TECHNISCHE MECHANIK, Band 14, Heft 3/4, (1994), 193-202
Manuskripteingang: 10. August 1994

Methods of Computer Algebra in the Theory of Shells

A. L. Smirnov, F. P. J. Rimrott

Asymptotic methods involving expansions in powers of certain parameters play a key role in the theory of plates and shells. Relevant equations can be obtained by a computer algebra approach in a form convenient for later asymptotic analysis.

Introduction

Engineering and applied mechanics problems are often solved by analytical or numerical methods. Since in the applications, known exact analytical solutions are rather rare, we search in most cases for approximate solutions. Among approximate analytical solutions, asymptotic methods based on expansions in powers of small or large parameters occupy a central place. When constructing an asymptotic solution in dynamic and buckling shell problems the complexity of the formulae increases drastically with the approximation number. That is why as a rule, one has to limit oneself to the first or the first two terms. At the same time the precision of the initial system allows one to obtain some more exact terms of the asymptotic series. The paper includes the first results of the application of computer algebra methods to derivation and asymptotic analysis of shell equations. The equations of the theory of plates and shells may be found in great detail, for example, in the monographs by Donnell (1976), Goldenveizer (1961, 1976), Grigolyuk and Kabanov (1978), Love (1944) and Novozhilov (1970). Studies of the equations of the theory of shells and plates, based on the asymptotic integration of the three-dimensional equations of elasticity theory, are developed by Goldenveizer (1976, 1979, 1982) and his pupils. A detailed review of this area may be found in Vaillancount and Smirnov (1993).

In this report we discuss the algorithm for transformation of the shell equations (see Grinkevich and Smirnov, 1994), which have been realized with *Mathematica* software (Wolfram, 1988). The aim of all these transformations is to represent the initial equations in a form convenient for later asymptotic analysis.

As a benchmark problem we consider linear equations describing the vibrations of a shell of revolution and use the following notation:

T_1, S_{21}, S_{12}, T_2	tangential forces
N_1, N_2	transverse forces
M_1, M_2, H_{21}, H_{12}	bending and twisting moments
ε_1, ω, ε_2	components of the tangential strain
κ_1, τ, κ	components of the bending strain
γ_1, γ_2	tangential shear strains
u, v	tangential components of displacement
w	deflection

The following functions are considered given:

R_1, R_2	curvature radii
B	distance to the axis of rotation

We use also the following constants: h - relative thickness, λ - frequency parameter, m - wave number, E - Young's modulus, ν - Poisson's ratio, R - characteristic radius.

After the substitution of the non-dimensional variables (dimensional variables are identified by an asterisk*)

$$\left(u,v,w,R_i,B,s\right)=\frac{1}{R}\left(u^*,v^*,w^*,R_i^*,s^*\right) \qquad \left(\varepsilon_i,\omega,\gamma_i\right)=\left(\varepsilon_i^*,\omega^*,\gamma_i^*\right)$$

$$h=\frac{1}{R\sqrt{12}}h^* \qquad \left(\kappa,\tau\right)=R\left(\kappa^*,\tau^*\right)$$

$$\left(T_i,S_{ij},N_i\right)=\frac{\left(1-\nu^2\right)}{Eh^*}\left(T_i^*,S_{ij}^*,N_i^*\right) \qquad \left(M_i,H_{ij}\right)=\frac{\left(1-\nu^2\right)}{REh^*}\left(M_i^*,H_{ij}^*\right)$$

we get the well-known system which consists of the following:

Formulae for shear strains versus displacements:

$$\gamma_1=-\frac{dw}{ds}-\frac{u}{R_1} \qquad \gamma_2 = -\frac{mw}{B}-\frac{u}{R_2}$$

