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# New Asymptotic Method for the Natural, Free and Forced Oscillations of Rectangular Plates with Mixed Boundary Conditions

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*We are concerned in this paper with the oscillation problem for rectangular plates with mixed boundary conditions by means of the new asymptotic method. The basic idea of the method presented may be described as follows. A parameter* $\varepsilon$ *is introduced into the boundary conditions in such a way that* $\varepsilon = 0$ *corresponds to the simple boundary problem and case* $\varepsilon = 1$ *corresponds to the problem under consideration. Then the* $\varepsilon$*-expansion of the solution is obtained. As a rule, just at point* $\varepsilon = 1$ *the expansion of the solution is divergent. Padé approximants may be used to remove this divergence.*

## 1 Introduction

Natural, free and forced vibrations of thin, homogeneous plates have been one of the most fundamental subjects in the study of the dynamics of structural members. Notable literature surveys of this subject were made by Young (1962), Leissa (1969, 1973), Timoshenko et al. (1974) and Gorman (1982). Many approaches, such as variational (Timoshenko et al., 1974), asymptotic (Kobayashi and Sonoda, 1986, 1991), numerical (Hughes et al., 1986) etc. are successfully used for solving this problem in the case of homogeneous boundary conditions. Sometimes an asymptotic approach presupposing the use of the WKB (Wentzel-Kramers-Brillouin) method may be very useful for the study of the vibration of plates and shells (Gristchak, 1990).

But above mentioned procedures, as a rule, work badly in the case of mixed boundary conditions. On the other hand, dynamic analysis of plates under mixed boundary conditions is of significant practical value: a lot of problems, arising in machine design, civil engineering etc. are of this type. The problems mentioned are usually solved using numerical methods, such as finite element procedures. Nevertheless, a numerical approach does not adequately fit the requirements of optimal structural design or any other kind of optimal structural design ideology. Then approximate analytical expressions, provided they are accurate enough, will be of great practical usefulness.

It seems that Bolotin's method may be advantageous for the above mentioned problematic. Bolotin (1961) proposed an effective asymptotic method for the investigation of linear continuous elastic system oscillations with complicated boundary conditions. Bolotin's method is also called the method of dynamic edge effects. The main idea of his approach is the separation of the continuous elastic system into two parts. In one of them - in the so-called interior zone - the solution may be expressed by trigonometric functions with unknown constants. In the other - in the dynamic edge effects zone - Bolotin used exponential functions. Then a matching procedure (along the edges of the unknown interior lines) allows one to obtain the unknown constants, and the complete solution of the problem may be written in relatively simple form. This approximate solution is very good for high frequency oscillations, but even for low frequency oscillations the error is not excessive (see Elishakoff, 1976; Andrianov and Krizhevsky, 1993, and references quoted therein). Unfortunately, for mixed boundary conditions Bolotin's method leads to a system of transcendental equations of higher order, and a solution of such a system is not an easy task.

Asymptotic procedures linked with Padé approximants (the so-called boundary conditions perturbation approach) was proposed by Andrianov (1991) and by Andrianov and Ivankov (1992, 1993), and successfully used for various static and dynamic plate problems. In the present paper we deal with natural and forced oscillations. The reader must have repeatedly wondered whether the asymptotic methods are of any practical use at all when computers are available? Is it not simple task to program the problem and solve it using, for example, a finite element procedure? The answer may be like this. Firstly the asymptotic methods are very useful in the preliminary stage of solving a problem even in cases where the principal aim is to obtain numerical results. An asymptotic analysis makes it possible to choose the best numerical method and gain an understanding of a vast body of numerical material, though not necessarily properly arranged. Secondly the asymptotic methods are especially effective in those regions of parameter values where machine computations are faced with serious difficulties. Moreover, the possibility exists of developing algorithms wherein smooth portions of solutions are obtained numerically, and the asymptotic approaches are applied to those parameter value regions where these solutions change drastically, say, within boundary layers (Steele, 1989). Therefore, it is quite proper to consider asymptotic and numerical methods not as competing, but as mutually complementary.

