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Buckling of Imperfect Sandwich Cones under Axial Compression - Equivalent-Cylinder Approach. Part I

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In order to simplify the problem, the equivalent-cylinder assumption has been adopted. The simplified governing equations of sandwich cones we obtain is different from the ones used in previous papers. They can be reduced into the equations of a corresponding sandwich cylinder if the buckle length parameter approaches zero. An analytical study has been carried out to determine the effect of axisymmetric shape imperfections on the compressive buckling strength of sandwich cones having isotropic facings and isotropic shear deformable cores. Buckling solutions are presented as a function of imperfection amplitude, wavelength, core shear flexibility coefficients and the small curvature ratio. The well known Koiter formula and circle have been obtained for the first time for sandwich cones.

1 Introduction

Circular conical shell structures are widely used in aerospace vehicles, such as in rockets and satellite components. Although these structures are commonly fabricated from metals, advanced composites are also gaining widespread usage because of their higher strength/weight and stiffness/weight ratios. Prior to the advent of composites, however, the development of sandwich constructions emerged because of the significant stiffness/weight improvements offered (Plantema, 1966). The buckling problem of such structures has to be considered in engineering. Most published results deal with sandwich cylinders and there is a little about sandwich cones (Sullins etc., 1969). The „equivalent-cylinder“ concept of Seide (1956) has been adopted generally as a practical expediency. The core is regarded as shear-deformable and the cones are treated as geometrically perfect in shape.

Up to this time, no Koiter-type imperfection analysis has been carried out on circular conical sandwich shells under axial compression, to assess the extent to which these structures are sensitive to geometric shape imperfections. In the present paper , the equivalent-cylinder assumption has again been adopted. The simplified governing equations for sandwich cones we get are different from the ones of former papers. They can be reduced into the equations of a corresponding sandwich cylinder if the small wave length parameter δ approaches zero. An analytical study has been carried out to determine the effect of axisymmetric shape imperfections on the compressive buckling strength of sandwich cones having isotropic facings and isotropic shear deformable cores. Buckling solutions are presented as a function of imperfection amplitude, wavelength, the core shear flexibility coefficients and a small material geometry parameter χ_c. The well known Koiter formula and circle have been obtained for the first time for sandwich cones in this paper.

The aim of present paper is to get some experience and useful information on the buckling of sandwich cones . It is only a first step on the road of studying the buckling of sandwich shells.

2 Formulation of Problem

Shell configuration

The axisymmetric imperfect sandwich cone geometry and coordinate system are shown in Figure 1. The cone geometry is characterized by its slant length, radius of the median surface R_c at the middle of slant, inner face thickness t_1, outer face thickness t_2, core thickness c and imperfection amplitude μ (see Figure 2). Let the reference surfaces be the median surface of the geometrically perfect cone, as defined by

$$h_1 = \frac{ht_2}{t_1 + t_2} \tag{1}$$

This definition has an advantage which can remove the coupling term in the constitutive relations. Thus the separation between the mid-surface of the inner and outer facings is

$$h = c + \frac{t_1 + t_2}{2} \tag{2}$$

The coordinate systems x, y, z or s, θ, z are measured with respect to the reference surface in the axial, circumferential and radial directions, respectively. The components of displacement u, v and w of a point on the perfect shell are displacements in the x, y, z or s, θ and z directions.

Restrictions and assumptions

The following restrictions apply to the analytical model (Tennyson and Chan, 1990):

(1) Both facings and the core are made of the same isotropic material
(2) The core and facings have uniform elastic properties
(3) Facings and core are of constant thickness throughout the shell wall, the two facings have different thicknesses
(4) There is no initial wrinkling in the facings, i.e. the separation between the facings is constant
(5) There is no failure of bonding between facings and core
(6) Facings are sufficiently thin (compared to the core) to be treated as membranes, i.e. the facings have in-plane stiffness but no flexural stiffness about their mid-surfaces
(7) The shell thickness is small compared with the radius of curvature R in the middle of slant
(8) The cone is long enough to ignore end boundary conditions
(9) There is no intercell buckling

The following assumptions are made in the analysis:

