TECHNISCHE MECHANIK, Band 15, Heft 4; (1995), 333-339
Manuskripteingang: 09.Oktober 1995

Distortional Hardening within a Cubic Yield Theory

R. Kreißig, G. Grewolls

Evolution equations for the coordinates of the state tensors of a cubic yield condition are formulated which result from a formal generalization of an approach by Danilov. The corresponding deformation law and the set of evolution equations are numerically integrated for selected loading paths in stress subspaces. Some of the experimentally observed effects are shown to be correctly described.

1 Introduction

In the presently available computation software for non-linear problems of solid mechanics (e. g. ABAQUS, ANSYS, MARC, PSU) plastic behavior of material is described based on a quadratic flow potential and for the most part in combination with the isotropic or the kinematic hardening rule. These restrictions are certainly sufficient in numerous cases, even if partially caused by missing material data. However, experimental studies of the last years show that the measured yield locus curves (in special subspaces, e.g. $\sigma_1, \sigma_2; \sigma, \tau$) may widely differ from the elliptic form, either in the initial state of the material (Taketa and Nasu, 1991) or as a result of hardening (Phillips and Tang, 1972; Khan and Wang, 1993). This requires abandoning the approach of an exclusively quadratic yield theory.

Wegener and Schlegel (1994) analyzed yield conditions of various types, e.g. based on the full integrity basis of second order tensors α (kinematic motion) and $\overline{\sigma} = \sigma - \alpha$ (see Boehler, 1987) or on tensorial internal variables of up to sixth order, with regard to the quality of their adaption to experiments published by Phillips and Tang (1972). Depending on the flexibility of the cubic term of the yield conditions some very good results have been achieved (Wegener, 1991; Wegener and Schlegel, 1994).

As for corresponding evolution equations, however, the situation looks quite different. Here only few proposals have been published (Voyiadjis and Foroozesh, 1990). Tests concerning their practicability have not been accessible so far. Therefore it is the aim of our numerical research to gain experience for the formulation of evolution equations which permit a close reproduction of experimentally observed effects.

2 Theoretical Foundations

Following the approach by Baltov and Sawczuk (1965) a yield function of third degree is formulated

$$f \equiv K_0 + K_{ijkl}(\sigma_{ij} - K_{ij})(\sigma_{kl} - K_{kl}) + K_{ijklmn}(\sigma_{ij} - K_{ij})(\sigma_{kl} - K_{kl})(\sigma_{mn} - K_{mn}) = 0 \tag{1}$$

Further the following assumptions and agreements are made:

- restriction to small strains
- plastic incompressibility
- additive decomposition of the strain rate tensor

$$\dot{\varepsilon} = \dot{\varepsilon}^{el} + \dot{\varepsilon}^{pl} \tag{2}$$

- application of the normality rule

$$\dot{\varepsilon}_{ij}^{pl} = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda} f_{ij} \tag{3}$$

Danilov (1971) proposed for the evolution of the coordinates of the state tensor of fourth order the formulation

$$K_{ijkl} = I_{ijkl}\left(\varepsilon_v^{pl}\right) + \int_0^{\varepsilon_v^{pl}} A_D\left(\bar{\varepsilon}_v^{pl}\right) \frac{d\varepsilon_{ij}^{pl}}{d\bar{\varepsilon}_v^{pl}} \frac{d\varepsilon_{kl}^{pl}}{d\bar{\varepsilon}_v^{pl}} d\bar{\varepsilon}_v^{pl} \tag{4}$$

which requires an isotropic initial state. The isotropic part of hardening is registered by the tensor I_{ijkl}, the free value of which is then a function of the plastic part of the equivalent strain, supposing plastic incompressibility.

Equation (4) can formally be applied to the coordinates of the other material tensors. The equations which are obtained in this way, completed by an analogous relation for K_0, form the basis of our present research (see Grewolls and Kreißig, 1995):

