TECHNISCHE MECHANIK, Band 15, Heft 4, (1995), 325-332
Manuskripteingang: 20. Oktober 1995

MHD Stability of Boundary Layer Flow over a Moving Flat Plate

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The purpose of the present study is to establish the characteristics of the MHD boundary layer flow of an incompressible, electrically conducting fluid over a continuously moving flat plate in the presence of a transverse magnetic field. The temporal neutral stability theory for wavelike disturbances of the Tollmien-Schlichting type are then presented for the velocity functions. The corresponding eigenvalue problem for the disturbance amplitude functions is also solved numerically. The neutral stability curves and critical Reynolds numbers are given for various values of magnetic field parameter.

1 Introduction

From a technological point of view, the study of the flow and heat transfer over a continuously moving flat plate in an electrically conducting fluid permeated by a uniform transverse magnetic field is of special interest and has many practical applications in manufacturing processes in industry. It appears that an understanding of the effect of an applied magnetic field on the flow and heat transfer is useful for the cooling process in the presence of an electrolytic bath. Many metallurgical processes, such as drawing, annealing and tinning of copper wires involve the cooling of continuous strips of filaments by drawing them through a quiescent fluid. In all these cases the properties of the final product depend to a great extent on the rate of cooling. By drawing such strips in an electrically conducting fluid subject to a magnetic field, the rate of cooling can be controlled and final products of desired characteristics can be achieved. Another interesting application of hydromagnetics to metallurgy lies in the purification of molten metals from non-metallic inclusions by the application of a magnetic field.

The first study of the magnetohydrodynamic flow over a stretching wall was conducted by Pavlov (1974), who obtained an exact solution of the momentum equation. Further analyses of this problem have been made by Chakrabarti and Gupta (1979), Vajravelu (1986), Kumari et al. (1990), Vajravelu and Rollins (1992), Andersson (1992), Watanabe and Pop (1995), and Andersson (1995).

In spite of the growing literature on magnetohydrodynamic flow over a moving plate and its obvious importance in polymer and electrochemical industry, it seems that the corresponding stability analysis has not received an adequate attention so far. To the best of our knowledge only Takhar et al. (1989) have investigated the linear stability of the flow of a viscous electrically conducting fluid in the presence of an applied magnetic field over a stretching sheet with respect to the three-dimensional disturbances of the Taylor-Görtler type.

The aim of the present study is to examine the stability of the magnetohydrodynamic boundary layer flow over a continuously moving flat plate for wavelike disturbances of the Tollmien-Schlichting type. The neutral stability curves and critical Reynolds numbers are given for various values of the magnetic field parameter.

2 Basic Equations

Consider a long flat plate, which issues from a slot and moves in an electrically conducting incompressible fluid (with electric conductivity σ), which is at rest. We assume that a uniform magnetic field of strength B_0 is imposed normal to the plate. Figure 1 shows the coordinate system and the flow configuration. The boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\sigma B_0}{\rho}u = \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

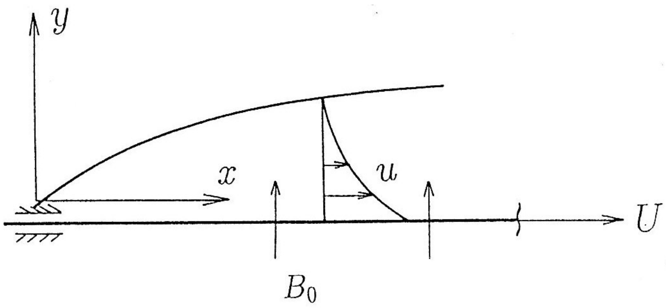

Figure 1. Physical Model and Coordinate System

where (x, y) are Cartesian coordinates (u, v) are the velocity components along the (x, y)-axes, and ρ and ν are the density and kinematic viscosity of the fluid, respectively. The boundary conditions are

$$y = 0 : \quad u = U \quad v = 0 \qquad y \to \infty : \quad u = 0 \tag{3}$$

Introducting the variables

$$\psi = \sqrt{\nu U x}\, f(x, y) \qquad \eta = y\sqrt{U / \nu x} \tag{4}$$

where ψ is the stream function defined in the usual way, equations (1) and (2) reduce to the following parabolic partial differential equation