(1) Displacements u, v and w are small compared with the shell thickness
(2) Strains ε_x, ε_y and ε_{xy} are small compared to unity (small strain theory)
(3) The core carries no in-plane stress
(4) Normal stiffness of the core is infinite so that instability associated with wrinkling of facings and other normal strain effects is not included. In practice, sandwich cones with honeycomb core will not fail by wrinkling of facings when there is no failure of bonding between facings and core (Plantema, 1966; Allen, 1967)
(5) The transverse normal stress is negligible, i.e. $\sigma_z = 0$

The above assumptions mean that the shell equations are of the Donnell-Mushtari-Vlassov (DMV) type. The DMV type equations are used here rather than more exact stability equations for the following reasons:

(a) A stress function F can be introduced, which leads to a reduction in the number of dependent variables in the analysis. In the DMV type analysis the four dependent variables are F, w, β_x and β_y ; in a more exact analysis the five dependent variables are u, v, w, β_x and β_y.

(b) The DMV type analysis yields sufficiently accurate results for almost all engineering applications. It is not accurate when applied to shells which buckle in an almost inextensional mode, such as shells with weak support at the edges. However, in most engineering applications, inextensional buckling modes may be eliminated by proper design of the structure which supports the shell.

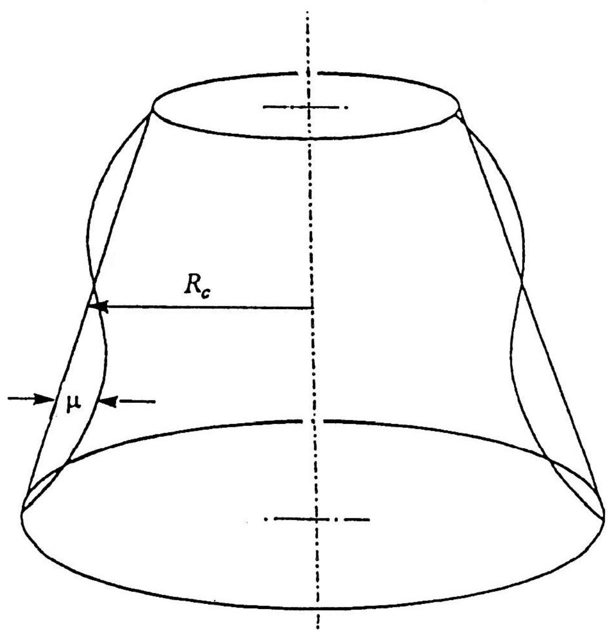

Figure 1. Circular cone with axisymmetric shape imperfections

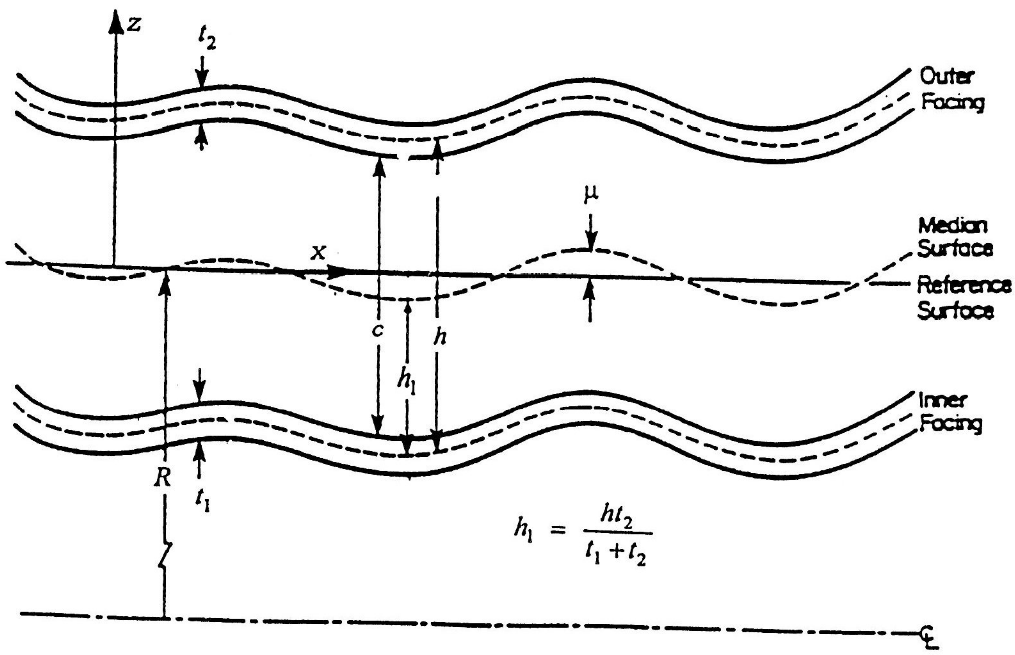

Figure 2. Geometrical definition of sandwich cone wall

3 The Sandwich-Oriented Theory of Cone

Strain-displacement relations

Using the nonlinear DMV strain-displacement relations, the relations between the displacement components u, v and w of the deformed median surface and the displacement components u_z, v_z and w_z of a point in the shell are

