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Modelling of Crack Tip Singularity

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Standard isoparametric elements are used as singular and transition elements in fracture mechanics computations by placing the side-nodes in the vicinity of a crack tip. Such elements are well known and have already been applied to several two- and three-dimensional problems. The present paper demonstrates a generalization of this method to higher-order strain singularities.

1 Introduction

Henshell and Shaw (1975) and Barsoum (1976, 1977) have independently shown that the inverse square root singularity characteristic of linear elastic fracture mechanics can be obtained in 2D and 3D isoparametric elements when the mid-side nodes in the vicinity of the crack tip are placed at the quarter point. Hibbit (1977) investigated the properties of these elements. Pu, Hussain and Lorensen (1978) developed 12-node quadrilateral isoparametric elements. It was shown that the $r^{-1/2}$ singularity of the strain field at the crack tip could be obtained by placing the two side nodes at 1/9 and 4/9 of the length of the side from the crack tip. These singular isoparametric elements can also be found in Akin (1982). Banks-Sills and Einav (1987) have used a nine-node, distorted, singular isoparametric element.

Lynn and Ingraffea (1978) have developed a transition element possessing a singularity of order $r^{-1/2}$ outside the element. In practical applications of this element the principal parameter affecting the accuracy of the stress intensity factor is the ratio of the singular element length to the crack length. As this ratio approaches a small value, the modelling capability of singular behaviour is lost because of the non-singular behaviour of the neighbouring elements. A better model is obtained by replacing the non-singular neighbouring elements with transition elements possessing the same order of singularity at the crack tip. The present paper shows that a singularity of order $r^{(1-m)/m}$ can be obtained by means of an nth-order isoparametric element.

2 Higher-Order Singular Isoparametric Elements

2.1 General Forming of Singular Isoparametric Elements

To create an element possessing a singularity of order $r^{(1-m)/m}$ consider a one-dimensional element which may form one side of a 2D or 3D nth-order isoparametric element. Such an element is shown in Figure 1.

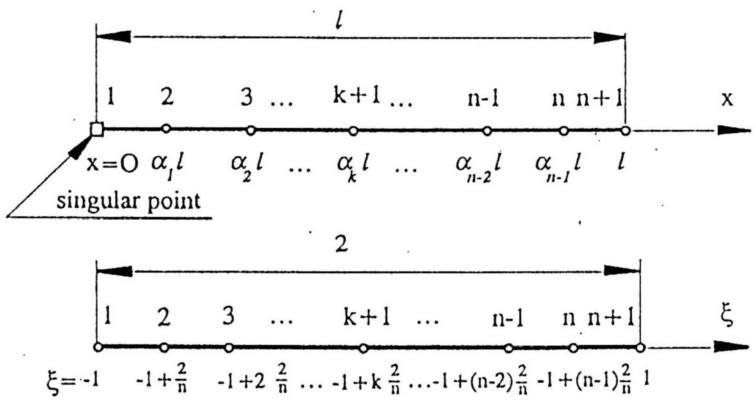


Figure 1. Mapping of the Coordinates

The nodes of the element, designated by 1, 2, ..., n+1 are mapped to $\xi = \pm 1$ on the ξ scale. The transformation is accomplished using the usual isoparametric shape functions, or the following polynomial interpolation:

$$x = a_0 + a_1\xi + a_2\xi^2 + \cdots + a_m\xi^m \qquad n \geq m \geq 2 \qquad (1)$$

As the element is isoparametric, the same shape functions or interpolations are used for the transformation of the displacement.

$$u = b_0 + b_1\xi + b_2\xi^2 + \cdots + b_n\xi^n \qquad n \geq 2 \qquad (2)$$

Substitution of x and ξ nodal values into equation (1) gives the following set of equations:

$$\begin{aligned}
0 &= a_0 - a_1 + a_2 - \cdots \pm a_m \\
\alpha_1 l &= a_0 + \left(-1+\frac{2}{n}\right)a_1 + \left(-1+\frac{2}{n}\right)^2 a_2 + \cdots + \left(-1+\frac{2}{n}\right)^m a_m \\
&\vdots \\
\alpha_k l &= a_0 + \left(-1+k\frac{2}{n}\right)a_1 + \left(-1+k\frac{2}{n}\right)^2 a_2 + \cdots + \left(-1+k\frac{2}{n}\right)^m a_m \\
&\vdots \\
l &= a_0 + a_1 + a_2 + \cdots + a_m
\end{aligned} \qquad (3)$$

To obtain a singular strain at $x = 0\ (\xi = -1)$, the reduced Jacobian, $dx/d\xi$, must vanish at $\xi = -1$. We obtain this derivative from equation (1).