$$\begin{aligned}
K_0 &= \overset{(0)}{K}_0 + \int_0^{\varepsilon_v^{pl}} A\left(\bar{\varepsilon}_v^{pl}\right) d\bar{\varepsilon}_v^{pl} \\
K_{ij} &= \overset{(0)}{K}_{ij} + \int_0^{\varepsilon_v^{pl}} B\left(\bar{\varepsilon}_v^{pl}\right) \frac{d\varepsilon_{ij}^{pl}}{d\bar{\varepsilon}_v^{pl}} d\bar{\varepsilon}_v^{pl} \\
K_{ijkl} &= \overset{(0)}{K}_{ijkl} + \int_0^{\varepsilon_v^{pl}} C\left(\bar{\varepsilon}_v^{pl}\right) \frac{d\varepsilon_{ij}^{pl}}{d\bar{\varepsilon}_v^{pl}} \frac{d\varepsilon_{kl}^{pl}}{d\bar{\varepsilon}_v^{pl}} d\bar{\varepsilon}_v^{pl} \\
K_{ijklmn} &= \overset{(0)}{K}_{ijklmn} + \int_0^{\varepsilon_v^{pl}} D\left(\bar{\varepsilon}_v^{pl}\right) \frac{d\varepsilon_{ij}^{pl}}{d\bar{\varepsilon}_v^{pl}} \frac{d\varepsilon_{kl}^{pl}}{d\bar{\varepsilon}_v^{pl}} \frac{d\varepsilon_{mn}^{pl}}{d\bar{\varepsilon}_v^{pl}} d\bar{\varepsilon}_v^{pl}
\end{aligned} \tag{5}$$

where

$$\varepsilon_v^{pl} = \int_0^{\varepsilon_v^{pl}} \sqrt{\frac{2}{3} d\bar{\varepsilon}_v^{pl} d\bar{\varepsilon}_v^{pl}}$$

is the plastic part of the equivalent strain which has the character of a scale variable and is not workconjugate with an equivalent stress based on equation (1).

In accordance with the yield condition (1), an isotropic initial state may occur. In this case all $\overset{(0)}{K}_{ij}$ are equal to zero, as where $\overset{(0)}{K}_{ijkl}$ and $\overset{(0)}{K}_{ijklmn}$ are equal to the isotropic tensors of corresponding orders. By using the relations

$$\frac{d}{dt} = \frac{d}{d\varepsilon_v^{pl}} \frac{d\varepsilon_v^{pl}}{dt} = \frac{d}{d\varepsilon_v^{pl}} \dot{\varepsilon}_v^{pl} \qquad \frac{d\varepsilon_{ij}^{pl}}{d\varepsilon_v^{pl}} = \frac{d\varepsilon_{ij}^{pl}}{dt} \frac{dt}{d\varepsilon_v^{pl}} = \frac{\dot{\varepsilon}_{ij}^{pl}}{\dot{\varepsilon}_v^{pl}}$$

the integral form of the evolution equations can be written in rate formulation as

$$\begin{aligned}
\dot{K}_0 &= A\left(\varepsilon_v^{pl}\right)\dot{\varepsilon}_v^{pl} \\
\dot{K}_{ij} &= B\left(\varepsilon_v^{pl}\right)\dot{\varepsilon}_{ij}^{pl} \\
\dot{K}_{ijkl} &= \frac{C\left(\varepsilon_v^{pl}\right)}{\dot{\varepsilon}_v^{pl}} \dot{\varepsilon}_{ij}^{pl} \dot{\varepsilon}_{kl}^{pl} \\
\dot{K}_{ijklmn} &= \frac{D\left(\varepsilon_v^{pl}\right)}{\left(\dot{\varepsilon}_v^{pl}\right)^2} \dot{\varepsilon}_{ij}^{pl} \dot{\varepsilon}_{kl}^{pl} \dot{\varepsilon}_{mn}^{pl}
\end{aligned} \tag{6}$$

where

$$\dot{\varepsilon}_{\mathrm{v}}^{pl} = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^{pl}\dot{\varepsilon}_{ij}^{pl}}$$

is the plastic part of the equivalent strain rate.

Since the strain-controlled numerical simulation requires to give the entire deformation rate, a new formulation of equation (3) and equation (6) is necessary. By use of equation (2), the linear elasticity law

$$\dot{\sigma}_{ij} = E_{ijkl}\dot{\varepsilon}_{kl}^{el} \tag{7}$$

and the consistency condition

$$\dot{f} = \frac{\partial f\left(\sigma_{ij}, K_{\dots}^{(k)}\right)}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \dot{\lambda}\sum_{k=0,2,4,6}\frac{\partial f\left(\sigma_{ij}, K_{\dots}^{(k)}\right)}{\partial K_{\dots}^{(k)}}q_{\dots}^{(k)} = 0 \tag{8}$$

