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A Modal Approximation Method of Frequency Modification

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By using the techniques of modal analysis, this paper discusses the reanalysis method in dynamic design for obtaining a desired natural frequency through mass modification, and derives the corresponding reanalysis formulas. Considering that the structure of a e. g. crane can be simplified essentially as an assembly of beam elements, the free vibration of a cantilever beam is discussed as a general method and the mode shape function of vibration is acquired. An example illustrating the general process with prediction of mass modification for the desired natural frequency is presented.

1 Introduction

The dynamic design of steel structures is often a process of redesign. Intending to achieve a predicted goal, engineers usually give the structure some modification in design, by redesign and reanalysis. In recent years the techniques of modal analysis have gained more extensive application. By means of these techniques, it is not necessary to reanalyse the original system, but by utilizing modal paramenters and physical paramenters only, through changing of mass, stiffness and damping, one can modify the dynamics so as to satisfy the requirements of design, and thereby realize economy and saving of time. The methods of redesign and reanalysis of structures are usually divided into three categories, namely the reanalysis method based on small modifications of the structure, based on local modifications or based on modal approximation. In this paper, only the reanalysis method of modal approximation in modal coordinates is discussed. Utilizing this method, the required modal paramenters of the original system can be obtained not only by the analytic method but also by the FEM and the experimental modal analysis method.

2 Reanalysis Method of Modal Approximation

In many systems in practical applications, their damping ratios are very small ($\varsigma < 0.01$), hence those systems may be approximated as undamped systems. Thus, the free vibration equation of this kind of system may be written as

$$[M]\{\ddot{x}(t)\} + [K]\{x(t)\} = 0 \tag{1}$$

If the mass and stiffness of the system are changed, then equation (1) becomes

$$[[M] + [\Delta m]]\{\ddot{x}(t)\} + [[K] + [\Delta k]]\{x(t)\} = 0 \tag{2}$$

rearranging the terms of equation (2) yields

$$[M]\{\ddot{x}(t)\} + [K]\{x(t)\} = -[\Delta m]\{\ddot{x}(t)\} - [\Delta k]\{x(t)\} \tag{3}$$

Let $\{f(t)\} = -[\Delta m]\{\ddot{x}(t)\} - [\Delta k]\{x(t)\}$, then equation (3) can be interpreted as the equation of motion of the original dynamic system under an exitation force $\{f(t)\}$.

Let $\{x(t)\} = \{X\}e^{j\omega t}$, then equation (3) becomes

$$\left[-\omega^2[M] + [K]\right]\{X\} = \left[\omega^2[\Delta m] - [\Delta k]\right]\{X\} \tag{4}$$

Consequently

$$\{X\} = H(\omega)\left[\omega^2[\Delta m] - [\Delta k]\right]\{X\} \tag{5}$$

where

$$H(\omega) = \begin{bmatrix} H_{11}(\omega) & \cdots & H_{1n}(\omega) \\ \vdots & & \vdots \\ H_{n1}(\omega) & \cdots & H_{nn}(\omega) \end{bmatrix} \tag{6}$$

$$H_{ik}(\omega) = \frac{X_k(\omega)}{F_i(\omega)} = \sum_{r=1}^{n} \frac{1}{\omega_r^2 - \omega^2} \phi_{ri}\phi_{rj} \tag{7}$$

If only the kth degree-of-freedom mass is modified by an amount Δm_k, then

$$[\Delta m] = [\Delta m]_k = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & \Delta m_k & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} \tag{8}$$

and equation (5) becomes

$$\{X\} = \omega^2 H(\omega)[\Delta m]_k\{X\} \tag{9}$$

Substitution of equations (6) and (8) into equation (9) yields

$$\begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix} = \begin{bmatrix} H_{11}(\omega) & \cdots & H_{1n}(\omega) \\ \vdots & & \vdots \\ H_{n1}(\omega) & \cdots & H_{nn}(\omega) \end{bmatrix} \cdot$$

$$\begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & \Delta m_k & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix} = \omega^2 \begin{Bmatrix} H_{1k}(\omega) \\ H_{2k}(\omega) \\ \vdots \\ H_{nk}(\omega) \end{Bmatrix} \Delta m_k X_k \tag{10}$$