$$\frac{dx}{d\xi} = a_1 + 2a_2\xi + \cdots + ma_m\xi^{m-1} \qquad (4)$$

For $\xi = -1$, and $dx/d\xi = 0$,

$$a_1 - 2a_2 + 3a_3 - \cdots \pm ma_m = 0 \qquad (5)$$

We need further equations, as the number of unknowns is $n + m\ (\alpha_1, \alpha_2, \cdots \alpha_{n-1}, a_0, a_1, \cdots, a_m)$. The higher derivatives of x must vanish at $\xi = -1$.

$$\left.\frac{d^2x}{d\xi^2}\right|_{\xi=-1} = \left.\frac{d^3x}{d\xi^3}\right|_{\xi=-1} = \cdots = \left.\frac{d^{(m-1)}x}{d\xi^{(m-1)}}\right|_{\xi=-1} = 0 \qquad (6)$$

By substituting $\xi = -1$, we obtain the following set of equations from equations (1) and (6):

$$\begin{aligned}
2a_2 - 6a_3 + \cdots \pm m(m-1)a_m &= 0 \\
&\vdots \\
(m-1)!\, a_{m-1} - m!\, a_m &= 0
\end{aligned} \qquad (7)$$

The last of equations (7) gives

$$a_{m-1} = ma_m \tag{8}$$

The solution of the other equations (7) gives the coefficients in general form

$$a_i = \binom{m}{i} a_m \qquad i = 1,2,\cdots,m \tag{9}$$

By means of equation (9) the last of equations (3) becomes

$$l = \binom{m}{0} a_m + \binom{m}{1} a_m + \binom{m}{2} a_m + \cdots + \binom{m}{m} a_m = a_m 2^m \tag{10}$$

From equation (10) we obtain

$$a_m = \frac{l}{2^m} \tag{11}$$

Substitution of equation (9) into equation (3) gives the coefficients in general form

$$\alpha_k l = \binom{m}{0} a_m + \left(-1+k\frac{2}{n}\right)\binom{m}{1} a_m + \left(-1+k\frac{2}{n}\right)^2 \binom{m}{2} a_m + \cdots + \left(-1+k\frac{2}{n}\right)^m \binom{m}{m} a_m = a_m 2^m \left(\frac{k}{n}\right)^m \tag{12}$$

Using equation (11) we obtain

$$\alpha_k l = \frac{l}{2^m} 2^m \left(\frac{k}{n}\right)^m \tag{13}$$

From equation (13) we have

$$\alpha_k = \left(\frac{k}{n}\right)^m \qquad k = 1,2,3,\ \cdots,\ n- \tag{14}$$

Equation (14) determines the positions of the side nodes. If we take into consideration the values of $k = 0$ and $k = 1$, they will be the equivalent of $\alpha_0 = 0$ and $\alpha_n = 1$. The exponent m determines the order of the strain singularity at the singular point (node 1) and n is the order of the element. The case $m = 1$ corresponds to the standard (non-singular) isoparametric element. For m and n the following relation is valid:

$$m \leq n \tag{15}$$

If we take into consideration only equation (4), it will be the equivalent of the case $m = 2$. That means the type of singularity is of order $r^{-1/2}$. If we take into consideration equations (4) and (6), they will be the equivalent of the case $m > 2$. That means the type of singularity is of order $r^{(1-m)/m}$. According to relation (15) an nth-order isoparametric element has $(n-1)$ different types of strain singularity depending on the placing of the side nodes.

In order to obtain a singular isoparametric element to be used at the crack tip or front, the strain must be singular. The required strain singularity is achieved by placing the side nodes into the corresponding position, according to equation (14).

It can be shown easily that by means if the above-mentioned technique we obtain the inverse square root singularity of the strain field at the crack tip. This is the required strain singularity in the calculation of the stress intensity factors of elastic fracture mechanics.

2.2 Investigation of Type of Singularity

Substitution of equations (9) and (11) into equation (1) gives the following expression:

$$x = \frac{l}{2^m}(1+\xi)^m \quad \text{or} \quad \xi = -1+2\left(\frac{x}{l}\right)^{1/m} \tag{16}$$

The strain ε_x in x-direction is then

$$\varepsilon_x = \frac{du}{dx} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial u}{\partial \xi}\frac{2}{m}\frac{1}{l^{1/m}}x^{(1-m)/m} \tag{17}$$

We have the displacement derivative from equation (2).

$$\frac{\partial u}{\partial \xi} = b_1 + 2b_2\xi + 3b_3\xi^2 + \cdots + nb_n\xi^{n-1} \tag{18}$$

This expression can be written in another form by inserting equation (16).

$$\frac{\partial u}{\partial \xi} = b_1 + 2b_2\left[-1+2\left(\frac{x}{l}\right)^{1/m}\right] + 3b_3\left[-1+2\left(\frac{x}{l}\right)^{1/m}\right]^2 + \cdots + nb_n\left[-1+2\left(\frac{x}{l}\right)^{1/m}\right]^{(n-1)} \tag{19}$$

Substitution equation (19) into equation (17) we have the strain ε_x.