where

$\dot{\lambda}q_{\dots}^{(k)} = \dot{K}_{\dots}^{(k)}$,

$q_{\dots}^{(k)}$ ... coordinates of tensor valued functions

$K_{\dots}^{(k)}$ ... coordinates of the state tensors of $0^{th}, 2^{nd}, 4^{th}$ and 6^{th} order

one obtains the deformation law

$$\begin{aligned}\dot{\sigma}_{ij} &= \left(E_{ijop} - \frac{E_{ijmn}E_{uwop}f_{mn}f_{uw}}{E_{qrst}f_{qr}f_{st} + V}\right)\dot{\varepsilon}_{op} \\ \dot{\boldsymbol{\sigma}} &= \mathbf{G}\left(\boldsymbol{\sigma}, \mathbf{K}, \varepsilon_{\mathrm{v}}^{pl}\right)\dot{\boldsymbol{\varepsilon}}\end{aligned} \tag{9}$$

and the evolution equations

$$\begin{aligned}\dot{K}_0 &= A\left(\varepsilon_{\mathrm{v}}^{pl}\right)\sqrt{\frac{2}{3}f_{ab}f_{ab}}\frac{E_{uwop}f_{uw}}{E_{qrst}f_{qr}f_{st} + V}\dot{\varepsilon}_{op} \\ \dot{K}_{ij} &= B\left(\varepsilon_{\mathrm{v}}^{pl}\right)\frac{E_{uwop}f_{uw}}{E_{qrst}f_{qr}f_{st} + V}f_{ij}\dot{\varepsilon}_{op} \\ \dot{K}_{ijkl} &= \frac{C\left(\varepsilon_{\mathrm{v}}^{pl}\right)}{\sqrt{\frac{2}{3}f_{ab}f_{ab}}}\frac{E_{uwop}f_{uw}}{E_{qrst}f_{qr}f_{st} + V}f_{ij}f_{kl}\dot{\varepsilon}_{op} \\ \dot{K}_{ijklmn} &= \frac{D\left(\varepsilon_{\mathrm{v}}^{pl}\right)}{\frac{2}{3}f_{ab}f_{ab}}\frac{E_{uwop}f_{uw}}{E_{qrst}f_{qr}f_{st} + V}f_{ij}f_{kl}f_{mn}\dot{\varepsilon}_{op} \\ \mathbf{K} &= \mathbf{H}\left(\boldsymbol{\sigma}, \mathbf{K}, \varepsilon_{\mathrm{v}}^{pl}\right)\dot{\boldsymbol{\varepsilon}}\end{aligned} \tag{10}$$

which contain now the so-called hardening function V.

$$V = -\sum_{k=0,2,4,6} \frac{\partial f\left(\sigma_{ij}, K_{\dots}^{(k)}\right)}{\partial K_{\dots}^{(k)}} q_{\dots}^{(k)} = -\left(\frac{\partial f}{\partial K_0} q_0 + \frac{\partial f}{\partial K_{ij}} q_{ij} + \frac{\partial f}{\partial K_{ijkl}} q_{ijkl} + \frac{\partial f}{\partial K_{ijklmn}} q_{ijklmn}\right)$$

$$= -\left[A\left(\varepsilon_v^{pl}\right)\sqrt{\frac{2}{3} f_{ab} f_{ab}} + B\left(\varepsilon_v^{pl}\right) f_{ij}\sigma_{ij} + \frac{C\left(\varepsilon_v^{pl}\right)}{\sqrt{\frac{2}{3} f_{ab} f_{ab}}} f_{ij}\sigma_{ij} f_{kl}\sigma_{kl} + \frac{D\left(\varepsilon_v^{pl}\right)}{\frac{2}{3} f_{ab} f_{ab}} f_{ij}\sigma_{ij} f_{kl}\sigma_{kl} f_{mn}\sigma_{mn}\right] \tag{11}$$

Equations (9) and (10) are valid for $f = 0$ and $f_{ij}\dot{\sigma}_{ij} > 0$. In the case of elasticity, for neutral loading ($f = 0$ and $f_{ij}\dot{\sigma}_{ij} = 0$) and for unloading ($f = 0$ and $f_{ij}\dot{\sigma}_{ij} < 0$) the above relations reduce to

$$\dot{\sigma}_{ij} = E_{ijop}\dot{\varepsilon}_{op}$$
$$\dot{K}_{\dots}^{(k)} = 0$$

In the deformation law (9) the coordinates of the stress rate tensor and of the deformation rate tensor are linearly related, independent of the degree of the yield condition. In the numerical simulations, the set of equations (9) and (10) is, in consideration of equation (11), explicitly integrated.