Formulae for strains versus displacements:

$$\varepsilon_1 = \frac{du}{ds}-\frac{w}{R_1} \qquad \varepsilon_2 = \frac{B'u}{B}+\frac{mv}{B}-\frac{w}{R_2}$$

$$\omega = B\frac{d}{ds}\left(\frac{v}{B}\right)-\frac{mu}{B}$$

$$\kappa_1 = -\frac{d\gamma_1}{ds} \qquad \kappa_2 = \frac{B'}{B}\gamma_1-\frac{m}{B}\gamma_2$$

$$\tau = -\frac{m}{B}\frac{dw}{ds}+\frac{mB'}{B^2}w-\frac{mu}{BR_1}+\frac{B}{R_2}\frac{d}{ds}\left(\frac{v}{B}\right)$$

Shell equilibrium equations:

$$\frac{dT_i}{ds}+\frac{B'}{B}\left(T_1-T_2\right)+\frac{m}{B}S_{12}-\frac{N_1}{R_1}+\lambda u = 0$$

$$\frac{dS_{21}}{ds}+\frac{B'}{B}\left(S_{12}-S_{21}\right)+\frac{m}{B}T_2-\frac{N_2}{R_2}+\lambda v = 0$$

$$\frac{T_1}{R_1}+\frac{T_2}{R_2}+\frac{dN_1}{ds}+\frac{B'}{B}N_1+\frac{m}{B}N_2+\lambda w = 0$$

$$\frac{dM_1}{ds}+\frac{B'}{B}\left(M_1-M_2\right)+\frac{m}{B}H_{12}+N_1 = 0$$

$$\frac{dH_{21}}{ds}+\frac{B'}{B}\left(H_{12}+H_{21}\right)-\frac{m}{B}M_2+N_2 = 0$$

Elasticity relations:

$$T_1 = \varepsilon_2 + \nu\varepsilon_1 \qquad T_2 = \varepsilon_1 + \nu\varepsilon_2$$

$$S_{21} = \frac{1-\nu}{2}\left(\omega + 2h^2\frac{\tau}{R_2}\right) \qquad S_{12} = \frac{1-\nu}{2}\left(\omega + 2h^2\frac{\tau}{R_1}\right)$$

$$M_1 = h^2(\kappa_1 + \nu\kappa_2) \qquad M_2 = h^2(\kappa_2 + \nu\kappa_1)$$

$$H_{21} = H_{12} = h^2(1-\nu)\tau$$

Boundary conditions:

$$u_1 = u_1^0 \quad \text{or} \quad T_i = T_1^0$$

$$u_2 = u_2^0 \quad \text{or} \quad S_1 + \frac{H}{R_2} = S_1^0 + \frac{H^0}{R_2}$$

$$w = w^0 \quad \text{or} \quad Q_1 - \frac{1}{A_2}\frac{\partial H}{\partial \alpha_2} = Q_1^0 - \frac{1}{A_2}\frac{\partial H^0}{\partial \alpha_2}$$

$$\gamma_1 = \gamma_1^0 \quad \text{or} \quad M_1 = M_1^0$$

Thus we obtain a system of 20 equations with the following variables:
$u, v, w, T_1, S_{21}, T_2, N_1, N_2, M_1, M_2, H_{21}, \varepsilon_1, \omega, \varepsilon_2, \kappa_1, \tau, \kappa_2, \gamma_1, \gamma_2$. Several specific cases may be analyzed

- axisymmetric vibrations of a circular cylinder $(m = B' = 0,\ R_1 = \infty)$
- nonaxisymmetric vibrations of a circular cylinder $(B' = 0,\ R_1 = \infty)$
- axisymmetric vibration of a shell of revolution $(m = 0,\ B' = 0)$
- general case $(m \neq 0,\ B' \neq 0)$

2 Transformation to Standard Form

As a result the linear theory of shells leads to the algebro-differential system

$$A\dot{X} = BX \tag{1}$$

where dim (A) = $\dim(B)$ = $[n \times n]$, $\dim(X)$ = $[n \times 1]$, with linear boundary conditions

$$(\Gamma_1\dot{X} + \Gamma_2 X)(s_i) = 0 \quad \text{for} \quad i = 1 \ \text{or}\ 2 \tag{2}$$

where Γ_1 and Γ_2 are square matrices.

