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Canonical Equations in Terms of Generalized Impulse Variables

F.P.J. Rimrott, B. Tabarrok

The Hamiltonian is expressed in terms of generalized displacements and generalized momenta, and so are the canonical equations. If generalized impulses and generalized extensions are used, complementary canonical equations result. Of special significance are generalized impulses that originate from the kinetic energy, in particular the impulse of the centrifugal force, which will be introduced and discussed by means of two simple examples.

1 Introduction

In the conventional formulation the Hamiltonian H must be expressed as function of generalized coordinates q and generalized momenta p, i.e.

$$H = H(q, p) \tag{1}$$

The canonical equations (Lanczos, 1986) then are

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \tag{2a}$$

and

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{2b}$$

2 A Conservative System

Expressed in conventional generalized displacements, the mechanical system of Figure 1 has a kinetic coenergy T^* of

$$T^* = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\,r^2\dot{\theta}^2 \tag{3}$$

Where the generalized displacments $q_1 = q_r = r$ and $q_2 = q_\theta = \theta$ are displacements of the mass m („mass focussed“ variables). The generalized momenta are

$$p_r = \frac{\partial T^*}{\partial \dot{r}} = m\dot{r} \tag{4a}$$

$$p_\theta = \frac{\partial T^*}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{4b}$$

Realizing that in Newtonian mechanics the kinetic energy $T\,(q, p)$ equals the kinetic coenergy $T^*\,(q,\,\dot{q})$, i.e.

$$T(q,p) = T^*(q,\dot{q})$$

we obtain for the kinetic energy T from equations (3) and (4)

$$T = \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} \tag{5}$$

The potential energy is

$$V = V(q) = \frac{1}{2}k_r(r - r_0)^2 + \frac{1}{2}k_\theta(\theta - \theta_0)^2 \tag{6}$$

with k_r in N/m and k_θ in Nm/rad.

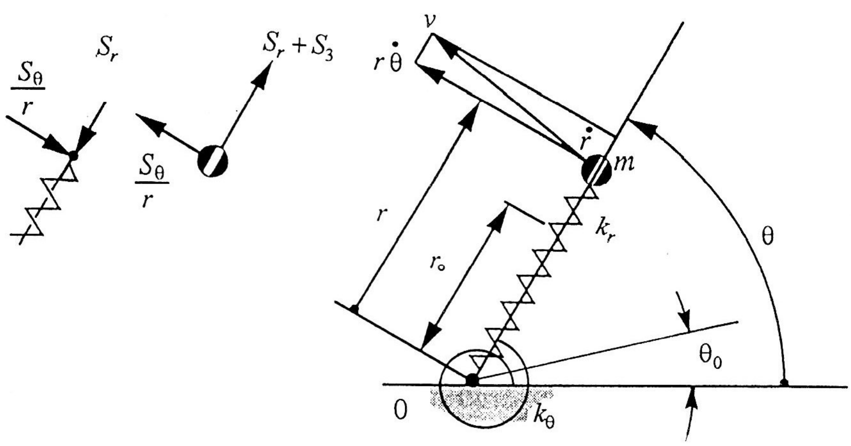

Figure 1. Horizontally Mounted Rotating System

The Hamiltonian (Lanczos, 1986) for a conservative system is simply

$$H = T + V \tag{7}$$

such that for the system of Figure 1

$$H = \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{1}{2}k(r - r_0)^2 + \frac{1}{2}k_\theta(\theta - \theta_0)^2 \tag{8}$$

where we have used r (instead of q_r) and θ (instead of q_θ) for convenience. The canonical equations (2) now provide us with

$$-\dot{p}_r = \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + k_r(r - r_0) \tag{9a}$$

$$\dot{q}_r = \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \tag{9b}$$

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = k_\theta(\theta - \theta_0) \tag{9c}$$

$$\dot{q}_\theta = \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \tag{9d}$$

Equations (9) and (4) result in the familiar equations of equilibrium

$$m\ddot{r} - mr\dot{\theta}^2 + k_r(r - r_0) = 0 \tag{10a}$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + k_\theta(\theta - \theta_0) = 0 \tag{10b}$$

3 Alternative Formulation

It can be shown (Tabarrok and Rimrott, 1994) that in the complementary alternative formulation the Hamiltonian H must be expressed as function of generalized impulses S and generalized extensions e, i.e.