3 Numerical Evaluation

For the numerical evaluation which is, according to the available experimental results, restricted to the subspaces σ_1, σ_2 and σ, τ, elastic isotropy and on the yield limit, an initial plastic isotropy have been assumed. An additional assumption is that a flow curve is known, taken from a specimen with rectangular cross-section, the axis of which is to be identical with the principal axis 1 in the case of initial orthotropy. This flow curve is approximated by the approach (Landgraf and Bergander, 1985)

$$\sigma_F\left(\varepsilon_v^{pl}\right) = \sigma_{FO}\left\{1 + \frac{c_0}{c_2}\left[\left(\varepsilon_v^{pl} + c_1\right)^{c_2} - c_1^{\,c_2}\right]\right\} \tag{12}$$

Further the R_1 value

$$R_1\left(\varepsilon_v^{pl}\right)\Big|_{\sigma_1} = \frac{d\varepsilon_2^{pl}}{d\varepsilon_3^{pl}} \tag{13}$$

is supposed to be known. It describes the ratio of plastic strain increments in the cross directions during an uniaxial tensile test and is equal to unity for plastic isotropy. The hardening is split up into four parts.

$$\sigma_F = \sigma_F\left(A\left(\varepsilon_v^{pl}\right), B\left(\varepsilon_v^{pl}\right), C\left(\varepsilon_v^{pl}\right), D\left(\varepsilon_v^{pl}\right), \varepsilon_v^{pl}\right) \tag{14}$$

Due to missing information from more complex experiments an arbitrary but consistent splitting had to be made. For simplification *B*, *C* and *D* were given independent of ε_v^{pl}. In the following numerical simulation of the uniaxial tensile test the function $A\left(\varepsilon_v^{pl}\right)$ was determined for the condition that for an assumed set of *B*, *C*, and *D* the given flow curve (12) is fulfilled. All further simulations with „arbitrary" loading paths are based on the variables $A\left(\varepsilon_v^{pl}\right)$, *B*, *C* and *D*, set in the described way. In the numerical computation all equations are used in a normalized form.

4 Examples and Conclusions

For the following examples an isotropic initial state has been assumed which is described by a purely quadratic yield condition.

Further all examples are based on equal material data:

$$\sigma_{FO} = 400\,\mathrm{N/mm^2},\; c_0 = 0.7,\; c_1 = 0.01,\; c_2 = 0.1$$

$$B = 4000.0\,\mathrm{N/mm^2},\; C = 150.0,\; D = -0.45\left(\mathrm{N/mm^2}\right)^{-1}$$

In each example the plastic equivalent strain has reached the value of 0.85 % at the end of the loading path. Figures 1 and 2 show in σ_1, σ_2 space and in σ, τ space respectively the initial yield locus curve (0) and the subsequent yield locus curve (1) at the end of the loading path for the simulation of the uniaxial tensile test described in paragraph 3.