$$\varepsilon_x = A_1x^{(1-m)/m} + A_2x^{(2-m)/m} + \cdots + A_nx^{(n-m)/m} \tag{20}$$

where

$$A_1 = C\left(b_1 - 2b_2 + 3b_3 - \cdots \pm nb_n\right)$$

$$A_2 = \frac{C}{l^{1/m}}2\left[2b_2 - 6b_3 + 12b_4 - \cdots \pm (n-1)nb_n\right]$$

$$\vdots$$

$$A_n = \frac{C}{l^{(n-1)/m}}2^{(n-1)}nb_n$$

$$C = \frac{2}{m}\frac{1}{l^{1/m}}$$

Equation (20) shows that the strain is singular at $x = 0\ (\xi = -1)$. The leading strain term contains $x^{(1-m)/m}$. As $x \to 0$, it represents the required strain singularity. Therefore, in principle, any derivative singularity from $x^{-1/2}$ to x^{-1} is available. Rice has shown that the singularity in strain at a crack tip is of order $x^{-1/(1+N)}$, where N is a hardening exponent varying between 1 and 0 for purely elastic to perfectly plastic response. The two exponents must be equal, so we obtain the expression

$$N = \frac{1}{m-1} \tag{21}$$

Equation (21) determines the connection between the type of material and the type of singularity. When $N = 1$, the material is linear elastic. In this case $m = 2$, so the type of singularity is $r^{-1/2}$. When $2 < m \le n$, we have a singularity of order $r^{(1-m)/m}$. In this case $N < 1$,which means a linear hardening material. As m and n are always integers, the exponent $(1-m)/m$ has discrete values, while the hardening exponent N varies continuously. Therefore in practice only discrete types of singularity are available by means of higher-order singular isoparametric elements.

2.3 An Example

Consider a side $(\eta = -1)$ of a 2D 4th-order isoparametric element, as shown in Figure 2.

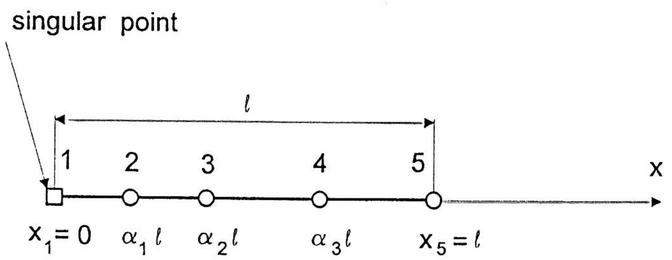


Figure 2. One Side $(\eta = -1)$ of a 2D 4th-order Element

We examine an inverse square root singularity of the element. For simplicity the length of the side is unity, $l = 1$. By substitution of $\eta = -1$, we have the localized shape functions

$$\begin{aligned}
N_1 &= \frac{1}{12}(1-\xi)\left(-8\xi^3+2\xi\right) \\
N_2 &= \frac{8}{12}\left(1-\xi^2\right)\left(4\xi^2-2\xi\right) \\
N_3 &= \frac{12}{12}\left(1-\xi^2\right)\left(1-4\xi^2\right) \\
N_4 &= \frac{8}{12}\left(1-\xi^2\right)\left(4\xi^2+2\xi\right) \\
N_5 &= \frac{1}{12}(1+\xi)\left(8\xi^3-2\xi\right)
\end{aligned} \tag{22}$$

As $n = 4$, $m = 2$, we obtain the expression, according to equation (14)

$$\alpha_k = \left(\frac{k}{n}\right)^m = \left(\frac{k}{4}\right)^2 \qquad k = 1,2,3 \tag{23}$$

By means of equation (23) the coordinates of the side nodes are:

$$x_2 = \alpha_1 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} \qquad x_3 = \alpha_2 = \left(\frac{2}{4}\right)^2 = \frac{4}{16} \qquad x_4 = \alpha_3 = \left(\frac{3}{4}\right)^2 = \frac{9}{16} \tag{24}$$

The relationship between the generalized position and nodal coordinates can be written in the following form:

$$x = \sum_{i=1}^{5} N_i x_i \tag{25}$$

Substitution of equations (22) and (24) into equation (25) gives

$$x = \frac{8}{12}(1-\xi^2)\left(4\xi^2-2\xi\right)\frac{1}{16} + \frac{12}{12}(1-\xi^2)\left(1-4\xi^2\right)\frac{4}{16} + \frac{8}{12}(1-\xi^2)\left(4\xi^2+2\xi\right)\frac{9}{16} + \frac{1}{12}(1+\xi)\left(8\xi^3-2\xi\right)1 \tag{26}$$

After reducing, we obtain equation (26) in the following form:

$$x = \frac{1}{24}\left(6\xi^2 + 12\xi + 6\right) = \frac{1}{2^2}(1+\xi)^2 \tag{27}$$

From equation (27) we have

$$\xi = -1 + 2\sqrt{x} \tag{28}$$

The displacement u can be written similar to equation (25).