## 2 Natural Oscillation

Let us consider the application of the present approach to the natural oscillation, i.e. oscillation in one of the eigenmodes, of a rectangular plate ( $-0.5\,a < x < 0.5\,a$; $-0.5\,b < y < 0.5\,b$). The plate is simply supported along $x = \pm\, a/2$ and subjected to mixed boundary conditions ("clamped - hinged"), symmetrical with respect to $y$.

The governing differential equation may be written as follows:

$$\nabla^4 W - \lambda W = 0 \tag{1}$$

where $\lambda = \omega^2 \rho h b^4 D^{-1}$, $\omega$ - natural frequency, $D = \dfrac{Eh^3}{12(1-\nu^2)}$

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$; $W = \overline{W}/b$; $N = \overline{N}/b$; $x = \overline{x}/b$; $y = \overline{y}/b$; $k = a/b$;
$\overline{W}$, $\overline{N}$, $\overline{y}$, $\overline{x}$ – initial values of variables.

The boundary conditions may be formalized as

$$W = 0 \text{ and } W_{xx} = 0 \qquad \text{when} \qquad x = \pm\, 0.5k \tag{2}$$

$$W = 0 \text{ and } \left(1-\overline{H}(x)\right)W_{yy} \pm \overline{H}(x)W_y = 0 \qquad \text{when} \qquad y = \pm\, 0.5 \tag{3}$$

where $\overline{H}(x) = H(x-\mu k)+H(-x-\mu k)$, with $H(x)$ as Heaviside function. Introducing now the parameter $\varepsilon$ into the boundary condition according to the aforementioned procedure one obtains

$$W = 0 \text{ and } W_{yy} = -\varepsilon\overline{H}(x)\left(W_{yy} \pm W_y\right) \qquad \text{when} \qquad y = \pm\, 0.5 \tag{4}$$

The case $\varepsilon = 0$ brings us the plate, simply supported along the boundary; the case $\varepsilon = 1$ corresponds to the problem under consideration (1) - (3). The intermediate values of $\varepsilon$ are related to mixed conditions of the "simple support - elastic clamping" kind with elastic support coefficient $\mu = \varepsilon/(1-\varepsilon)$. Eigenvalue $\lambda$ and eigenfunction $W$ are presented by $\varepsilon$ -based expansions:

$$\lambda = \sum_{i=0}^{\infty}\lambda_i\varepsilon^i \qquad W = \sum_{i=0}^{\infty}W_i\varepsilon^i \tag{5}$$

Substituting series (5) into the governing boundary problem (1), (2), (4) and splitting it with respect to powers of $\varepsilon$, one obtains the recurrent sequence of boundary problems

For $\varepsilon^0$:

$$\nabla^4 W_0 - \lambda_0 W_0 = 0$$

$$W_0 = 0 \text{ and } W_{0xx} = 0 \qquad \text{while} \qquad x = \pm\, 0.5\,k$$

$$W_0 = 0 \text{ and } W_{0yy} = 0 \qquad \text{while} \qquad y = \pm\, 0.5$$

For $\varepsilon^j$:

$$\nabla^4 W_j - \lambda_0 W_j = \sum_{i=1}^{j}\lambda_i W_{j-1}$$

$$W_j = 0 \text{ and } W_{jxx} = 0 \qquad \text{while} \quad x = \pm\, 0.5\,k$$

$$W_j = 0 \text{ and } W_{jyy} = \mp \overline{H}(x)\sum_{i=0}^{j-1}\left(-W_{iy}\right) \qquad \text{while} \quad y = +\, 0.5$$

$$j = 1, 2, 3,...$$

Eliminating the nonuniformity of asymptotic expansions (we used for this purpose a routine asymptotic procedure (Nayfeh, 1973)), we obtain the expression for the eigenvalue in form of a truncated perturbation expansion.