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{1}{2} f \frac{\partial^2 f}{\partial \eta^2} + x\left(\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial x \partial \eta}\frac{\partial f}{\partial \eta}\right) - mx\frac{\partial f}{\partial \eta} = 0 \tag{5}$$

subject to the boundary conditions

$$\eta = 0 : \quad \frac{\partial f}{\partial \eta} = 1 \quad \text{and} \quad \frac{1}{2} f + x\frac{\partial f}{\partial x} = 0 \qquad \eta \to \infty : \quad \frac{\partial f}{\partial \eta} = 0 \tag{6}$$

where mx is called the magnetic parameter and is defined by

$$N = mx = \sigma B_0^2\, x / (\rho U) \tag{7}$$

Further, we put $X = mx$, and use the difference-differential method to solve equations (5) and (6). Since this method has been described at length in the literature (see e.g. Watanabe and Pop, 1995), for details are omitted. Thus, the solution of equations (5) and (6) can be expressed as

$$\frac{df_i}{d\eta} = 1 + \int_0^\eta P(\eta)\int_0^\eta \frac{Q(\eta)}{P(\eta)} d\eta d\eta - \left\{1 + \int_0^\eta P(\eta)\int_0^\eta \frac{Q(\eta)}{P(\eta)} d\eta d\eta\right\}\frac{R(\eta)}{R(\infty)} \tag{8}$$

$$f_i = \int_0^\eta \frac{df_i}{d\eta} d\eta \tag{9}$$

where

$$P(\eta) = \exp\left[-\int_0^\eta \left\{\frac{1}{2} f_i + \frac{i}{6}\left(11 f_i - 18 f_{i-1} + 9 f_{i-2} - 2 f_{i-3}\right)\right\} d\eta\right] \tag{10}$$

$$Q(\eta) = \left\{\frac{i}{6}\left(11\frac{df_i}{d\eta} - 18\frac{df_{i-1}}{d\eta} + 9\frac{df_{i-2}}{d\eta} - 2\frac{df_{i-3}}{d\eta}\right) + ih\right\}\frac{df_i}{d\eta} \tag{11}$$

$$R(\eta) = \int_0^\eta P(\eta)d\eta \tag{12}$$

The coefficient of skin friction is defined as

$$C_f = \frac{\tau_w}{\rho U^2} \tag{13}$$

where τ_w is the skin friction at the plate and is given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{14}$$

with μ being the dynamic viscosity. Using equation (4), we obtain

$$C_f \mathrm{Re}_x^{1/2} = \left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\eta=0} = -\left\{1 + \int_0^\infty P(\eta)\int_0^\eta \frac{Q(\eta)}{P(\eta)}d\eta d\eta\right\}\frac{1}{R(\infty)} \tag{15}$$

where $\mathrm{Re}_x = Ux/\nu$ is the local Reynolds number.

3 Stability of MHD Boundary Layer

We now consider the linear stability problem of the MHD boundary layer flow over a moving flat plate, in which small disturbances of the Tollmien-Schlichting type occur. The stream function of the wavelike disturbances is expressed as

$$\psi = \nu \mathrm{Re}\,\phi(\eta)\exp\left[i\left(\bar{\alpha}x - \bar{\beta}t\right)\right] \tag{16}$$

where $\bar{\alpha}$ is the real wave number, $\bar{\beta}$ is the complex frequency, t is the time and $i = \sqrt{-1}$. The non-dimensional disturbance amplitude function ϕ is in general complex and the physical quantities correspond to the real part of complex functions. It should be mentioned that functions of wavelike disturbances of the Tollmien-Schlichting type (16) were previously used by Watanabe (1978, 1986) and Watanabe et al. (1995) to study the stability of MHD boundary layer flow along a fixed flat plate, and the stability of free convection flow from a vertical permeable flat plate, respectively.