For a certain natural frequency modified to ω_s, the modification Δm_k must satisfy the following equation:

$$\begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix} = \omega_s^2 \begin{Bmatrix} H_{1k}(\omega_s) \\ H_{2k}(\omega_s) \\ \vdots \\ H_{nk}(\omega_s) \end{Bmatrix} \Delta m_k X_k \tag{11}$$

Retaining only X_k , equation (11) becomes

$$X_k = \omega_s^2 H_{kk}(\omega_s) \cdot \Delta m_k \cdot X_k$$

The modification Δm_k at the desired frequency ω_s can be evaluated from the simple scalar equation

$$\Delta m_k = \left(\omega_s^2 H_{kk}(\omega_s)\right)^{-1} \tag{12}$$

where $$H_{kk}(\omega_s) = \sum_{r=1}^{n} \frac{\Phi_{rk}^2}{\omega_r^2 - \omega_s^2}$$

Similarly, through changing the stiffness, we can find the follow result:

$$\Delta k_k = -\left(H_{kk}(\omega_s)\right)^{-1} \tag{13}$$

3 Sensitivity Analysis in Redesign

For the above-mentioned undamped multi-degree-of-freedom discrete system, through mass modification in a certain coordinate, its natural frequency can be modified. After all, the problem is how to choose the location of mass modification to yield the desired change of dynamic paramenters. This problem can be solved by using sensitivity analysis. The scope of sensitivity analysis is very extensive. Only the sensitivity analysis of the eigenvalue is discussed in the following.

3.1 Sensitivity Analysis of Mass Modification of Undamped System

Let $\{x(t)\} = \{X\}e^{jwt}$, then from equation (1)

$$-\omega^2[M] + [K] = 0 \tag{14}$$

postmultiplying equation (14) by $[\Phi]$ yields

$$-\omega^2[M][\Phi] + [K][\Phi] = 0 \tag{15}$$

The partial differential equation of equation (15) with respect to m_{ij} is

$$-\frac{\partial\omega^2}{\partial m_{ij}}[M][\Phi] - \omega^2\frac{\partial[M]}{\partial m_{ij}}[\Phi] - \omega^2[M]\frac{\partial[\Phi]}{\partial m_{ij}} + \frac{\partial[K]}{\partial m_{ij}}[\Phi] + [K]\cdot\frac{\partial[\Phi]}{\partial m_{ij}} = 0 \tag{16}$$

Premultiplying this equation by $[\Phi]^T$ yields

$$-\frac{\partial\omega^2}{\partial m_{ij}}[\Phi]^T[M][\Phi] - \omega^2[\Phi] - \omega^2[\Phi]^T\frac{\partial[M]}{\partial m_{ij}}[\Phi] + [\Phi]^T[K]\frac{\partial[\Phi]}{\partial m_{ij}} = 0 \tag{17}$$

Since $$[\Phi]^T[M][\Phi] = [I], \frac{\partial[K]}{\partial m_{ij}} = 0$$

then, from equation (17), we obtain

$$-2\omega\frac{\partial\omega}{\partial m_{ij}} - \omega^2[\Phi]^T\frac{\partial[M]}{\partial m_{ij}}[\Phi] \tag{18}$$

Also, since $$-\omega^2[M] + [K] = 0$$

then from equation (18), we obtain

$$\frac{\partial\omega}{\partial m_{ij}} = -\frac{1}{2}\omega\Phi_{ri}\Phi_{rj} \tag{19}$$

When $i = j$ and $\omega = \omega_r$

$$\frac{\partial\omega_r}{\partial m_{ij}} = -\frac{1}{2}\omega_r\cdot\Phi_{ri}^2 \tag{20}$$