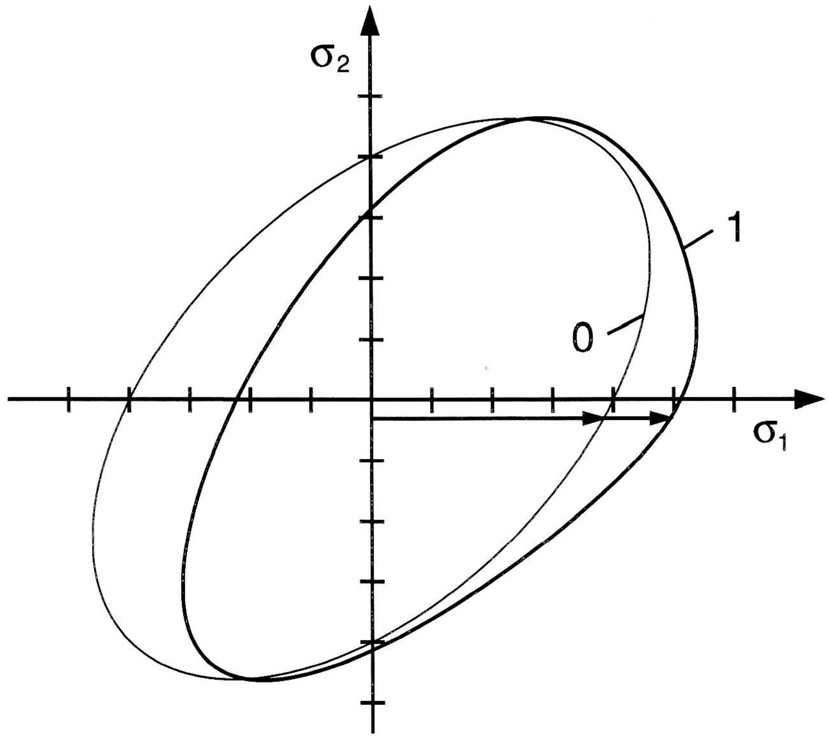

Figure 1. Numerical Simulation of the Uniaxial Tensile Test, Yield Locus Curve in σ_1, σ_2 Space

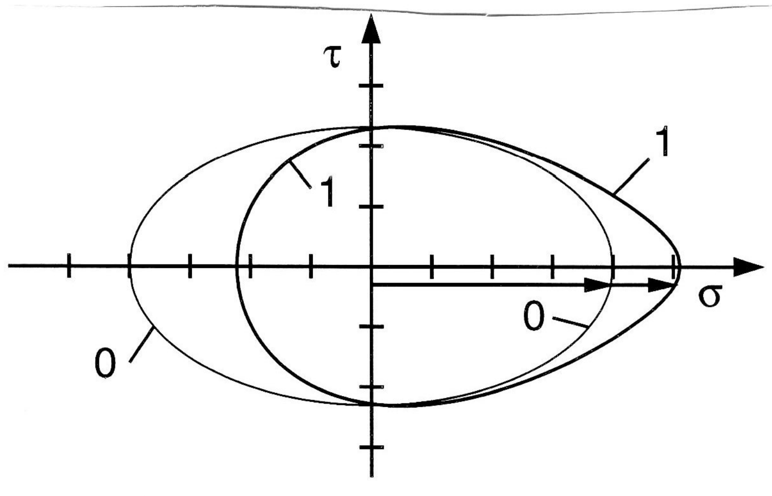

Figure 2. Numerical Simulation of the Uniaxial Tensile Test, Yield Locus Curve in σ, τ Space

After pure shear loading of an initially isotropic specimen the yield locus curve in Figure 3 is obtained.

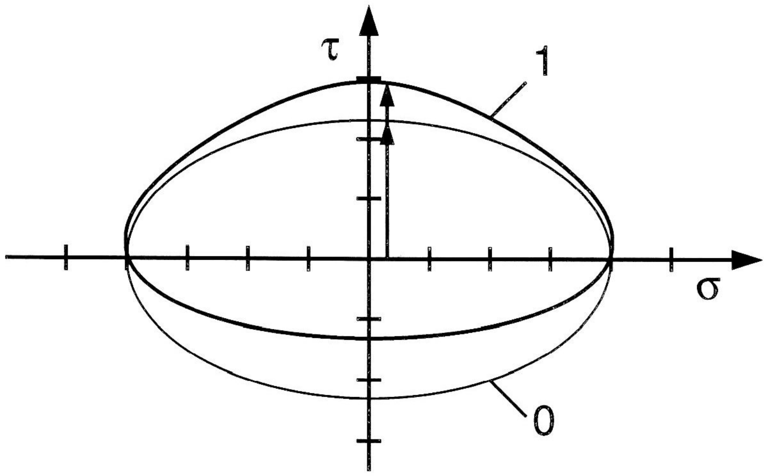

Figure 3. Numerical Simulation of Pure Shear Loading

Figure 4 shows the yield locus curves which result from the simulation of a so-called hook path. Tension with equal rates in 1- and in 2-direction as first loading is followed by uniaxial tension in 1-direction.