As in Watanabe (1978, 1986), we find that ϕ is given by the following ordinary differential equation

$$(f' - \beta)\left(\phi'' - \alpha^2\phi\right) - f'''\phi = \frac{-i}{\alpha \mathrm{Re}}\left\{\phi'''' - \left(2\alpha^2 + N\right)\phi'' + \alpha^4\phi\right\} \tag{17}$$

subject to the boundary conditions

$$\phi = \phi'(0) = \phi(\infty) = \phi'(\infty) = 0 \tag{18}$$

where $\mathrm{Re} = \left(\mathrm{Re}_x\right)^{1/2}$ is the modified local Reynolds number and primes denote differentiation with respect to η. The non-dimensional quantities used in the above equation are defined as

$$\alpha = \bar{\alpha}\delta \qquad \beta = \delta^2\bar{\beta}/(\nu \mathrm{Re}) = \beta_r + i\beta_i$$
$$\delta = x/\mathrm{Re} \qquad N = \frac{\sigma B_0^2\delta^2}{\rho\nu} = mx \qquad \mathrm{Re} = \left(\mathrm{Re}_x\right)^{1/2} \tag{19}$$

where δ is the boundary layer thickness, β_r is the wave propagation velocity in x-direction and β_i is the temporal exponential amplification factor. The subscripts r and i denote the real and imaginary parts of the complex functions. Note that neutral stability corresponds to $\beta_i = 0$ and downstream amplification (unstable flow) occurs for $\beta_i > 0$.

The problem of stability of MHD flow over a moving wall has been thus reduced to an eigenvalue problem, to find ϕ, β_r and β_i from equation (17) with the boundary conditions (18) for given values of Re, α and N.

4 Results and Discussion

Equations (8) and (9) were solved numerically for some values of the parameter N using Simpson's rule. Table 1 contains values of $f''(0)$ which is related to the skin friction coefficient C_f given by equation (15). In order to verify the proper treatment of the present problem, we compare the results obtained for $N = 0$ (there is no applied magnetic field) with those reported by Ingham and Pop (1987). Thus, $C_f \mathrm{Re}_x^{1/2} = -0.4438$ from Pop (1987), while our result is $C_f \mathrm{Re}_x^{1/2} = -0.44375$, see Table 1 for $N = 0$. Therefore, the present results are in excellent agreement with those from Pop (1987).

N	$f''(0)$
0.0	-0.44375
0.1	-0.50939
0.2	-0.57286
0.3	-0.63421
0.4	-0.69348
0.5	-0.75070

Table1. Values of $f''(0)$

Figure 2 represents the velocity component along the plate for some values of N. It shows as expected, the retarding effect of magnetic drag on the boundary layer. The distributions of the first- and the second-order derivatives of the velocity along the plate are shown in Figures 3 and 4. These profiles are important for calculating staibility curves of the known boundary layer velocity f'. This problem has been solved in a similar way as that described by Watanabe et al. (1995). The details are presented there and need not be repeated here.