$$u = \sum_{i=1}^{5} N_i u_i \tag{29}$$

By substitution we obtain the following expression:

$$u = \frac{1}{12}u_1(1-\xi)\left(-8\xi^3+2\xi\right) + \frac{8}{12}u_2(1-\xi^2)\left(4\xi^2-2\xi\right) + \frac{12}{12}u_3(1-\xi^2)\left(1-4\xi^2\right) + \frac{8}{12}u_4(1-\xi^2)\left(4\xi^2+2\xi\right) + \frac{1}{12}u_5(1+\xi)\left(8\xi^3-2\xi\right) \tag{30}$$

Rearranging of equation (30) gives

$$u = u_3 + \frac{1}{6}\left(u_1 - 2u_2 + 2u_4 - u_5\right)\xi + \frac{1}{6}\left(-u_1 + 16u_2 - 30u_3 + 16u_4 - u_5\right)\xi^2 + + \frac{2}{3}\left(-u_1 + 2u_2 - 2u_4 + u_5\right)\xi^3 + \frac{2}{3}\left(u_1 - 4u_2 + 6u_3 - 4u_4 + u_5\right)\xi^4 \tag{31}$$

According to equation (2), we can write

$$u = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + b_4\xi^4 \tag{32}$$

where

$$b_0 = u_3$$

$$b_1 = \frac{1}{6}\left(u_1 - 2u_2 + 2u_4 - u_5\right)$$

$$\mathrm{b}_2 = \frac{1}{6}\left(-u_1 + 16u_2 - 30u_3 + 16u_4 - u_5\right)$$

$$\mathrm{b}_3 = \frac{2}{3}\left(-u_1 + 2u_2 - 2u_4 + u_5\right)$$

$$b_4 = \frac{2}{3}\left(u_1 - 4u_2 + 6u_3 - 4u_4 + u_5\right)$$

Using equation (28), we obtain equation (32) in another form.

$$u = b_0 + b_1\left(2\sqrt{x}-1\right) + b_2\left(2\sqrt{x}-1\right)^2 + b_3\left(2\sqrt{x}-1\right)^3 + b_4\left(2\sqrt{x}-1\right)^4 \tag{33}$$

The strain (17) is then

$$\varepsilon_x = \frac{du}{dx} = \frac{1}{\sqrt{x}}\left(b_1 - 2b_2 + 3b_3 - 4b_4\right) + 4\left(b_2 - 3b_3 + 6b_4\right) + 12\sqrt{x}\left(b_3 - 4b_4\right) + 32xb_4 \tag{34}$$

In equation (34) ,the strain ε_x has a singularity of order $x^{-1/2}$, when x tends to zero. The two other types of singularity can be written similarly in the case of this element.

3 Higher-Order Isoparametric Transition Elements

3.1. General Forming of Transition Elements

The general forming of transition elements is similar to the forming of a singular element. But the transition element possesses a singularity of order $r^{(1-m)/m}$ outside the element. To develop a transition element, consider a one-dimensional element which may also form a side of a 2D or 3D nth-order isoparametric element. Figure 3 shows such an element.

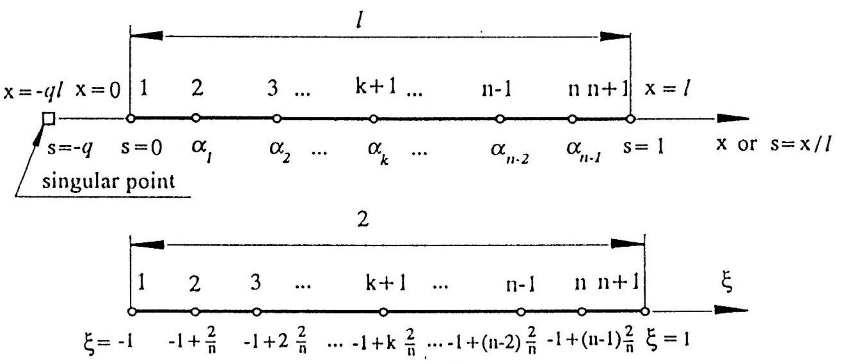


Figure 3. Element Coordinate Mapping

The nodes of the element, designated by 1,2, ..., $n+1$ are mapped to $\xi = \pm 1$ on the ξ scale. The transformation is realised by means of the usual isoparametric shape functions, or in general by the following polynomial interpolation:

$$x = a_0 + a_1\xi + a_2\xi^2 + \cdots + a_m\xi^m \qquad n \geq m \geq 2 \tag{35}$$

Substitution of x and ξ nodal values into equation (35) gives a set of simultaneous equations.