$$u_z(s,\theta,z) = u(s,\theta) + z\beta_s(s,\theta) \qquad v_z(s,\theta,z) = v(s,\theta) + z\beta_\theta(s,\theta) \qquad w_z(s,\theta,z) = w(s,\theta) \tag{3}$$

where β_s and β_θ can be interpreted physically as the components of change of slope of the normal to the undeformed median surface. (3) may be called a 5-variable mathematical model of sandwich cones.

When the thickness of the facings is sufficiently small compared with the core, and when the transverse core shear strain is small, the strains for the mid-surfaces of the facings can be approximated by the strains at the facings and the core interface. By this assumption, the facings are in effect considered to be membranes.

In this way, we can get the strain-displacement relations as follows:

$$E_s = \varepsilon_s + z\kappa_s \qquad E_\theta = \varepsilon_\theta + z\kappa_\theta \qquad E_{s\theta} = \varepsilon_{s\theta} + z\kappa_{s\theta} \qquad E_{sz} = \varepsilon_{sz} + \beta_s \qquad E_{\theta z} = \varepsilon_{\theta z} + \beta_\theta \tag{4}$$

where the median surface strains and changing ratios of curvature are given by

$$\varepsilon_s = u_{,s} + \frac{1}{2}(w_{,s})^2 \qquad \varepsilon_\theta = \frac{1}{s\cos\varphi} v_{,\theta} + \frac{u}{s} + \frac{w}{s}\tan\varphi + \frac{1}{2s^2\cos^2\varphi}(w_{,\theta})^2$$

$$\varepsilon_{s\theta} = v_{,s} - \frac{v}{s} + \frac{1}{s\cos\varphi} u_{,\theta} + \frac{1}{s\cos\varphi} w_{,s} w_{,\theta} \tag{5}$$

$$\kappa_s = \beta_{s,s} \qquad \kappa_\theta = \frac{1}{s\cos\varphi}\beta_{\theta,\theta} + \frac{\beta_s}{s} \qquad \kappa_{s\theta} = \beta_{\theta,s} - \frac{\beta_\theta}{s} + \frac{1}{s\cos\varphi}\beta_{s,\theta} \tag{6}$$

$$\varepsilon_{sz} = w_{,s} \qquad \varepsilon_{\theta z} = \frac{1}{s\cos\varphi} w_{,\theta} \tag{7}$$

$$(:)_{,s} = \frac{\partial(:)}{\partial s} \qquad (:)_{,\theta} = \frac{\partial(:)}{\partial\theta} \tag{8}$$

We assume that the median surface displacements u and v do not vary in z direction. The intial stress-free lateral deviation term w_0 of the median surface is assumed to be small, but finite values are also permissible provided that

$$|w_0| \le R_c \qquad |w_{0,s}| \le 1 \tag{9}$$

Shallow shell theory requires that the radius of curvature of the shape imperfection cannot be excessively small

$$R_c|w_{0,ss}| \le 0\,(1) \tag{10}$$

The modified strain-displacement relations for the median surface including the initial imperfections are

$$\varepsilon_s = u_{,s} + \frac{1}{2}(w_{,s})^2 + w_{,s}w_{0,s}$$

$$\varepsilon_\theta = \frac{1}{s\cos\varphi}v_{,\theta} + \frac{u}{s} + \frac{w}{s}\tan\varphi + \frac{1}{2s^2\cos^2\varphi}(w_{,\theta})^2 + \frac{1}{s^2\cos^2\varphi}w_{,\theta}w_{0,\theta} \tag{11}$$

$$\varepsilon_{s\theta} = v_{,s} - \frac{v}{s} + \frac{1}{s\cos\varphi}u_{,\theta} + \frac{1}{s\cos\varphi}w_{,s}w_{,\theta} + \frac{1}{s\cos\varphi}w_{,s}w_{0,\theta} + \frac{1}{s\cos\varphi}w_{0,s}w_{,\theta}$$

$$\varepsilon_{sz} = w_{,s} \qquad \varepsilon_{\theta z} = \frac{1}{s\cos\varphi}w_{,\theta} \tag{12}$$

$$\kappa_s = \beta_{s,s} \qquad \kappa_\theta = \frac{1}{s\cos\varphi}\beta_{\theta,\theta} + \frac{\beta_s}{s} \qquad \kappa_{s\theta} = \beta_{\theta,s} - \frac{\beta_\theta}{s} + \frac{1}{s\cos\varphi}\beta_{s,\theta} \tag{13}$$