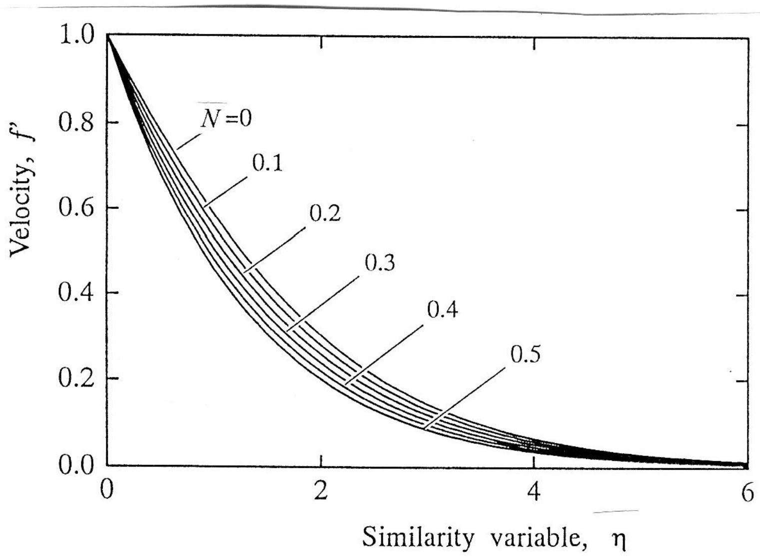

Figure 2. Velocity Profiles of the Basic Flow

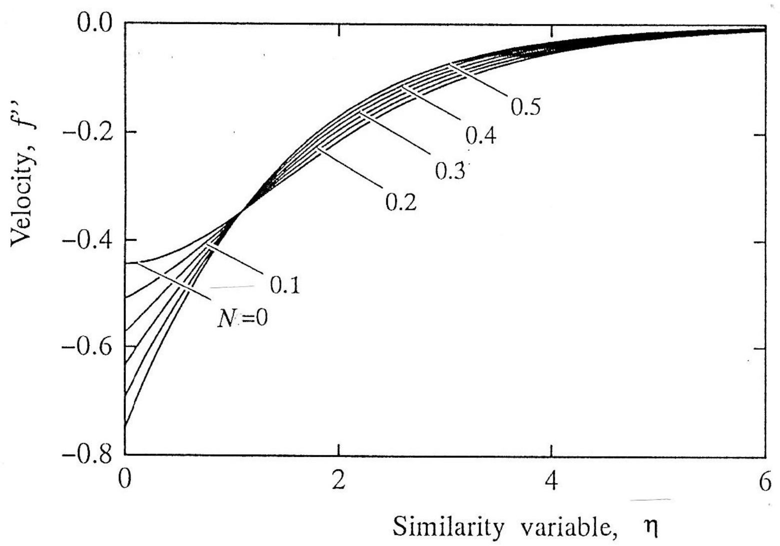

Figure 3. First-order Derivative of the Velocity Profiles of the Basic Flow

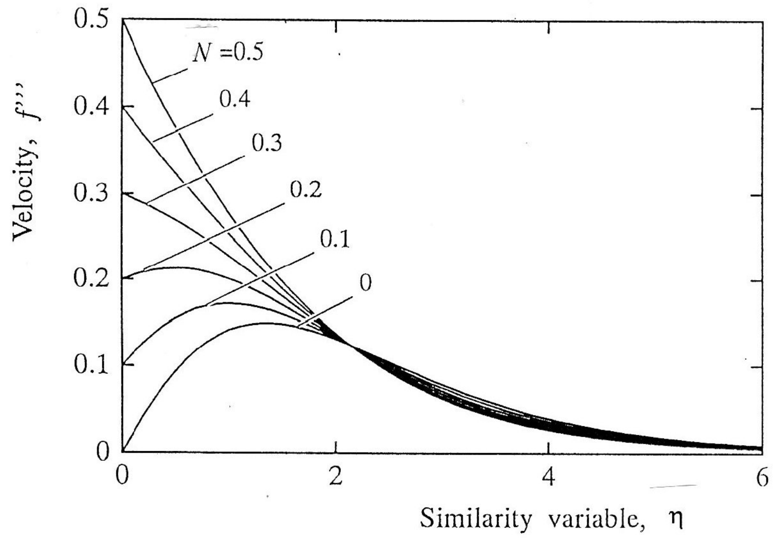

Figure 4. Second-order Derivative of the Velocity Profiles of the Basic Flow

Figures 5 and 6 show the effect of Re on β_r and β_i, for $N = 0.0$ (non-magnetic field) and 0.05 respectively. It is seen from these figures that both β_r and β_i curves increase with Re. On the other hand, we notice from Figure 6 that for $N = 0$, the flow becomes completely stable $(\beta_i < 0)$ when $\text{Re} < 2218$, while for $N = 0.05$ it becomes completely stable when $\text{Re} < 3245$.