$$\begin{aligned}
0 &= a_0 - a_1 + a_2 - \cdots \pm a_m \\
\alpha_1 l &= a_0 + \left(-1 + \frac{2}{n}\right)a_1 + \left(-1 + \frac{2}{n}\right)^2 a_2 + \cdots + \left(-1 + \frac{2}{n}\right)^m a_m \\
&\vdots \\
\alpha_k l &= a_0 + \left(-1 + k\frac{2}{n}\right)a_1 + \left(-1 + k\frac{2}{n}\right)^2 a_2 + \cdots + \left(-1 + k\frac{2}{n}\right)^m a_m \\
&\vdots \\
l &= a_0 + a_1 + a_2 + \cdots + a_m
\end{aligned} \tag{36}$$

To obtain a singular strain at $x = -ql$, outside the element, the reduced Jacobian, $dx/d\xi$, must vanish. We obtain further equations in a similar way. The derivatives of x must vanish, too. The value of variable ξ is unknown at the singular point. We denote it by $\bar{\xi}$. So we have

$$\left.\frac{dx}{d\xi}\right|_{\xi=\bar{\xi}} = \left.\frac{d^2x}{d\xi^2}\right|_{\xi=\bar{\xi}} = \left.\frac{d^3x}{d\xi^3}\right|_{\xi=\bar{\xi}} = \cdots = \left.\frac{d^{(m-1)}x}{d\xi^{(m-1)}}\right|_{\xi=\bar{\xi}} = 0 \tag{37}$$

From condition (37) we obtain the following set of equations:

$$\begin{aligned}
& a_1 + 2a_2\bar{\xi} + 3a_3\bar{\xi}^2 + \cdots + ma_m\bar{\xi}^{(m-1)} = 0 \\
& 2a_2 + 6a_3\bar{\xi} + \cdots + m(m-1)a_m\bar{\xi}^{(m-2)} = 0 \\
& \vdots \\
& (m-1)!a_{m-1} + m!a_m\bar{\xi} = 0
\end{aligned} \tag{38}$$

The last of equation (38) gives

$$\bar{\xi} = -\frac{a_{m-1}}{ma_m} \tag{39}$$

This value of ξ means the point of singularity outside the element. From equations (38), we obtain the coefficients in general form.

$$a_i = \binom{m}{i} a_m \left(\frac{a_{m-1}}{ma_m}\right)^{(m-i)} \qquad i = 1,2, \cdots, m \tag{40}$$

Substitution of equation (40) into the last of equations (36) gives the expression

$$l = a_0 - \binom{m}{0} a_m \left(\frac{a_{m-1}}{ma_m}\right)^m + a_m\left(\frac{a_{m-1}}{ma_m} + 1\right)^m \tag{41}$$

From equation (41) it is clear that

$$a_0 = l + \binom{m}{0} a_m \left(\frac{a_{m-1}}{m a_m} \right)^m - a_m \left(\frac{a_{m-1}}{m a_m} + 1 \right)^m \tag{42}$$

By means of equations (40) and (42) we obtain equation (35) in the form

$$x = l - a_m \left(\frac{a_{m-1}}{m a_m} + 1 \right)^m + a_m \left(\frac{a_{m-1}}{m a_m} + \xi \right)^m \tag{43}$$

The point of singularity is located outside the element, at $x = -ql \left(\xi = -\frac{a_{m-1}}{m a_m} \right)$, as shown in Figure 3. Substitution of these values into equation (43) gives the following expression:

$$-ql = l - a_m \left(\frac{a_{m-1}}{m a_m} + 1 \right)^m \tag{44}$$

Substitution of equation (44) into equation (43) results in

$$x = -ql + a_m \left(\frac{a_{m-1}}{m a_m} + \xi \right)^m \tag{45}$$

From equation (44) we can obtain the following:

$$a_{m-1} = m l^{1/m} a_m^{(m-1)/m} (1+q)^{1/m} - m a_m \tag{46}$$

By means of equation (46) we find

$$x = -ql + a_m \left[a_m^{-1/m} l^{1/m} (1+q)^{1/m} - 1 + \xi \right]^m \tag{47}$$

From equation (47) the coefficient a_m can be obtained by setting $x = 0$ and $\xi = -1$.

$$a_m = l \left[\frac{(1+q)^{1/m} - q^{1/m}}{2} \right]^m \tag{48}$$

Substitution of equations (40) and (42) into equation (36) gives the coefficients in general form.

$$\alpha_k l = a_m \left(\frac{a_{m-1}}{m a_m} - 1 + k \frac{2}{n} \right)^m - ql \tag{49}$$

By means of equations (46) and (48) the desired strain singularity occurs when

$$\alpha_k = \left[\frac{(n-k) q^{1/m} + k (1+q)^{1/m}}{n} \right]^m - q \qquad k = 1, 2, \cdots, n-1 \tag{50}$$

If we take into consideration the values of $k = 0$ and $k = n$, they will be the equivalent of $\alpha_0 = 0$ and $\alpha_n = 1$. The exponent m determines the order of the strain singularity and n is the order of the element. Between m and n the same relation is valid, as it was in the case of the singular elements.

$$m \leq n \tag{51}$$

Equation (50) relates the location of the element side nodes $\alpha_k l$ to the singular point ql outside the element domain, at which the $r^{(1-m)/m}$ singularity is found.