<u>Constitutive relations</u>

(a) *Core relations* Assuming the core only resistant to transverse shear and not to carry any in-plane stresses, the stress-strain relations for the core are

$$\tau_{sz} = G\varepsilon_{sz} \qquad \tau_{\theta z} = G\varepsilon_{\theta z} \qquad \sigma_s = \sigma_\theta = \tau_{s\theta} \tag{14}$$

The transverse shear stresses can then be expressed in terms of the median surface strains.

$$\tau_{sz} = G(\varepsilon_{sz} + \beta_s) \qquad \tau_{\theta z} = G(\varepsilon_{\theta z} + \beta_\theta) \tag{15}$$

(b) *Facing relations* The in-plane stress-strain relations for thc isotropic facings in a plane stress state are

$$\sigma_s = \frac{E_{f_i}}{1-\nu_i^2}(\varepsilon_s + \nu_i\varepsilon_\theta) \qquad \sigma_\theta = \frac{E_{f_i}}{1-\nu_i^2}(\varepsilon_\theta + \nu_i\varepsilon_s) \qquad \tau_{s\theta} = \frac{E_{f_i}}{2(1+\nu_i)}\varepsilon_{s\theta} \tag{16}$$

<u>Shell forces and moments</u>

An equivalent system of force and moment resultants is considered to be acting at the median surface of an element of the shell as defined by the following relations:

(a) Stress resultants

$$[N_s \ N_\theta \ N_{s\theta}]^T = \int_{h_1 - t_1/2}^{h_2 + t_2/2} [\sigma_s \ \sigma_\theta \ \tau_{s\theta}]^T dz \tag{17}$$

(b) Transverse shearing stress resultants

$$[Q_s \ Q_\theta]^T = \int_{h_1 - t_1/2}^{h_2 + t_2/2} [\tau_{sz} \ \tau_{\theta z}]^T dz \tag{18}$$

(c) Moment resultants

$$[M_s \ M_\theta \ M_{s\theta}]^T = \int_{h_1 - t_1/2}^{h_2 + t_2/2} [\sigma_s \ \sigma_\theta \ \tau_{s\theta}]^T z dz \tag{19}$$

where the integration is taken across the whole shell wall and h_1 and h_2 are the distances of the middle surfaces of the inner and outer facings, respectively, from the median surface of the total shell wall.

After performing the above integration, we have

$$[N] = [A][\varepsilon] + [B][\kappa] \qquad [\mathrm{M}] = [B][\varepsilon] + [D][\kappa]$$

$$Q_s = Gc\left(\varepsilon_{sz} + \beta_s\right) \approx Gh\left(\varepsilon_{sz} + \beta_s\right) \qquad Q_\theta = Gc\left(\varepsilon_{\theta z} + \beta_\theta\right) \approx Gh\left(\varepsilon_{\theta z} + \beta_\theta\right) \approx Gh\left(\varepsilon_{\theta z} + \beta_\theta\right) \tag{20}$$

where

$$[N] = \left[N_s \;\; N_\theta \;\; N_{s\theta}\right]^T \qquad [M] = \left[M_s \;\; M_\theta \;\; M_{s\theta}\right]^T$$

$$\varepsilon = \left[\varepsilon_s \;\; \varepsilon_\theta \;\; \varepsilon_{s\theta}\right]^T \qquad [\kappa] = \left[\kappa_s \;\; \kappa_\theta \;\; \kappa_{s\theta}\right]^T \tag{21}$$

$$[A] = t_1[Q_1] + t_2[Q_2] \qquad [B] = h_2 t_2[Q_2] - h_1 t_1[Q_1] \qquad [D] = \left(h_1^2 + \frac{t_1^2}{12}\right)[\,Q_1] + \left(h_2^2 + \frac{t_2^2}{12}\right)[\,Q_2] \tag{22}$$

$$[Q_i] = \frac{E_{f_i}}{1-\nu_i^2}\begin{bmatrix} 1 & \nu_i & 0 \\ \nu_i & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu_i}{2} \end{bmatrix} \tag{23}$$

In the process of above integration, the assumption of linear variation of transverse shear stresses across the facings has been used, since the transverse shear stresses in the facings are equal to the transverse shear stresses in the core at the facings, and vanish on the free surface.