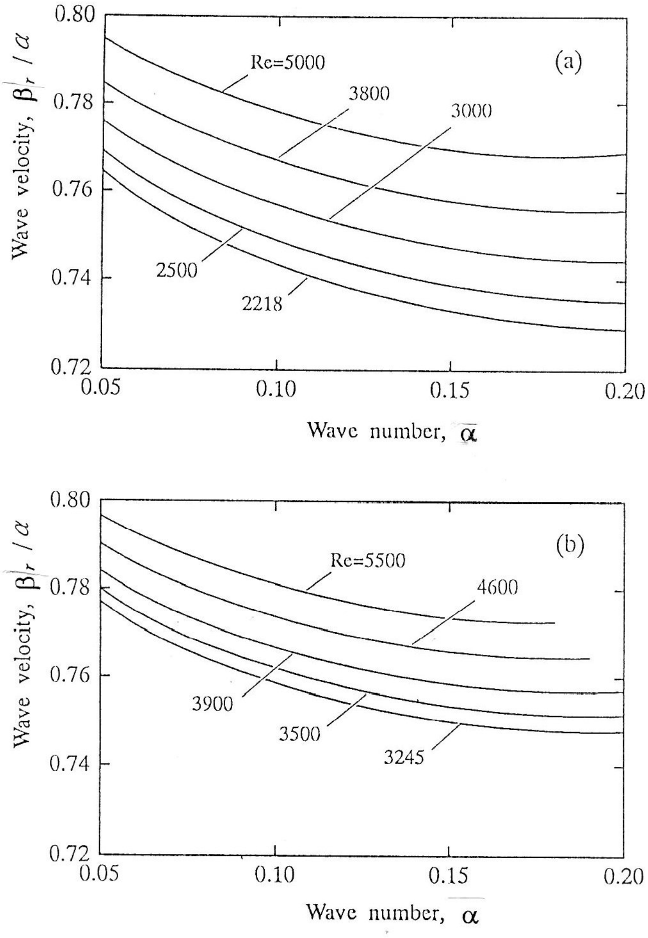

Figure 5. Wave Propagation Velocity (a) $N = 0.0$ and (b) $N = 0.05$

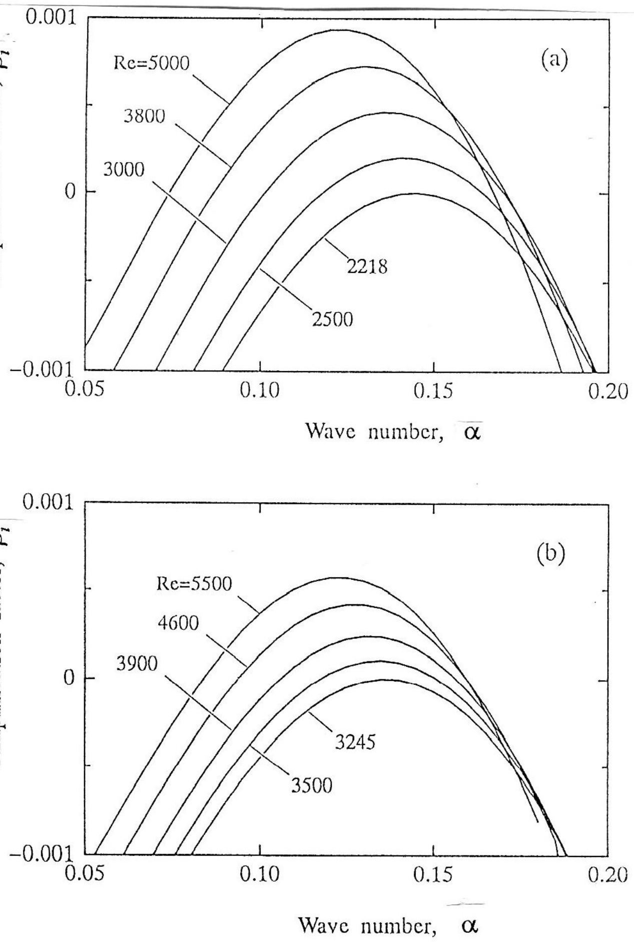

Figure 6. Temporal amplification factor (a) $N = 0.0$ and (b) $N = 0.05$

Figure 7 illustrates the neutral stability curves for the velocity profiles shown in Figure 3. These curves are plotted against Re and for several values of N. As is evident from this figure, the critical Reynolds number Re_c increases as the magnetic parameter N increases. It is also observed that the wave number α decreases as N is increased.