3.2 Special Cases of the General Mapping

It can be seen easily that we obtain the singular and non-singular elements as special cases of this general mapping.

3.2.1 Non-singular Isoparametric Elements

In order to obtain the non-singular elements, in equation (50) we have to choose $m = 1$ and $q = 0$. Then

$$\alpha_k = \frac{k}{n} \tag{52}$$

So we have from equations (46) and (48)

$$\begin{aligned} a_{m-1} &= a_0 = \frac{l}{2} \\ a_m &= a_1 = \frac{l}{2} \end{aligned} \tag{53}$$

From equation (35) or (47) it is clear that

$$x = \frac{l}{2}(1+\xi) \tag{54}$$

Figure 4 shows the behaviour of the transformation of equation (54) on a $\xi - x$ plane.

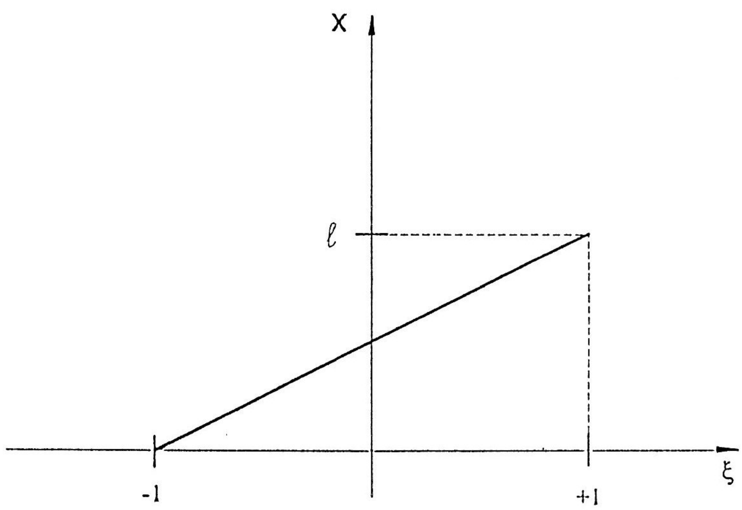


Figure 4. Behaviour of Linear Mapping

3.2.2 Singular Isoparametric Elements

Choosing $q = 0$ and $m \geq 2$ in equation (50), we have

$$\alpha_k = \left(\frac{k}{n}\right)^m \tag{55}$$

This expression is the same as equation (14). From equation (48) we can write

$$a_m = \frac{l}{2^m} \tag{56}$$

Equation (56) is the same as equation (11). By means of equation (47) we obtain

$$x = \frac{l}{2^m}(1+\xi)^m \tag{57}$$

Equation (57) is the same as equation (16). The character of the transformation of equation (57) is presented on a $\xi - x$ plane in Figure 5.

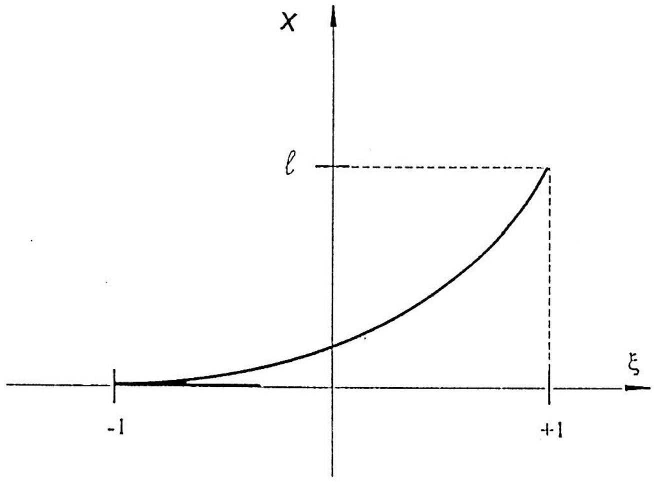


Figure 5. Character of General Mapping for Singular Elements

3.2.3 Transition Elements

Choosing $q \neq 0$ and $m \geq 2$, equation (50) shows the location α_k of the element side nodes. Substitution of equation (48) into equation (47) gives the following relation:

$$x = -ql + \frac{l}{2^m}\left\{(1+q)^{1/m} + q^{1/m} + \xi\left[(1+q)^{1/m} - q^{1/m}\right]\right\}^m \tag{58}$$

Figure 6 represents the character of the transformation of equation (58) on a $\xi - x$ plane.