$$H = H(S, e) \tag{11}$$

The canonical equations then are

$$\dot{e}_j = \frac{\partial H}{\partial S_j} \tag{12a}$$

and

$$-\dot{S}_j = \frac{\partial H}{\partial e_j} \tag{12b}$$

Let us use the system of Figure 1 to illustrate the alternative formulation. The variables S are generalized impulses. They are "spring focussed" variables. Since there is a spring (in radial direction) in the system, we select $S_1 = S_r$ with

$$\dot{S}_1 = \dot{S}_r = -k_r(r - r_0) \tag{13}$$

In addition there is a torsional spring. As a consequence we select an $S_2 = S_\theta$ with

$$\dot{S}_2 = \dot{S}_\theta = -k_\theta(\theta - \theta_0) \tag{14}$$

Both $\dot{S}_r$ and $\dot{S}_\theta$ are associated with the systems potential energy V. These is a third force acting (which derives from the kinetic energy), viz. the centrifugal force. The third variable then is S_3 with

$$\dot{S}_3 = mr\dot{\theta}^2 \tag{15}$$

We conclude that there are 3 variables in the alternative formulation, viz.

$$S_r,\ S_\theta,\ S_3 \tag{16}$$

Now in the alternative formulation equilibrium must be satisfied a priori. In equation (5) we let

$$p_r = S_r + S_3 = m\dot{r} \tag{17a}$$

$$p_\theta = S_\theta = mr^2\dot{\theta} \tag{17b}$$

such that

$$T = \frac{1}{2}\frac{(S_r + S_3)^2}{m} + \frac{1}{2}\frac{S_\theta^2}{mr^2} \tag{18}$$

4 Generalized Extensions

For the Hamiltonian (11) one needs generalized extensions e. They may be obtained here simply by inspection. For the translational spring k_r we have

$$e_r = r - r_0 \tag{19a}$$

For the torsional spring k_θ we have

$$e_\theta = \theta - \theta_0 \tag{19b}$$

Both will be needed in the potential energy expression V. For the centrifugal force S_3 we have

$$e_3 = r \tag{19c}$$

which is needed in the kinetic energy expression (18).

In a more formal fashion, the generalized extensions e_r and e_θ are obtained via the potential coenergy. The potential coenergy V^* expressed in terms of variables (13) and (14) is

$$V^* = \frac{1}{2}\frac{\dot{S}_r^2}{k_r} + \frac{1}{2}\frac{\dot{S}_\theta^2}{k_\theta} \tag{20}$$

The generalized extensions are obtained by writing $e_j = -\dfrac{\partial V^*}{\partial \dot{S}_j}$ (Tabarrok and Rimrott, 1994) and result in

$$e_r = -\frac{\partial V^*}{\partial \dot{S}_r} = -\frac{\dot{S}_r}{k_r} = -\frac{-k_r(r-r_0)}{k_r} = r - r_0 \tag{21a}$$

and

$$e_\theta = -\frac{\partial V^*}{\partial \dot{S}_\theta} = -\frac{\dot{S}_\theta}{k_\theta} = -\frac{k_\theta(\theta-\theta_0)}{k_\theta} = \theta - \theta_0 \tag{21b}$$

In order to find e_3 we first bring the kinetic energy expression into a form that resembles the expression for the potential coenergy V^*, i. e. we introduce the centrifugal force $\dot{S}_3$. Then one can express the kinetic energy (18) as

$$T = \frac{1}{2}\frac{(S_r + S_3)^2}{m} + \frac{1}{2}\frac{\dot{S}_3^2}{m\dot{\theta}^2} \tag{22}$$

where $m\dot{\theta}^2$ is the "stiffness" of the centrifugal force field. For the extension e_3 we form

$$e_3 = \frac{\partial T}{\partial \dot{S}_3} = \frac{\dot{S}_3}{m\dot{\theta}^2} = \frac{mr\dot{\theta}^2}{m\dot{\theta}^2} = r \tag{23}$$

Equation (19c) or (23) entered into the kinetic energy expression (18) gives

$$T(S,e) = \frac{1}{2}\frac{(S_r + S_3)^2}{m} + \frac{1}{2}\frac{S_\theta^2}{me_3^2} \tag{24}$$

Equations (19a, b) or (21a, b) entered into the potential energy expression (6) results in

$$V(e) = \frac{1}{2}k_r e_r^2 + \frac{1}{2}k_\theta e_\theta^2 \tag{25}$$

The Hamiltonian H for the system of Figure 1 expressed as function of S and e is consequently

$$H = T + V = \frac{1}{2}\frac{(S_r + S_3)^2}{m} + \frac{1}{2}\frac{S_\theta^2}{m e_3^2} + \frac{1}{2}k_r e_r^2 + \frac{1}{2}k_\theta e_\theta^2 \tag{26}$$