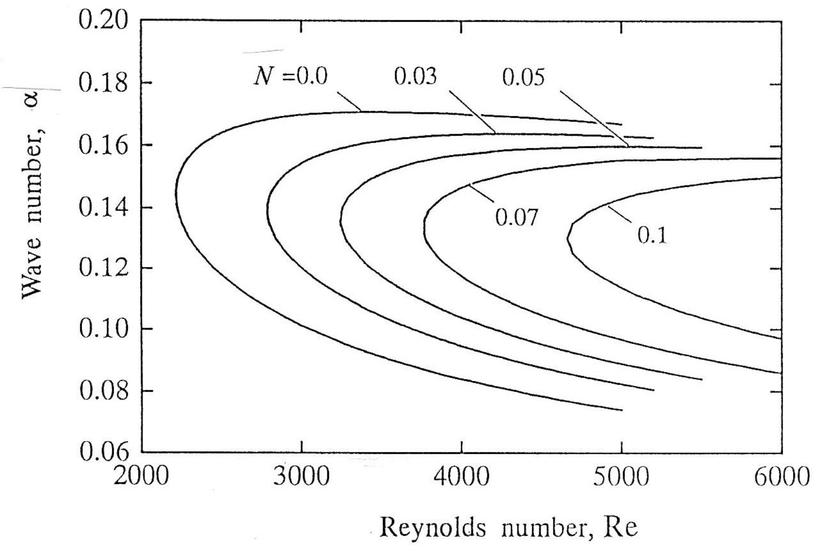

Figure 7. Neutral Stability Curves

Finally, Figure 8 shows the variation of the critical Reynolds number Re_c as a function of N. As is evident from this figure, the critical Reynolds number increases with increasing the magnetic parameter N. The reverse situation occurs for the corresponding problem of a fixed flat plate, see Watanabe (1978).

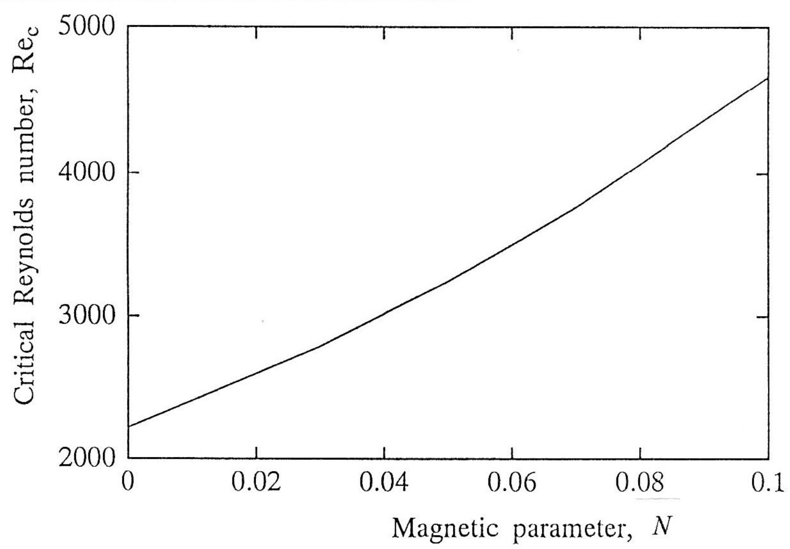

Figure 8. Variation of the Critical Reynolds Number

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