$$\lambda = \pi^4\alpha^2 + 4\pi^2 n^2 \gamma_{mm}\varepsilon + \left\{4\pi^2 n^2 \gamma_{mm}\left(1 - \frac{\gamma_{mm}}{\pi^2\alpha}\left[\frac{\pi\beta_1}{2}\cot^{(-1)^j}\frac{\pi\beta_1}{2} + \frac{n^2}{\alpha_m} - \frac{3}{2}\right]\right)\right.$$

$$\left. -2\frac{n^2}{\alpha}\sum_{\substack{\left\{\begin{smallmatrix} i=1,3,5,\cdots \\ i=2,4,6,\cdots\end{smallmatrix}\right\} \\ i\neq m}}^{\infty} \gamma_{im}^2 \left[\alpha_{1i}\cot^{(-1)^j}\frac{\alpha_{1j}}{2} + \left\{\begin{matrix}\varphi_{1i}\cot^{(-1)^i}\frac{\varphi_{1i}}{2} \\ \beta_{1i}\cot^{(-1)^i}\frac{\beta_{1i}}{2}\end{matrix}\right\}\right]\right\}\varepsilon^2 \qquad \left\{\begin{matrix} i^2 > m^2 + n^2/k^2 \\ i^2 < m^2 + n^2/k^2 \end{matrix}\right\} \tag{6}$$

$$m, n = 1,2,3,...$$

Here $\alpha = n^2 + \left(m^2/k^2\right)$ $\qquad \beta_1 = \left(2(m/k)^2 + n^2\right)^{1/2}$ $\qquad \beta_z = \left((m/k)^2 + 2n^2\right)^{1/2}$

$\alpha_{1i} = \pi\left(\left(i^2 + m^2\right)/k^2 + n^2\right)^{1/2}$ $\qquad \beta_{1i} = \pi\left(\left(i^2 + m^2\right)/k^2 + n^2\right)^{1/2}$ $\qquad \varphi_{1i} = \pi\left(\left(i^2 - m^2\right)/k^2 - n^2\right)^{1/2}$

$$\gamma_{im} = \begin{cases} 2(0.5-\mu) - \dfrac{(-1)^m}{\pi m}\sin 2\pi\mu m & \text{when } i = m \\ \dfrac{4}{\pi}\dfrac{1}{m^2 - i^2}\left[\left\{\begin{matrix} i \\ m\end{matrix}\right\}\sin\pi\mu i\,\cos\pi\mu m - \left\{\begin{matrix} m \\ i\end{matrix}\right\}\sin\pi\mu m\,\cos\pi\mu i\right] & \text{when } i \neq m \end{cases}$$

Then the eigenfunction $W$ may be obtained. Its expression is very lengthy and is therefore not presented here. We shall use our method for the first eigenvalue of governing eigenvalue problem, because it is most strongly depended on boundary conditions. A truncated series (6) does not give a good result in comparison with exact solutions, which may be obtained for particular cases, therefore we shall use as method of improvement of the power series the method of Padé approximants (PA).

Let us introduce the PA definition (Baker and Graves-Morris, 1981). For an expansion given by

$$F(\varepsilon) = \sum_{i=0}^{\infty} c_i \varepsilon^i \tag{7}$$

the fractional-rational function $F(\varepsilon)\,[m/n]$ becomes

$$F(\varepsilon)\,[m/n] = \left(\sum_{i=0}^{m} a_i\varepsilon^i\right)\left(\sum_{i=0}^{n} b_i\varepsilon^i\right)^{-1} \tag{8}$$

and represents the PA of expansion (7), if a McLaurin expansion of $F(\varepsilon)$ shows the coincidents of its coefficience with corresponding ones of expansion (8) up to terms of $(m+n+1)$th order. The features of the PA are: it possesses uniqueness while $m$ and $n$ are chosen; it performs meromorphic continuation of function; for

its definition from the source expansion (7) the linear algebraic problem arises (Baker and Graves-Morris, 1981). For the case $\mu = 0.0$ the exact eigenvalue is known (Timoshenko et. al., 1974). For $k = m = n = 1$ we have $\lambda = (1.7050\pi)^4$, PA yields (for $\varepsilon = 1$) $\lambda = (1.7081\pi)^4$, showing less then 0.2 % discrepancy.
The results of numerical calculations for the mixed boundary conditions are displayed in Figures 1 and 2. Data obtained by our method are shown as continuous line. The dashed line displays the data obtained by the integral equations method (Keer and Stahl, 1972), the dot-and-dash line represents numerical results by Hamada and Ota (1963) and the dotted line describes the experimental results by the same authors.