We denote as $\dim(A)$ the size of a matrix $A = [a_{ij}]$, for example, $\dim(A) = [n \times n]$. All coefficients and variables are functions in s, $s \in [s_1, s_2]$ by default. We denote by a dot the derivative with respect to s, for example $\dot{A} = [\dot{a}_{ij}]$. We write $A = 0$, *if* $\forall\, i,j,s \quad a_{ij}(s) \equiv 0$ and $|A(s)| = 0$, if this equality is valid for each s.

The analytical solution of system (1) may be found only in special cases. In the general case one can apply the methods of numerical or asymptotic integration. For this purpose it is better to represent equation (1) in the standard form

$$\dot{X} = CX \tag{3}$$

with the homogeneous boundary conditions

$$x_i(s_j) = 0 \qquad i = 1,\cdots,n \qquad j = 1 \text{ or } 2 \tag{4}$$

We propose the algorithm of such transformation in the following form:
Let rank $(A) = k$, where $k \le n$. If $k = n$ then $|A| \ne 0$ and the substitution $C = A^{-1}B$ solves the problem. If $k < n$ we denote $m = n - k$ and transform (1) into the following form:

$$\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where

$$\begin{aligned} &\dim(A_1) = \dim(B_1) = [k \times k] && \dim(A_2) = \dim(B_2) = [k \times m] \\ &\dim(B_3) = [m \times k] && \dim(B_4) = [m \times m] \\ &\dim(X_1) = [k \times 1] && \dim(X_2) = [m \times 1] \end{aligned}$$

We note that the last m equations do not contain derivatives.

If $|B_4| \ne 0$ one can express X_2 in terms of X_1 from the matrix equation $B_3X_1 + B_4X_2 = 0$ and obtain

$$X_2 = -B_4^{-1}B_3X_1 = DX_1 \tag{5}$$

If $|B_4| = 0$, we try to make the transposition of the variables $x_i^* = x_j$ with $i,j = 1,\cdots,n,$ such that $|B_4| \ne 0$. If for any transposition $|B_4| = 0$ then the initial system is non-definite. Finally we substitute equation (5) into the first k equations and get a system of differential equations of the order k for X_1

$$\begin{aligned} A^*\dot{X}_1 &= B^*X_1 \\ A^* &= A_1 + A_2D \\ B^* &= B_1 + B_2D - A_2\dot{D} \end{aligned} \tag{6}$$

and equations for the variables (5).

If $|A^*| = 0$, one can repeat the procedure, taking system (6) as the initial one, until after some step $|A^*|$ becomes non-zero. If there exist k variables x_{i_l} with $l = 1,\cdots,k$, such that

$$c_{ij} = 0 \qquad \text{for} \qquad i \neq i_l, \qquad j \neq i_l$$

then equation (3) splits into two independent systems of the orders k and n - k.

We may formulate this condition in terms of the graph theory. We consider matrix C as a vertex incidence matrix for some graph and introduce a Boolean matrix $\widetilde{C}$ as

$$\forall i,j \qquad \widetilde{C}_{ij} = \begin{cases} 1 \text{ if } c_{ij} \neq 0 \\ 0 \text{ if } c_{ij} = 0 \end{cases}$$