Also, since $$\omega_r = 2\pi f_r \tag{21}$$

$$\frac{\partial f_r}{\partial m_{ij}} = -\frac{1}{4\pi}\omega_r\Phi_{ri}^2 \tag{22a}$$

3.2 Sensitivity Analysis of Stiffness Modification of Undamped System

Similarly, from the partial differentials of equation (15) with respect to K_{ij}, we obtain the following relation:

$$\frac{\partial f_r}{\partial K_{ij}} = -\frac{1}{2\pi\omega_r}\Phi_{ri}\Phi_{rj} \tag{22b}$$

It can be seen from the above equation that the sensitivity of eigenvalues to mass and stiffness is principally dependent on the coefficients of the mode shape matrix. Therefore, if the eigenvalues and the related mode shape vectors are obtained, then the sensitivity analysis of dynamic modification can be performed so as to determine the optimum modification point and thus a fastest modification scheme can be formed, and blindness of analysis and design would be avoided.

The structure of e. g. a crane can be simplified essentially as an assembly of beam elements. For beam elements under different support conditions, after their cross-section dimensions and lengths have been determined, their mode shape vectors can be obtained directly analytically. By utilizing the above-mentioned method to simulate on a computer, and according to a predicted requirement to perform dynamic modification of the structure, the requirement of its dynamic properties can be satisfied. Without loss of generalization, a cantilever beam is selected as example for discussion of a generalized method of dynamic modification.

4 Dynamic Behavior Modification of Cantilever Structure

A suitable coordinate system for an undamped uniform cantilever system is shown in Figure 1.

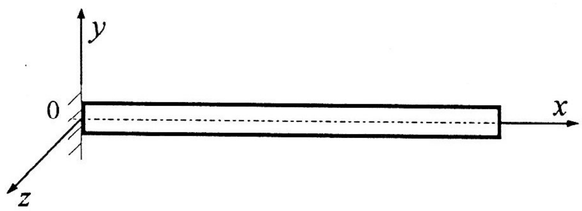

Figure 1. Cantilever Beam Configuration

Its transverse vibration equation can be written as

$$\frac{\partial^4 y(x, t)}{\partial x^4} + \frac{pA}{EI_z}\frac{\partial^2 y(x, t)}{\partial t^2} = 0 \tag{23}$$

Let $y(x, t) = \Phi(x) \cdot T(t)$, and substituting into equation (23), results in

$$-\frac{\ddot{T}(t)}{T(t)} = \frac{EI_z}{pA}\frac{\Phi^{IV}(x)}{\Phi(x)} \tag{24}$$

Then let

$$-\frac{\ddot{T}(t)}{T(t)} = \frac{EI_z}{pA}\frac{\Phi^{IV}(x)}{\Phi(x)} = \omega^2$$

and let the boundary conditions of the cantilever beam be

$$\begin{aligned} &x = 0, \quad \Phi(0) = 0 \quad \Phi'(0) = 0 \\ &x = L, \quad \Phi''(L) = 0 \quad \Phi'''(L) = 0 \end{aligned}$$

From orthogonality, all mode shape functions of this beam can be obtained as

$$\left.\begin{aligned} \Phi_1(x) &= 0.51677(\cos a_1 x - \mathrm{ch} a_1 x - 0.734\sin a_1 x + 0.734\mathrm{sh} a_1 x) \\ \Phi_2(x) &= 0.518203(\cos a_2 x - 1.018\sin a_2 x + 1.018\mathrm{sh} a_2 x) \\ \Phi_3(x) &= 0.505909(\cos a_3 x - \mathrm{ch} a_3 x - 0.999\sin a_3 x + 0.999\mathrm{sh} a_3 x) \\ \Phi_4(x) &= 0.52897(\cos a_4 x - \mathrm{ch} a_4 x - \sin a_4 x + \mathrm{sh} a_4 x) \\ &\cdots\cdots\cdots \end{aligned}\right\} \tag{25}$$

where

$$\begin{aligned} a_1 &= 1.\ 8751040704141 \\ a_2 &= 4.\ 6940911332876 \\ a_3 &= 7.\ 8547574382376 \\ a_4 &= 10.\ 995540735266 \\ &\cdots\cdots\cdots \end{aligned}$$