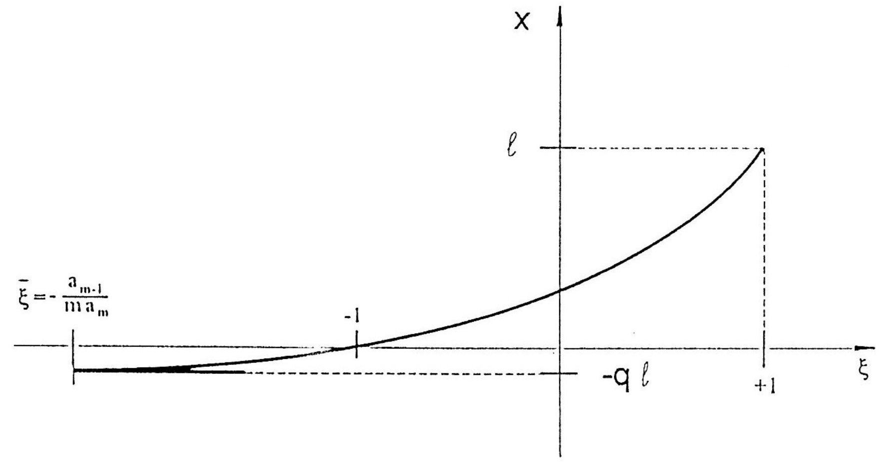


Figure 6. Character of General Mapping for Transition Elements

It becomes apparent that such a transition element can be easily constructed by simply shifting the side nodes to some properly calculated points. Equation (52) is the upper and equation (55) is the lower limit of equation (50) for the same values of m and n. Using this general mapping, it can be shown easily that we obtain the form in the case of the quadratic elements, which was demonstrated by Lynn and Ingraffea, (1978).

4 Some Numerical Examples

The isoparametric singular and transition elements are demonstrated by application of two simple examples.

4.1 Example 1

The example considered is that of a plate under tension that contains a crack of length $2a$ perpendicular to the direction of loading. The thickness is assumed to be unity.

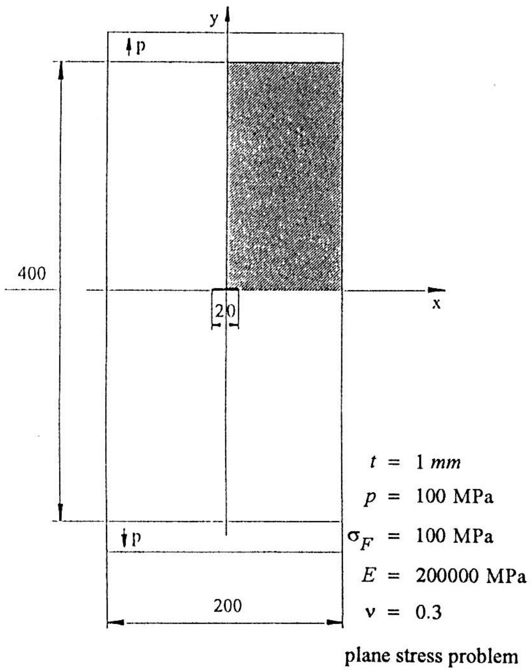


Figure 7. Centrally Cracked Plate

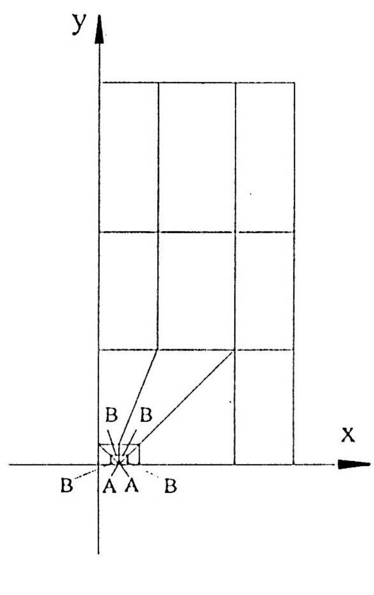


Figure 8. Mesh for Centrally Cracked Plate

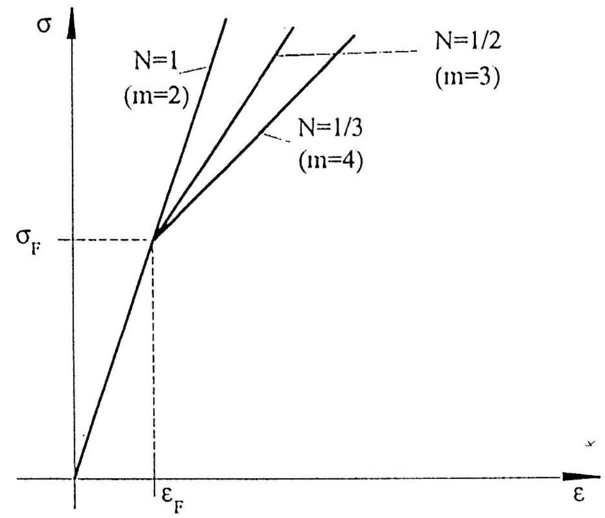


Figure 9. Types of Material Employed

The centrally cracked plate is shown in Figure 7. Figure 8 shows the finite element mesh employed in the solution. The mesh consists of 20 isoparametric elements. In the first calculations all elements are assumed to be non-singular 4th-order (quartic) isoparametric elements. In the second calculations the elements A at the crack tip were singular isoparametric elements. In the third calculations the elements A were singular and the elements B were transition elements. The material properties assumed for analysis are included in Figure 9. The J-integral values, obtained from the present numerical studies, are listed in Table 1 in the case of $p = 100$ MPa. Table 2 contains the J-integral values for $p = 130$ MPa.