After that we transform matrix $\widetilde{C}$ to block form, where each block corresponds to some connected component. To effect this we may simply use the algorithm proposed by Warshall (1962) for the transitive closure of binary relations. This algorithm is of the order of $O(n^3)$. If matrix $\widetilde{C}$ consists of l blocks it means that the system splits into l systems. Returning to the shell equations we show that in all cases considered, the initial system is transformed to an 8-th order system in u, v, w, T_1, S_{21}, N_1, M_1, γ_1. Moreover in axisymmetric cases the system splits into two: one in u, w, T_1, N_1, M_1, γ_1 and the other in v and S_{21}. For example, in the case of axisymmetric vibrations of a circular cylinder $(m = 0)$ the transformation to the standard form gives us

$$\dot{X}_1 = CX_1$$

where

$$X_1 = (u,\ v,\ w,\ T_1,\ N_1,\ M_1,\ \gamma_1,\ S_{21})^T$$

and

$$c_{13} = -\nu \qquad c_{14} = -1 \qquad c_{28} = \frac{2}{(2h^2+1)(\nu-1)} \qquad c_{37} = 1 \qquad c_{41} = \lambda$$

$$c_{53} = \lambda + \nu^2 - 1 \qquad c_{54} = \nu \qquad c_{65} = 1 \qquad c_{76} = \frac{1}{h^2} \qquad c_{82} = \lambda\frac{2h^2+1}{4h^2+1}$$

This system agrees, for example, with that of Goldenveizer et al. (1979).

3 Transformation to Zero Boundary Conditions

At the same time we transform the boundary conditions. Since the boundary conditions have the same form for each edge we consider only one equation (2), say

$$\Gamma_1\dot{X} + \Gamma_2 X = 0$$

Rewriting it in the form

$$\Gamma_1^1\dot{X}_1 + \Gamma_1^2\dot{X}_2 + \Gamma_2^1 X_1 + \Gamma_2^2 X_2 = 0$$

we obtain due to equation (5)

$$\tilde{\Gamma}_1\dot{X}_1 + \tilde{\Gamma}_2 X_1 = 0$$

where

$$\tilde{\Gamma}_1 = \Gamma_1^1 + \Gamma_1^2 D$$

$$\tilde{\Gamma}_2 = \Gamma_2^1 + \Gamma_2^2 D + \Gamma_1^2\dot{D}$$

We assume that the initial system is reduced to

$$\dot{X} = CX$$

where $|C| \neq 0$, and

$$\tilde{\Gamma}_1\dot{X} + \tilde{\Gamma}_2 X = 0$$

To apply numerical or asymptotic methods to the analysis of such a system, it is often convenient to use variables for which the boundary conditions are homogeneous, i. e. have the form (4). For this purpose we try to construct the nonsingular linear substituion in the form $Y = FX$. By virtue of the system

$$\tilde{\Gamma}_1\dot{X} + \tilde{\Gamma}_2 X = (\tilde{\Gamma}_1 C + \tilde{\Gamma}_2)X = \Gamma X$$

we rewrite the boundary conditions on the left and the right edge correspondingly in the form

$$\Gamma^L X\big|_{s=s_1} = 0 \qquad \dim\left(\Gamma^L\right) = [l \times n]$$

$$\Gamma^R X\big|_{s=s_2} = 0 \qquad \dim\left(\Gamma^R\right) = [r \times n]$$

We assume that rank $\left(\Gamma^L\right) = l$ and rank $\left(\Gamma^R\right) = r$. Otherwise we have to exclude linear dependent rows.

Let us construct the matrix $F = \left(\frac{\Gamma^L}{\Gamma^R}\right)$ and calculate is rank k.

We do not consider the case $k > n$ where only the trivial solution of the initial system is possible. The other two cases are:

1. $k = n$ and
2. $k < n$

In the first case $|F| \neq 0$ and the substitution $Y = FX$ solves the problem. In the second case we construct the matrix F in the following form: The first l lines are the rows of the matrix Γ^L, the next $k-l$ lines are the rows of the matrix Γ^R (linearly independent of the rows of the matrix Γ^L) and the last $n-k$ lines are the rows of the identity matrix (linearly independent of previous rows). As a result (after transposition of the variables, if necessary) the matrix F has a form

$$F = \begin{bmatrix} \Gamma_l^L & \Gamma_{k-l}^L & \Gamma_{n-k}^L \\ \Gamma_l^{R1} & \Gamma_{k-1}^{R1} & \Gamma_{n-k}^{R1} \\ 0 & 0 & E_{n-k} \end{bmatrix}$$