The above represention of forces and moment resultants can be rewritten in detail.

$$M_s = \frac{E_f}{1-\nu^2}\left[\left(h_2 t_2 - h_1 t_1\right)\left(\epsilon_s + \nu\, \epsilon_\theta\right) + \Gamma\left(\kappa_s + \nu\kappa_\theta\right)\right]$$

$$M_\theta = \frac{E_f}{1-\nu^2}\left[\left(h_2 t_2 - h_1 t_1\right)\left(\varepsilon_\theta + \nu\varepsilon_s\right) + \Gamma\left(\kappa_\theta + \nu\kappa_s\right)\right] \qquad M_{s\theta} = \frac{E_f}{1-\nu^2}\left[\left(h_2 t_2 - h_1 t_1\right)\varepsilon_{s\theta} + \Gamma\kappa_{s\theta}\right] \tag{24}$$

$$N_s = \frac{E_f}{1-\nu^2}\left[\left(t_1 + t_2\right)\left(\varepsilon_s + \nu\varepsilon_\theta\right) + \left(h_2 t_2 - h_1 t_1\right)\left(\kappa_s + \nu k_\theta\right)\right] \quad N_\theta = \frac{E_f}{1-\nu^2}\left[\left(t_1 + t_2\right)\left(\varepsilon_\theta + \nu\varepsilon_s\right) + \left(h_2 t_2 - h_1 t_1\right)\left(\kappa_\theta + \nu\kappa_s\right)\right]$$

$$N_{s\theta} = \frac{E_f}{1-\nu^2}\left[\varepsilon_{s\theta} + \left(h_2 t_2 - h_1 t_1\right)\kappa_{s\theta}\right] \qquad Q_s = Gh\left(\varepsilon_{sz} + \beta_s\right) \qquad Q_\theta = Gh\left(\varepsilon_{\theta z} + \beta_\theta\right) \tag{25}$$

$$\Gamma = h_1^2 t_1 + \frac{t_1^3}{12} + h_2^2 t_2 + \frac{t_2^3}{12} \qquad \nu_1 = \nu_2 = \nu \tag{26}$$

With the definition (1) of the median surface, the coupling between in-plane strains and rotation of the normal displacements, and stress and moment resultants simplify to

$$N_s = \frac{A}{1-\nu^2}\left(\varepsilon_s + \nu\varepsilon_\theta\right) \qquad N_\theta = \frac{A}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_s\right) \qquad N_{s\theta} = \frac{A}{2(1+0)}\,\varepsilon_{s\theta} \tag{27}$$

$$M_s = D(\kappa_s + \nu k_\theta) \qquad M_\theta = D(\kappa_\theta + \nu\kappa_s) \qquad M_{s\theta} = D\frac{1-\nu}{2}\kappa_{s\theta} \tag{28}$$

where the in-plane stiffness A and bending stiffness D are given by:

$$A = E_f(t_1 + t_2) = E_f t \qquad D = \frac{E_f}{1-\nu^2}\left(\frac{h^2 t_1 t_2}{t_1 + t_2} + \frac{t_1^3 + t_2^3}{12}\right) \approx \frac{E_f h^2 t_1 t_2}{(1-\nu^2)t} \tag{29}$$

Equilibrium equations

From Novozhilov's theory (1953), we have the following equations for conical shells:

(a) Equilibrium of horizontal forces

$$(sN_s)_{,s} + \frac{1}{\cos\varphi}N_{s\theta,\theta} - N_\theta = 0 \qquad (sN_{s\theta})_{,s} + \frac{1}{\cos\varphi}N_{\theta,\theta} + N_{s\theta} = 0 \tag{30}$$