## 3 Forced Oscillations

Let us consider the application of the present approach to the forced oscillation problem of the rectangular plate dealt with previously. We will use a new asymptotic technique (in comparison with above written procedure) in this section. The governing differential equation may be written as follows:

$$\nabla^4 W - \lambda_f W = q\cos\frac{\pi m x}{k}\cos\pi n y \qquad m, n = 1, 3, 5, \cdots \tag{9}$$

Here $q = q_0 b^3 / D$ with $q_0$ as amplitude of forcing pressure, $\lambda_f = \omega_f^2 \rho h b^4 D^{-1}$, $\omega_f$ – forcing frequency. The boundary condition may be formalized as in equations (2) and (3). Introducing the parameter $\varepsilon$ into the boundary condition according to the aforementioned procedure one obtains equation (4). Let us represent the bending moments, distributed along the sides $y = \pm 0.5$

$$M_{yy} = \sum_{m=1,3,5}^{\infty} A_m \cos\frac{\pi m x}{k} \qquad \text{for } y = \pm 0.5 \tag{10}$$

Statisfying the boundary conditions (4), one obtains the following infinite linear algebraic system with the coefficients $A_m$ as the unknowns:

$$A_i = \varepsilon \sum_{m=1,3,5,\cdots}^{\infty} \gamma_{im} A_m \left[1 - \frac{1}{2\lambda^{1/2}}\left(\alpha_{1m}\tanh\frac{\alpha_{im}}{2}{}_{1m} / 2 + \begin{Bmatrix} \varphi_{im}\tan\frac{\varphi_{im}}{2} \\ \beta_{im}\tanh\frac{\beta_{im}}{2} \end{Bmatrix}\right)\right] + \varepsilon\gamma_{im} Q \qquad i = 1, 3, 5, \ldots \tag{11}$$

Here $Q = \left(q\pi(-1)^{(n-1)/2}\right) / \left(\pi^4\alpha^2 - \lambda\right)$

$$\gamma_{im} = \begin{cases} 2\left(0.5 - \mu - \frac{1}{2\pi m}\right) & \text{when } i = 1 \\ \frac{4}{\pi}\frac{1}{\left(m^2 - i^2\right)}\left[i\sin\pi\mu i\cos\pi\mu m - m\sin\pi\mu m\cos\pi\mu i\right] & \text{when } i \neq 1 \end{cases}$$

Let us apply the perturbation technique to the system (11), representing $A_i$ as an $\varepsilon$-expansion.

$$A_i = \sum_{j=0}^{\infty} A_{i(j)}\varepsilon^j \tag{12}$$

Substituting equation (12) into system (11) and splitting it according to powers of $\varepsilon$, one obtains the recurrence formulas for $A_i$.