The matrix F is nonsingular and we can express old variables X through new ones

$$X = F^{-1}Y \tag{7}$$

Substituting equation (7) into the initial system

$$F^{-1}Y + F^{-1}\dot{Y} = CF^{-1}Y$$

we obtain

$$\dot{Y} = F\left(CF^{-1} - F^{-1}\right)Y = C'Y$$

For the first case considered the boundary conditions for Y are always homogeneous. In general, for the second case, $n-k$ conditions are typically not homogeneous, but sometimes, for example, if the omitting row of the matrix Γ^R coincides with a row of the matrix Γ^L the corresponding condition is also homogeneous.

4 Transformation to Higher Order Equations

For asymptotic analysis it is useful to reduce the number of variables and equations in system (3). Definitely, the orders of the derivatives come up. Let us use k of n variables. We denote as α_i with $i = 1,\cdots,k$, the maximal orders of derivatives of the variables, and $\alpha^* = \max \alpha_i$. Obviously $\sum_{i=1}^{k}\alpha_i = n$. For example, let us take the first k variables. Evaluating the derivatives in equation (3) with respect to s we get

$$\ddot{X} = \dot{C}X + C\dot{X} = \left(\dot{C} + C^2\right)X$$

If we denote

$$C^{\{1\}} = C \qquad\qquad C^{\{2\}} = \dot{C} + C^2$$

then

$$\dddot{X} = \left(\dot{C}^{\{2\}} + C^{\{2\}}\cdot C\right)X = C^{\{3\}}X$$

Repeating the differentiation α^* times we get

$$X^{(l)} = \left(\dot{C}^{\{l-1\}} + C^{\{l-1\}}C\right)X = C^{\{l\}}X \qquad l \in \left[1,\alpha^*\right]$$

As a result we obtain α^* systems, each of which consists of n equations. We select α_1 equations for the derivatives of x_1, α_2 equations for the derivatives of x_2, etc. In other words we have a system of n equations

$$Z = GX$$

where

$$Z = \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \vdots \\ x_1^{(\alpha_1)} \\ x_2 \\ \vdots \\ x_k^{(\alpha_k)} \end{bmatrix} \qquad G = \begin{bmatrix} c_{11}^1 & c_{12}^1 \cdots c_{1n}^1 \\ c_{11}^2 & c_{12}^2 \cdots c_{1n}^2 \\ \vdots & \vdots \ddots \vdots \\ c_{11}^{\alpha_1} & c_{12}^{\alpha_1} \cdots c_{1n}^{\alpha_1} \\ c_{21}^1 & c_{22}^1 \cdots c_{2n}^1 \\ \vdots & \vdots \ddots \vdots \\ c_{k1}^{(\alpha_k)} & c_{k2}^{\alpha_k} \cdots c_{kn}^{\alpha_k} \end{bmatrix}$$

and the coefficients c_{ij}^m are the elements of the matrix $C^{\{m\}}$. We may exclude $n - k$ variables, transforming G into the form

$$\begin{bmatrix} G_1 & 0 \\ G_2 & G_3 \end{bmatrix}$$

where $\dim(G_1) = [k \times k], \dim(G_2) = [(n-k) \times k]$ and $\dim(G_3) = [(n-k) \times (n-k)]$. If we denote the matrix of the transformation as S, system (8) may be written as $SZ = (SG)X$. Now the k equations contain only the variables $x_1, \cdots, x_k$ and their derivatives. For example, for a cylindrical shell the initial system of equations may be transformed to one equations in w and its derivatives,

$$-h^2\left(\frac{d^6w}{ds^6} + \lambda\frac{d^4w}{ds^4}\right) + b\frac{d^2w}{ds^2} + \lambda(\lambda-1)w = 0$$

where

$$b(s,\lambda) = \lambda + \nu^2 - 1$$

which agrees with the corresponding equation in Goldenveizer et al. (1979) and gives us some additional terms.