Then the corresponding natural frequencies can be obtained as follows:

lst order:

$$\omega_1 = 63.20 \text{ rad/s} \quad \text{or} \quad f_1 = 10.06 \text{ Hz}$$

2nd order:

$$\omega_2 = 396.06 \text{ rad/s} \quad \text{or} \quad f_2 = 63.03 \text{ Hz}$$

3rd order:

$$\omega_3 = 1108.97 \text{ rad/s} \quad \text{or} \quad f_3 = 176.5 \text{ Hz}$$

4th order:

$$\omega_4 = 2173.3 \text{ rad/s} \quad \text{or} \quad f_4 = 345.86 \text{ Hz}$$

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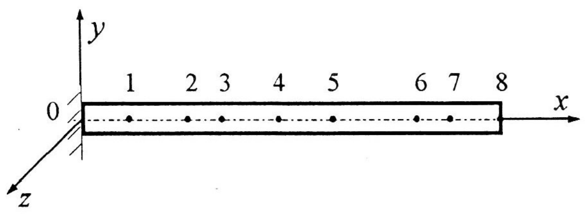

Figure 2. Discrete Model of Cantilever Beam

For the 8 points of the cantilever beam shown in Figure 2, the mode shape vectors corresponding to different natural frequencies can be obtained from equation (25). The results of sensitivity analysis of mass modification by using equation (22a) are shown in Table 1.

	$\frac{\partial f_1}{\partial m_{ii}}$	$\frac{\partial f_2}{\partial m_{ii}}$	$\frac{\partial f_3}{\partial m_{ii}}$	$\frac{\partial f_4}{\partial m_{ii}}$	$\frac{\partial f_5}{\partial m_{ii}}$	$\frac{\partial f_6}{\partial m_{ii}}$	$\frac{\partial f_7}{\partial m_{ii}}$	$\frac{\partial f_8}{\partial m_{ii}}$
1	-. 0056	-. 9539	-13. 4087	-67. 6687	-167. 1289	-264. 3944	-274. 5519	-163. 3318
2	-. 1268	-10. 8201	-49. 6100	-16. 8856	-48. 6510	-233. 3372	-84. 2908	-89. 4720
3	-. 3099	-16. 2550	-21. 1560	-28. 4629	-143. 9189	-. 0622	-333. 9741	-118. 3367
4	-1. 0809	-12. 5523	-17. 0389	-29. 8900	-145. 3215	-. 0539	-303. 7628	-118. 2962
5	-1. 7116	-4. 9425	-38. 7939	-13. 7247	-51. 3977	-235. 9675	- 84. 9744	-89. 1762
6	-2. 9891	-. 6176	-9. 2376	-70. 9490	-132. 5019	-95 .6856	-5. 7339	-79. 2800
7	-3. 5029	-3. 8544	-. 2371	-27. 4863	-205. 3789	-228. 1382	-136. 7220	
8	-5. 3723	-33. 8541	-90. 3472	-193. 5521	-316. 6131	-469. 7640	-653. 2208	-866. 7722

Table 1. Sensitivity Analysis

From Table 1 we see that if we attempt to change the natural frequencies of the structure by means of mass modification, then for frequency modifications of all frequencies from lst to 8th order, the most efficient point is point 8. Through sensitivity analysis for the cantilever beam, it follows that after determining the optimum modification point, the mass modification can be performed as desired. The process can be carried through as follows:

From equation (12)

$$\Delta m_k \left[\sum_{r=1}^{n} \frac{\Phi_{rk}^2}{(\frac{\omega_r}{\omega})^2 - 1} \right] = 1$$

Also,

$$\because \quad \omega = 2\pi f$$

$$\therefore \quad \Delta m_k \left[\sum_{r=1}^{n} \frac{\Phi_{rk}^2}{(\frac{f_r}{f})^2 - 1} \right] = 1$$

Thus

$$\Delta m_k \frac{\Phi_{1k}^2}{(\frac{f_1^2}{f}) - 1} + \Delta m_k \frac{\Phi_{2k}^2}{(\frac{f_2^2}{f}) - 1} + \cdots \Delta mk \frac{\Phi_{nk}^2}{(\frac{f_n^2}{f}) - 1} = 1 \qquad (26)$$

Let us transfer only the lst order natural frequency to $f_1^* = 8.05$ Hz.

Considering that the influencies of higher components upon the lst order mass modification are not large, we may let $n = 8$ in equation (26), and also on the basis of the number of points in the structure at which mass modification is permissible, let $k = 8$, we can obtain the amount of mass modification on the 8th point as

$$\Delta m_8 = 0.5197 \text{ kg}$$

Thus increasing the mass at point 8 by 0.5197 kg will satisfy the requirement of abovementioned modification. In other words, increasing total mass of the beam by 14 % will cause the lst order natural frequency to decrease by 20 %.

After this kind of modification, we would like to find out the new behavior of the modified system. Substituting Δm_8 into equation (26), we can obtain the natural frequencies of all orders, namely:

lst order: $\omega_1^* = 50.57964432872$ rad/s $= 8.05$Hz

2nd order: $\omega_2^* = 329.6554397674$ rad/s

3rd order: $\omega_3^* = 984.8079022841$ rad/s

4th order: $\omega_4^* = 1959.118472330$ rad/s

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Substituting ω_r^* and Δm_8 into equation (11) and letting X_k be an arbitrary constant, we can obtain the corresponding mode shape vectors of the modified system .

i	1	2	3	4	5	6	7	8
Φ^*_{1i}	-.604E-03	-.190E-01	-.891E-01	-.335E+00	-.716E+00	-.145E+01	-.222E+01	-.374E+01
Φ^*_{2i}	-.485E-03	-.162E-01	-.786E-01	-.314E+00	-.687E+00	.144E+01	.222E+01	.382E+01
Φ^*_{3i}	-.526E-03	-.168E-01	-.800E-01	.312E+00	.683E+00	.143E+01	.222E+01	-.383E+01
Φ^*_{4i}	-.505E-03	-.167E-01	.802E-01	.313E+00	-.683E+00	-.143E+01	-.221E+01	.384E+01
Φ^*_{5i}	-.517E-03	.167E-01	.799E-01	-.313E+00	-.683E+00	.143E+01	.221E+01	-.384E+01
Φ^*_{6i}	-.510E-03	.167E-01	-.800E-01	-.313E+00	.684E+00	-.143E+01	-.221E+01	.384E+01
Φ^*_{7i}	-.514E-03	.167E-01	-.800E-01	.313E+00	-.684E+00	.143E+01	.221E+01	-.384E+01
Φ^*_{8i}	-.512E-03	-.167E-01	-.800E-01	-.313E+00	-.684E+00	.143E+01	-.221E+01	.384E+01

Table 2. Mode Shapes.

Then according to further requirements of frequency modification, we can carry out the corresponding sensitivity analysis, and utilize equation (26) to find the modification of the mass matrix. Therefore, the process of regulation of frequency characteristics of the whole structure system is a repetitive process.

5 Conclusions

For other beam elements of different boundary conditions, cross-section areas and lengths, after obtaining the mode shape function, we can apply the methods discussed above to carry out directly the frequency modification on a computer and reconstruct the FRF matrix. If accompanied by FEM, or experimental modal analysis, the application scope of the method presented in this paper can be further extended.

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