(b) Equilibrium of vertical forces

$$\frac{1}{s}\left[(sQ_s)_{,s} + \frac{1}{\cos\varphi}Q_{\theta,\theta}\right] - N_s(\chi_s + \chi_s^0) - 2N_{s\theta}(\chi_{s\theta} + \chi_{s\theta}^0) - N_\theta(\chi_\theta + \chi_\theta^0) - \frac{N_\theta}{s}\tan\varphi - q_n = 0 \tag{31}$$

where

$$\chi_s = -w_{,ss} \qquad \chi_\theta = -\left(\frac{1}{s}w_{,s} + \frac{1}{s^2\cos^2\varphi}w_{,\theta\theta}\right) \qquad \chi_{s\theta} = \frac{1}{s\cos\varphi}\left(w_{,s\theta} - \frac{1}{s}w_{,\theta}\right) \tag{32}$$

$$\chi_s^0 = -w_{0,ss} \qquad \chi_\theta^0 = -\left(\frac{1}{s}w_{0,s} + \frac{1}{s^2\cos^2\varphi}w_{0,\theta\theta}\right) \qquad \chi_{s\theta}^0 = \frac{1}{s\cos\varphi}\left(w_{0,s\theta} - \frac{1}{s}w_{0,\theta}\right) \tag{33}$$

(c) Equilibrium of moments

$$sQ_s = (sM_s)_{,s} + \frac{1}{\cos\varphi}M_{s\theta,\theta} - M_\theta \qquad Q_\theta = \frac{1}{s\cos\varphi}M_{\theta,\theta} + \frac{1}{s}(sM_{s\theta})_{,s} + \frac{1}{s}M_{s\theta} \tag{34}$$

An Airy stress function F is defined such that it will satisfy the equilibrium equations (30) in horizontal direction identically, i.e.

$$N_s = \frac{1}{s}F_{,s} + \frac{1}{s^2\cos^2\varphi}F_{,\theta\theta} \qquad N_\theta = F_{,ss} \qquad N_{s\theta} = -\frac{1}{s\cos\varphi}\left(F_{,s\theta} - \frac{1}{s}F_{,\theta}\right) \tag{35}$$

By equations (13), (28) and (34), the shear resultants can be expressed in terms of shear angles as

$$\begin{aligned} Q_s &= D\left[\beta_{s,ss} + \frac{1}{s}\beta_{s,s} - \frac{\beta_s}{s^2} + \frac{1-\nu}{2}\frac{1}{s^2\cos^2\varphi}\beta_{s,\theta\theta}\right] + D\left[\frac{1+\nu}{2}\frac{1}{s\cos\varphi}\beta_{\theta,s\theta} - \frac{3-\nu}{2}\frac{1}{s^2\cos\varphi}\beta_{\theta,\theta}\right] \\ Q_\theta &= D\left[\frac{1+\nu}{2}\frac{1}{s\cos\varphi}\beta_{s,s\theta} + \frac{3-\nu}{2}\frac{1}{s^2\cos\varphi}\beta_{s,\theta}\right] + D\left[\frac{1-\nu}{2}\left(\beta_{\theta,ss} + \frac{1}{s}\beta_{\theta,s} - \frac{\beta_\theta}{s^2}\right) + \frac{1}{s^2\cos^2\varphi}\beta_{\theta,\theta\theta}\right] \end{aligned} \tag{36}$$

Since the shear forces are carried by the core, replacing the shear forces in equation (36) by constitutive relations (20) of the core, we have

$$Gc\left(w_{,s}+\beta_s\right) = D\left[\beta_{s,ss}+\frac{1}{s}\beta_{s,s}-\frac{\beta_s}{s^2}+\frac{1-\nu}{2}\frac{1}{s^2\cos^2\varphi}\beta_{s,\theta\theta}\right]+D\left[\frac{1+\nu}{2}\frac{1}{s\cos\varphi}\beta_{\theta,s\theta}-\frac{3-\nu}{2}\frac{1}{s^2\cos\varphi}\beta_{\theta,\theta}\right] \tag{37}$$

$$Gc\left(\frac{1}{s\cos\varphi}w_{,\theta}+\beta_\theta\right) = D\left[\frac{1+\nu}{2}\frac{1}{s\cos\varphi}\beta_{s,s\theta}+\frac{3-\nu}{2}\frac{1}{s^2\cos\varphi}\beta_{s,\theta}\right]+D\left[\frac{1-\nu}{2}\left(\beta_{\theta,ss}+\frac{1}{s}\beta_{\theta,s}-\frac{\beta_\theta}{s^2}\right)+\frac{1}{s^2\cos^2\varphi}\beta_{\theta,\theta\theta}\right]$$