5 Canonical Equations

The canonical equations (12), with the Hamiltonian (26), now result in

$$\dot{e}_r = \frac{\partial H}{\partial S_r} = \frac{S_r + S_3}{m} \tag{27a}$$

$$-\dot{S}_r = \frac{\partial H}{\partial e_r} = k_r e_r \tag{27b}$$

$$\dot{e}_\theta = \frac{\partial H}{\partial S_\theta} = \frac{S_\theta}{m e_3^2} \tag{27c}$$

$$-\dot{S}_\theta = \frac{\partial H}{\partial e_\theta} = k_\theta e_\theta \tag{27d}$$

$$\dot{e}_3 = \frac{\partial H}{\partial S_3} = \frac{S_r + S_3}{m} \tag{27e}$$

$$-\dot{S}_3 = \frac{\partial H}{\partial e_3} = -\frac{S_\theta^2}{m e_3^3} \tag{27f}$$

Equations (27) represent the equations of motion already. They involve only first derivatives. If we want to eliminate some variables we can express the equations of motion in terms of second derivatives. Then equations (27a, b) result in

$$\frac{\ddot{S}_r}{k_r} + \frac{S_r + S_3}{m} = 0 \tag{28a}$$

Equations (27c, d, f) give

$$\frac{\ddot{S}_\theta}{k_\theta} + \frac{\dot{S}_3^{2/3}}{m^{1/3} S_\theta^{1/3}} = 0 \tag{28b}$$

and from equations (27e, f) we obtain

$$\frac{S_\theta^{2/3}}{3m^{1/3}\dot{S}_3^{4/3}}\ddot{S}_3 - \frac{2\dot{S}_\theta}{3m^{1/3} S_\theta^{1/3}\dot{S}_3^{1/3}} + \frac{S_r + S_3}{m} = 0 \tag{28c}$$

For both, equations (28b) and (28c), we used

$$e_3 = \frac{S_\theta^{2/3}}{m^{1/3}\dot{S}_3^{1/3}} \tag{29}$$

which is obtained by rearranging equation (27f). Equations (28) represent the (three) equations of motion in impulse coordinates S. They correspond to the (two) equations (10) of motion in displacement coordinates q.

6 Kepler's Problem

As a second example, we shall have a look at Kepler's problem (Figure 2). In the conventional formulation, its Hamiltonian

$$H = T + V = \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} - \frac{\mu m}{r} \tag{30}$$

with $q_r = r$ and $q_\theta = \theta$ for convenience. The conventional canonical equations result in

$$-\dot{p}_r = \frac{\partial H}{\partial r} = \frac{\mu m}{r^2} \tag{31a}$$

$$\dot{q}_r = \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} - \frac{p_\theta^2}{mr^3} \tag{31b}$$

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \tag{31c}$$

$$\dot{q}_\theta = \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \tag{31d}$$

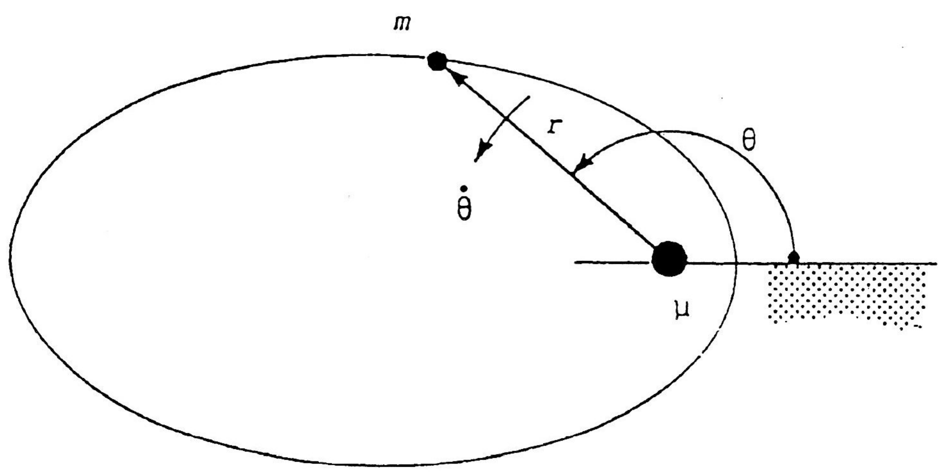

Figure 2. Kepler's Problem in Displacement Coordinates

Eliminating the generalized momenta p one obtaines the well-known second order differential equations