J-integral [N / mm]			
m	No special elements	Singular elements	Singular + transition elements
2	1.6259	1.6310	1.6461
3	1.8038	1.8321	1.8339
4	2.0158	2.1103	2.1174

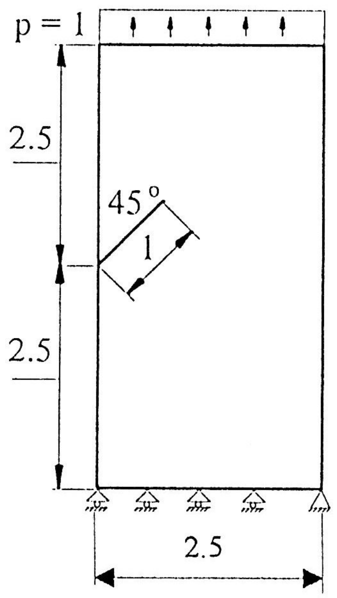
Table 1. J-integral Values in the case of $p = 100$ MPa

J-integral [N / mm]			
m	No special elements	Singular elements	Singular + transition elements
3	4.1395	4.2131	4.2174
4	5.3953	5.6027	5.6122

Table 2. J-integral Values for $p = 130$ MPa

4.2 Example 2

The problem considered is that of a plate under tension that contains an inclined edge crack, as shown in Figure 10. The finite element mesh is presented in Figure 11. The employed mesh consists of 18 cubic isoparametric elements. The type of material is linear elastic.



p = 1 MPa

E = 200000 MPa

ν = 0.3

plane stress problem

Figure 10. An Inclined Edge Crack

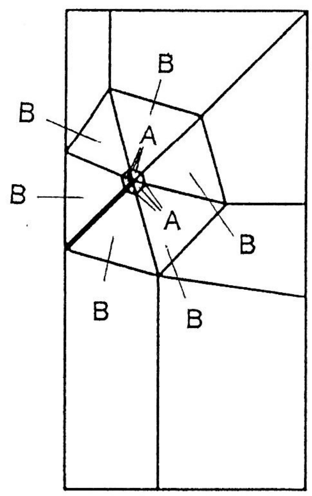


Figure 11. Mesh for the Inclined Edge Crack

In the first calculations the elements A were cubic, singular isoparametric elements. In the second calculations the elements A were singular and elements B transition elements. The K_I and K_{II} stress intensity factors calculated from J-integral are listed in Table 3 for different types of Gauss integration. Table 4 shows the same factors calculated from the displacement field for different types of Gauss integration. Table 5 contains the values obtained by Pu, Hussain and Lorensen and Bowie (1978). Pu, Hussain and Lorensen used another finite element mesh, and calculated the stress intensity factors from the displacement field. Bowie obtained his results by a conformal mapping technique.

	Gauss integration	$K_I\left[\mathrm{N\ mm}^{-3/2}\right]$	$K_{II}\left[\mathrm{N\ mm}^{-3/2}\right]$
Singular elements	3×3	1.667	1.126
Singular + transition elements	3×3	1.723	1.016
Singular elements	4×4	1.658	1.051
Singular + transition elements	4×4	1.752	0.873

Table 3. Stress Intensity Factors from J-integral

	Gauss integration	$K_I\left[\mathrm{N\,mm}^{-3/2}\right]$	$K_{II}\left[\mathrm{N\,mm}^{-3/2}\right]$
Singular elements	3×3	1.719	0.946
Singular + transition elements	3×3	1.88	0.981
Singular elements	4×4	1.786	0.952
Singular + transition elements	4×4	1.952	1.002

Table 4. Stress Intensity Factors from Displacement Field

	Gauss integration	$K_I\left[\mathrm{N\,mm}^{-3/2}\right]$	$K_{II}\left[\mathrm{N\,mm}^{-3/2}\right]$
Pu, Hussain, Lorensen	3×3	1.89	0.95
	4×4	1.83	0.92
Bowie	-	1.86	0.88

Table 5. Stress Intensity Factors

5 Conclusion

Through an extension of general mapping of element coordinates a simple scheme of forming of higher order singular and transition elements is presented. It has been demonstrated that the elements possess a strain singularity of order $r^{(1-m)/m}$. The transition elements contain the non-singular and the singular elements as extreme cases, too. The forming of elements is simple and the formulas are easily programmable. It was demonstrated that an nth-order element has $(n-1)$ different types of singularity. These elements should have wide applications to the analysis of 2D or 3D problems where fracture is investigated.

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