$$A_{i(0)} = 0$$

$$A_{i(1)} = \gamma_{im} / Q$$

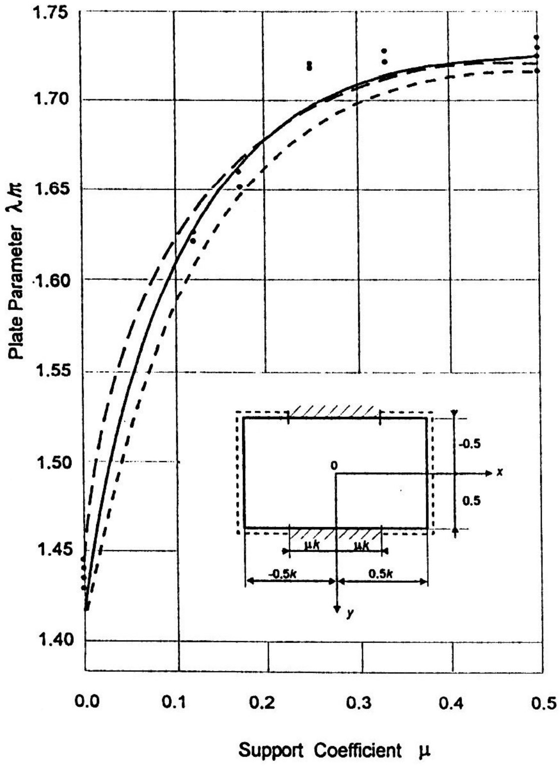


Figure 1. Plate Parameter versus Width

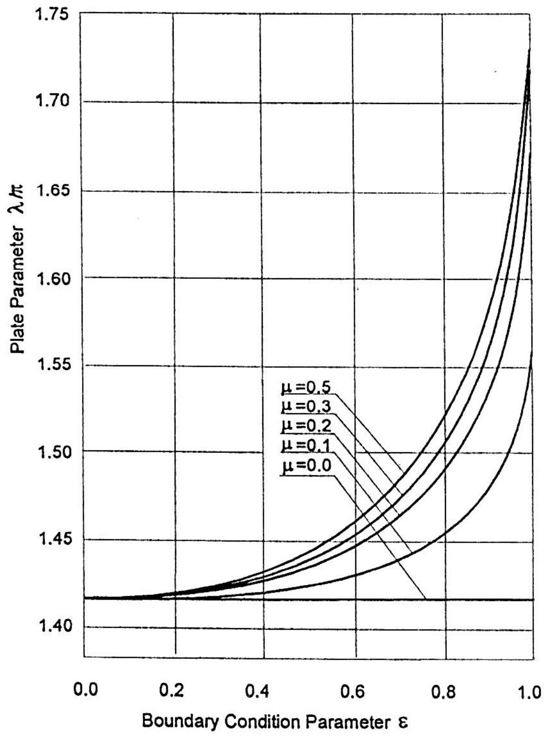


Figure 2. Plate Parameter versus Boundary Conditions

$$A_{1(p)} = \varepsilon \sum_{m=1,3,5,\cdots}^{\infty} \gamma_{1m} A_{m(p-1)} \left[ 1 - \frac{1}{2\lambda^{1/2}} \left( \alpha_{1m} \tanh \frac{\alpha_{1m}}{2} + \begin{Bmatrix} \varphi_{1m} \tan \frac{\varphi_{1m}}{2} \\ \beta_{1m} \tanh \frac{\beta_{1m}}{2} \end{Bmatrix} \right) \right] \tag{13}$$

The truncated perturbation expansion (kepping three initial non-zero terms) may be PA transformed.

$$A_i[1/1](\varepsilon) = \varepsilon \left( a_0 + a_1 \varepsilon \right) \left( b_0 + b_1 \varepsilon \right)^{-1} \tag{14}$$

where $a_0 = A_{i(1)}$, $b_0 = 1$, $a_1 = A_{i(2)} + b_1 A_{i(1)}$, $b_1 = -A_{i(3)} / A_{i(2)}$. An analysis was carried out for the square plate, with $m = n = 1$ and $\lambda = 0.225\pi^4$. The expansion (13) for the $A_i$ was truncated after ten (initial) terms for $\varepsilon = 1$. The deflection and bending moments in the center of the plate were calculated on the basis of expression (14) (see Figures 3 and 4). As the problem of frcc oscillations, with preseribed initial conditions, may be reduced to the forced one, it is not necessary to study free oscillations at this point.

## 4 Concluding Remarks

Various problems of statics, dynamics and stability of plates and shells, subjected to mixed boundary conditions, with prescribed initial conditions, may be solved effectively on the basis of the approach presented.