5 Analysis of Asymptotically Small Terms

The linear shell theory equatins contain as naturally small parameters the shell thickness h and sometimes the wave number parameter m which may be equal to 0 (axisymmetric case) or large. Besides that the initial equations may also contain some other small parameters. For example, if we consider a rotating sthell the relative angular velocity Ω is a small parameter. The parameter h is considered as the main one and the other of small parameters are represented in the form h^{-p}, where $p \geq 0$. Our aim is to propose an algorithm which keeps only such terms in the shell equation coefficients which are the main ones at least for some values of parameters p and q. Let us start with the case when the equation coefficients depend only on the parameters p and q and look like $\Sigma_i a_i p^{\alpha_i} q^{\beta_i}$, where $a_i \sim 1$ and α_i, β_i are real (not necessarily integer or positive).

The proposed algorithm is the following:

1. We plot the set of points $\{\alpha_i,\ \beta_i\}$ on the plane $(\alpha_i,\ \beta_i)$ and transfer the coordinate origin in such a way that all the points have nonnegative coordinates. This means that we eliminate the lower power in p and q. For example, the polynomial

$$pq^2 + pq^3 + p^2 + p^3q^{-1} + p^4q + p^4q^2 + p^5 + p^6q^{-2}$$

is transformed to

$$pq^{-2}\left(q^4 + q^5 + pq^2 + p^2q + p^3q^3 + p^3q^4 + p^4q^2 + p^5\right)$$

2. Then we construct the convex hull of the point set $\{\alpha_i,\ \beta_i\}$.

3. Finally we choose such points on the convex hull of the point set which are seen from the coordinate origin. For the polynomial considered it would be

$$pq^2 + p^2 + p^3q^{-1} + p^6q^{-2}$$

It may be proved that we have kept all terms which are the main ones for any values of the parameters p and q.

The case of three and more parameters is much more difficult. To our knowledge there is an algorithm of the order of $n \cdot \log n$, which determines the convex hull for the point set in 3-D. In the plane case we use a simple algorithm of the order of n^2. As an example we write one of the coefficients of the shell equations (shell of revolution, general case), which depends on two small parameters h and m^{-1}. Before simplification it is

$$-B'\left(8B^2h^4\nu^2 - 8B^2h^4\nu^2 - 8B^2h^4 + 6B^2h^2R^2 + B^2\nu^2R^4 + 2h^4m^2(\nu-1)\right)\nu R^2$$

$$-4h^4m^2(\nu-1)R^2 - 4h^4m^2\nu^2R^2 + 4h^4m^2R^2 + h^2m^2(\nu-1)\nu R^4 - h^2m^2(\nu-1)R^4$$

$$-B^2R^4 - h^2m^2(\nu-1)R^4 - 2h^2m^2\nu^2R^4 + 2h^2m^2R^4) / \left(8B^3h^4R + 6B^3h^2R^3 + B^3R^5\right)$$

and after the simplification

$$-\left(B'\left(B^2\left(\nu^2-1\right) + h^2m^2(1-\nu)(4+\nu)\right)\right) / \left(B^3R\right)$$

6 Conclusions

The ultimate goal of the proposed approach is to develop an algorithm for the asymptotic integration of the shell equations considered in this paper. The problem is complicated by the numerous small and large parameters $(h,\ m,\ \Omega)$. Using computer algebra algorithmus we aim to obtain the highest terms for the asymptotic expansions of the solutions.

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Addresses: Professor A. Smirnov, Department of Theoretical and Applied Mechanics, St. Petersburg State University, St. Petersburg, Russia 198 904; Professor F.P.J. Rimrott, Department of Mechanical Engineering, University of Toronto, Toronto, Ontario, Canada M5S 1A4.