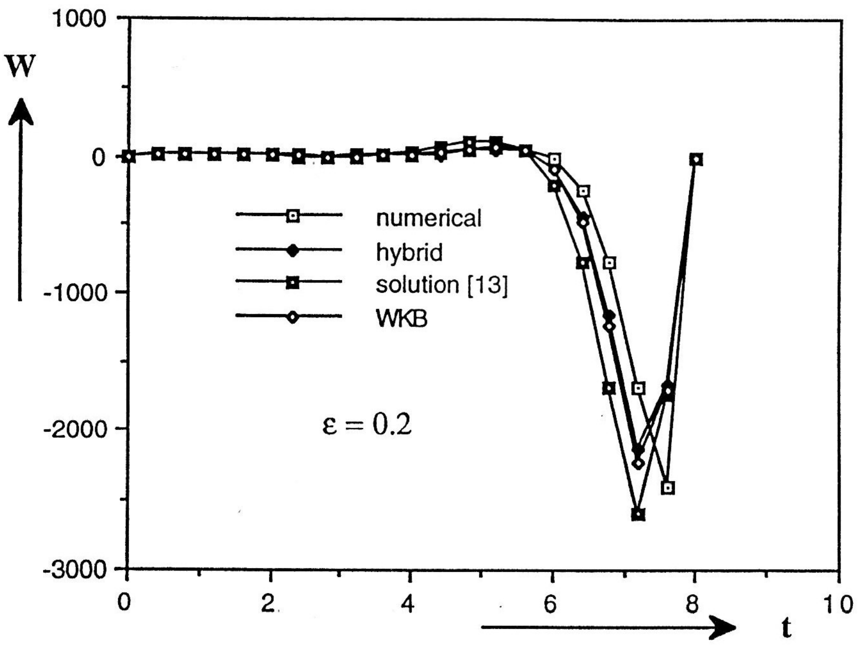

Figure 5. Hybrid WKB-Galerkin Solution for the Parameter $\varepsilon = 0.2$

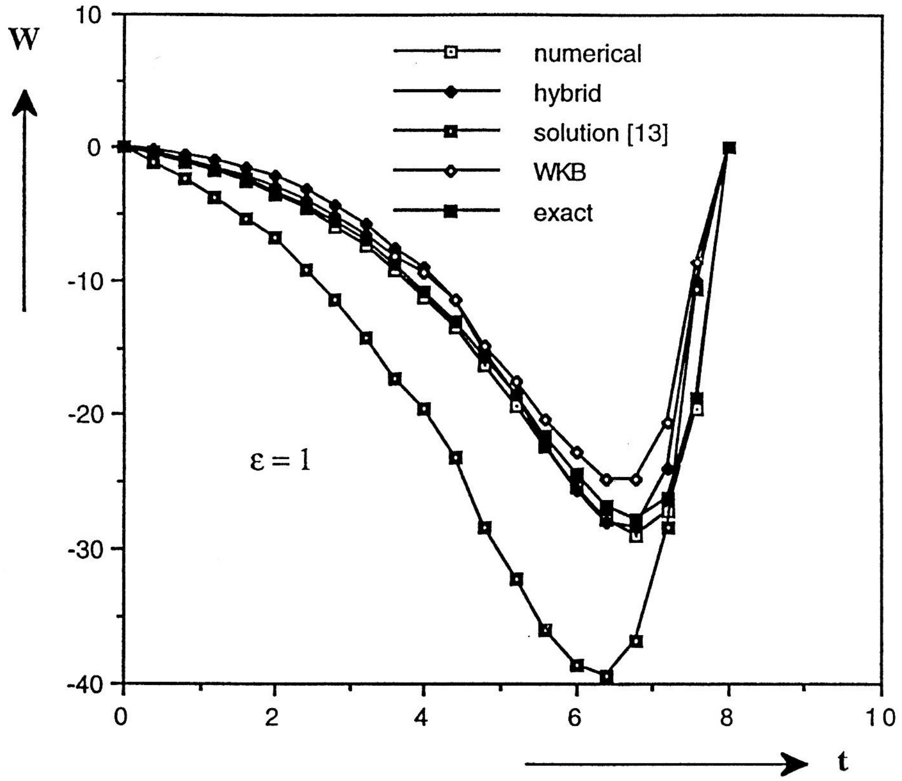

Figure 6. Hybrid WKB-Galerkin Solution for the Parameter $\varepsilon = 1$

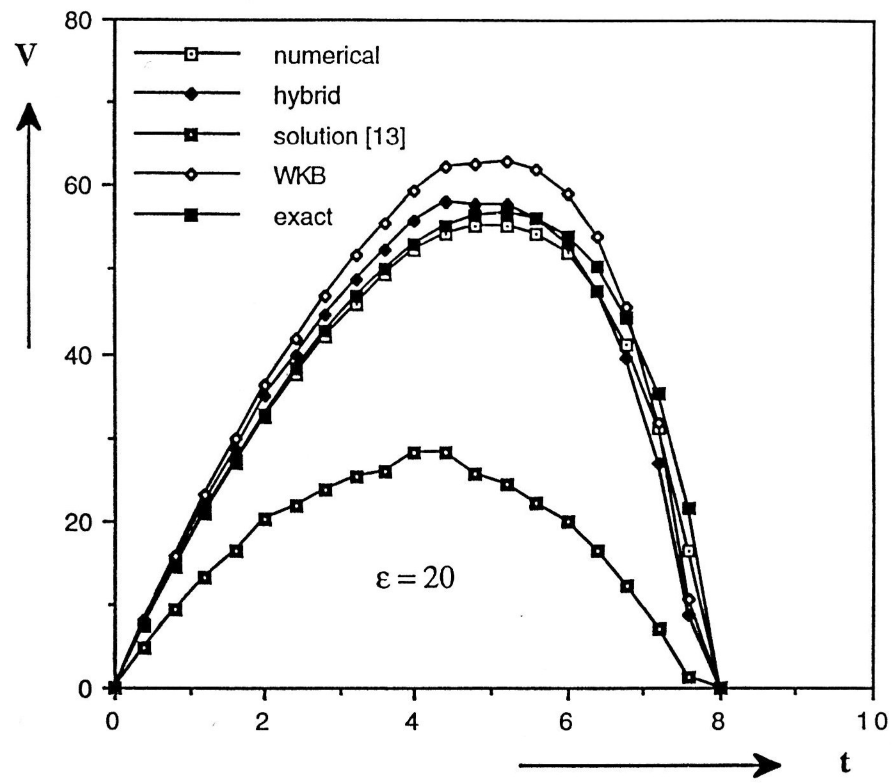

Figure 7. Hybrid WKB-Galerkin Solution for the Parameter $\varepsilon = 20$

We note that the hybrid WKB-Galerkin method is accurate up even to large values of parameter ε in comparisation with the one-term WKB-solution.

5 Concluding Remarks

Special attention is paid in this paper to the analysis of mechanical systems whose behavior is described by differential equations in complex form. Various mechanical problems of buckling and vibrations of structures, especially with variable configuration and rigidity, may be solved effectively on the basis of the approximate analytical-numerical approach presented. The proposed WKB-Galerkin method can also be used for more complicated problems such as linear systems with mixed boundary conditions or investigations into the initial postbuckling behavior of nonhomogeneous system.

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