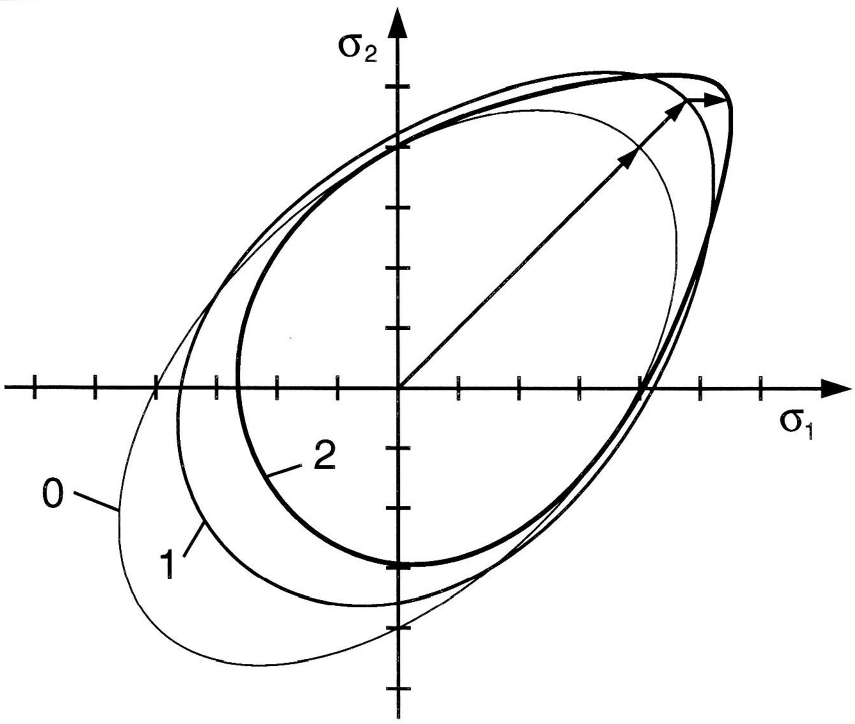

Figure 4. Numerical Simulation of the Hook Path

As can be seen from the examples shown above and from other examples as well, the numerical simulations lead to a distortion of the yield surface which is in accordance with the experimental observation of sharpening of the curvature in the direction of preloading and flattening in the opposite direction.

This applies not to the modelling of the R-value which is as a rule dependent on the plastic equivalent strain and which is used mainly in metal forming for the description of plastic anisotropy. During the simulation of the uniaxial tensile test (see Figures 1 and 2) based on equation (3) and equations (5) or (6) respectively R_1 keeps the value which occurs at reaching the yield limit for the first time because no rotation of f_{ij} and with that of $\dot{\varepsilon}_{ij}^{pl}$ occurs at the points of intersection of the yield surface with the σ_1 axis.

Since most experiments confirmed the validity of the normality rule (3), the „expanded Danilov-formulation" (6) is to be completed by an additional term in order to allow the modelling of a variable R-value. Additional systematic research concerning the cross effect is under way.

Literature

1. Baltov, A.; Sawczuk, A.: A Rule of Anisotropic Hardening, Acta Mechanica, 1, (1965), 81-92.

2. Boehler, J. P. (Ed.): Application of Tensor Functions in Solid Mechanics (CISM No. 292), Wien, New York, Springer-Verlag, (1987).

3. Danilov, V. L.: K formulirovke zakona deformacionnogo uproΦnenija, Mechanika tverdogo tela 6, (1971), 146-150.

4. Grewolls, G.; Kreißig, R.: Numerische Untersuchungen zur Distorsionsverfestigung bei Verwendung einer Fließbedingung dritten Grades, ZAMM, 75, (1995), 185-186.

5. Khan, A. S.; Wang, X.: An Experimental Study of Subsequent Yield Surface after Finite Shear Pre-straining, Int. J. of Plasticity, 9, (1993), 889-905.

6. Landgraf, G.; Bergander, H.: Ob odnom napravlenii v Φislennom re enii zadaΦ neuprugich plastinok i oboloΦek, Uspechi Mechaniki, 8, (1985), 3-38.

7. Phillips, A.; Tang, J. L.: The Effect of Loading Path on the Yield Surface at Elevated Temperatures, Int. J. Solids Struct., 8, (1972), 463-474.

8. Taketa, T.; Nasu, Y.: Evaluation of Yield Function Including Effects of Third Stress Invariant and Initial Anisotropy, J. Strain Anal., 26, (1991), 47-53.

9. Voyiadjis, G. Z.; Foroozesh, M.: Anisotropic Distortional Yield Model, J. Appl. Mech., 57, (1990),537-547.

10. Wegener, K.: Zur Berechnung großer plastischer Deformationen mit einem Stoffgesetz vom Überspannungstyp, Braunschweiger Schriften zur Mechanik, 2, Mechanik Zentrum, TU Braunschweig, (1991).

11. Wegener, K.; Schlegel, M.: Zur Darstellung von Fließpotentialen, ZAMM, 74, (1994), T 329-330.

Address: Professor Dr.-Ing. Reiner Kreißig and Dipl.-Ing. Gerald Grewolls, Institut für Mechanik, Technische Universität Chemnitz-Zwickau, D-09107 Chemnitz