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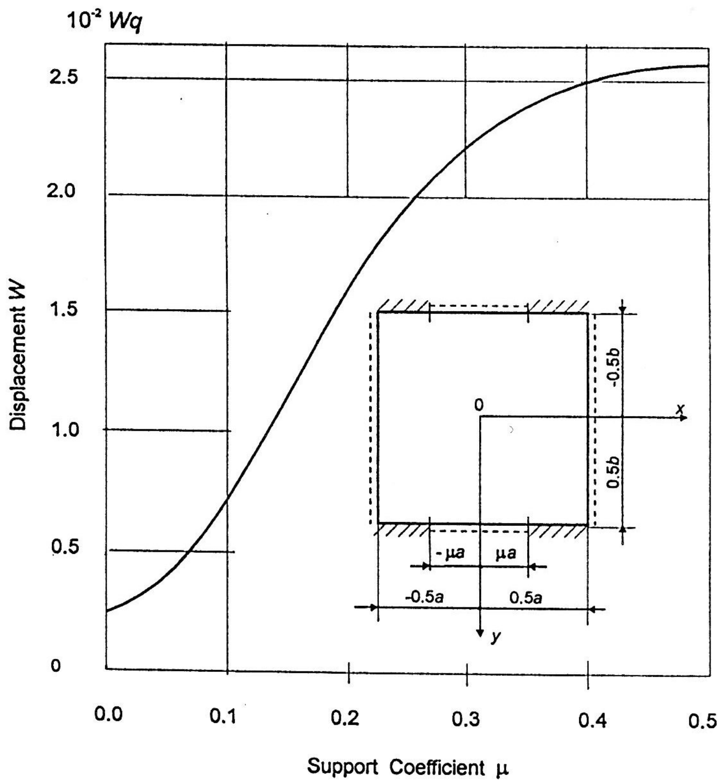


Figure 3. Deflection versus Support Length

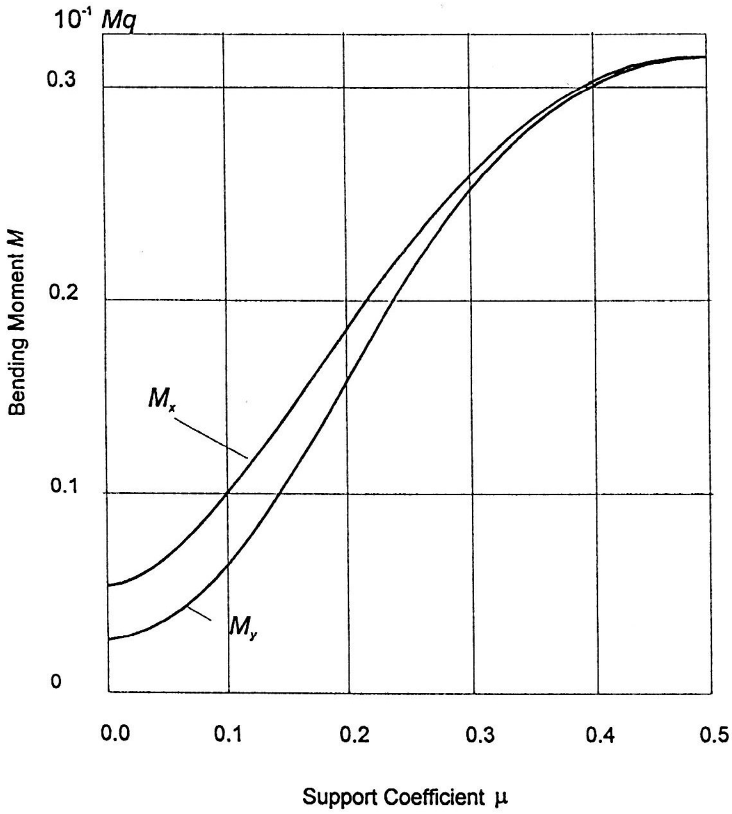


Figure 4. Bending Moments versus Support Length

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