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\mu m}{r^2} = 0 \tag{32a}$$

$$mr\ddot{\theta}^2 + 2m\dot{r}\dot{\theta} = 0 \tag{32b}$$

In the alternative formulation (Rimrott and Förster, 1994) the Hamiltonian becomes

$$H = T + V = \frac{1}{2}\frac{(S_r + S_3)^2}{m} + \frac{1}{2}\frac{S_\theta^2}{m e_3^2} - \frac{\mu m}{e_r} \tag{33}$$

with $S_\theta = m r^2 \dot{\theta}$ = angular equilibrium (integrated) (34a)

$S_r + S_3 = m\dot{r}$ = radial equilibrium (integrated) (34b)

$\dot{S}_3 = m r \dot{\theta}^2$ = centrifugal equilibrium (34c)

as shown in Figure 3. The alternative canonical equations (Tabarrok and Rimrott, 1994) result in

$$\dot{e}_r = \frac{\partial H}{\partial S_r} = \frac{S_r + S_3}{m} \tag{35a}$$

$$-\dot{S}_r = \frac{\partial H}{\partial e_r} = \frac{\mu m}{e_r^2} \tag{35b}$$

$$\dot{e}_\theta = \frac{\partial H}{\partial S_\theta} = \frac{S_\theta}{m e_3^2} \tag{35c}$$

$$-\dot{S}_\theta = \frac{\partial H}{\partial e_\theta} = 0 \tag{35d}$$

$$\dot{e}_3 = \frac{\partial H}{\partial S_3} = \frac{S_r + S_3}{m} \tag{35e}$$

$$-\dot{S}_3 = \frac{\partial H}{\partial e_3} = -\frac{S_\theta}{m e_3^3} \tag{35f}$$

The elimination of the generalized extensions e leads to the second order differential equations

$$\frac{\sqrt{-\mu m}}{2\dot{S}_r^{3/2}}\ddot{S}_r + \frac{S_r}{m} = 0 \tag{36a}$$

$$S_\theta = \text{constant} \tag{36b}$$

$$\frac{S_\theta^{2/3}}{3m^{1/3}\dot{S}_3^{4/3}}\ddot{S}_3 + \frac{S_r + S_3}{m} = 0 \tag{36c}$$

Equations (36) represent the (three) equations of motion in impulse coordinates S. The correspond to the (two) equations (32) of motion in displacement coordinates q.

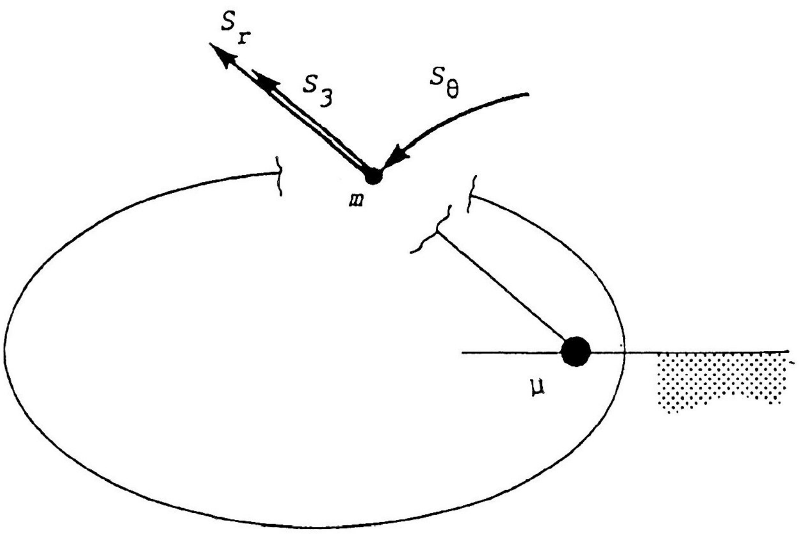

Figure 3. Kepler's Problem in Impulse Coordinates

7 Conclusion

In the foregoing the role played by the centrifugal force in the complementary generalized impulse formulation has been investigated by means of two examples. It has been shown that the centrifugal impulse represents an additional variable, and that a centrifugal extension is associated with the centrifugal impulse.

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Addresses: Professor Dr. F. P. J. Rimrott, Department of Mechanical Engineering, University of Toronto, Toronto, Ontario, Canada M5S 1A4; Professor Dr. B. Tabarrok, Department of Mechanical Engineering, University of Victoria, P.O. Box 3055, Victoria, B.C., Canada V8W 3P6