Substituting equation (36) into equation (31), we have

$$D\left[L_{\beta_s}(\beta_s)+L_{\beta_\theta}(\beta_\theta)\right] - \frac{\tan\varphi}{s}F_{,ss} + L(F,w+w_0) + q_n = 0 \tag{38}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s^2\cos^2\varphi}\frac{\partial^2}{\partial\theta^2} \tag{39}$$

$$L_{\beta_s}[:] = \frac{1}{s}\left(\nabla^2-\frac{2}{s}\frac{\partial}{\partial s}+\frac{1}{s^2}\right)\frac{\partial[:]}{\partial s} \qquad L_{\beta_\theta}[:] = \frac{1}{s\cos\varphi}\left(\nabla^2-\frac{2}{s}\frac{\partial}{\partial s}+\frac{2}{s}\frac{\partial}{\partial s}+\frac{1}{s^2}\right)\frac{\partial[:]}{\partial\theta} \tag{40}$$

$$L(A,B) = \left(\frac{1}{s}A_{,s}+\frac{1}{s^2\cos^2\varphi}A_{,\theta\theta}\right)B_{,ss}+A_{,ss}\left(\frac{1}{s}B_{,s}+\frac{1}{s^2\cos^2\varphi}B_{,\theta\theta}\right)-2\frac{1}{s^2\cos^2\varphi}\left(\frac{1}{s}A_{,s\theta}-A_{,\theta}\right)\left(\frac{1}{s}B_{,s\theta}-B_{,\theta}\right) \tag{41}$$

Compatibility equation

Since we do not use displacemental shell equations, compatibility has to be considered. After eliminating u and v from the median surface strain-displacement relations, an equation for the lateral deflection w in terms of the median surface strains ε_s, ε_θ and $\varepsilon_{s\theta}$ is obtained.

$$\frac{1}{s}\left(s\,\varepsilon_{\theta,s}\right)_{,s} + \frac{1}{s}\left(\varepsilon_\theta-\varepsilon_s\right)_{,s} - \frac{1}{s\cos\varphi}\varepsilon_{s\theta,s\theta} + \frac{1}{s^2\cos^2\varphi}\varepsilon_{s,\theta\theta} - \frac{1}{s^2\cos\varphi}\varepsilon_{s\theta,\theta} = \chi_{s\theta}^2 - \chi_s\chi_\theta - \frac{1}{s\cos\varphi}\chi_s \tag{42}$$

The final form of the compatibility equation in terms of the Airy stress function F and lateral deflection w is

$$\frac{1}{A}\nabla^4F = \frac{1}{2}L(w,w+2w_0) + \frac{\tan\varphi}{s}w_{,ss} \tag{43}$$

Equations (37) and (38) are equilibrium equations and equation (43) is a compatibility equation of the shell. They represent the governing equations of the problem.

(a) *Non-shear deformable core* $(G\to\infty)$ In this case, we have

$$\beta_s = -w_{,s} \qquad \beta_\theta = -\frac{1}{s\cos\varphi}w_{,\theta} \qquad L_{\beta_s}(\beta_s)+L_{\beta_\theta}(\beta_\theta) = -\nabla^4 w \tag{44}$$

and the equilibrium equations (37) and (38) become

$$D\nabla^4 + \frac{\tan\varphi}{s}F_{,ss} = q_n + L(F,w+w_0) \tag{45}$$

Equation (45) and the compatibility equation (43) are exactly the same as those for homogeneous isotropic shells except for the stiffness terms A and D. Thus we arrive at the following conclusion: all previous analytical results for isotropic cones can be immediately applied to sandwich cores with non-shear deformable cores when the appropriate in-plane stiffness A and bending stiffness D terms are used.

(b) *Isotropic cylinder-no core* (c=0) By setting c=0, in (29), we have in-plave stiffness A and bending stiffness D as follows:

$$A = Et \qquad D = \frac{Et^3}{12\left(1-\nu^2\right)} \tag{46}$$

In this case, the previous equilibrium and compatibility equations reduce to those of an isotropic cone.

(c) *Sandwich circular plate* $\left(s \to r,\ s\cos\theta \to r,\ \varphi\right)$ In this case, equations (37), (38) and (43) become

$$\begin{aligned} Gh\left(w_{,r}+\beta_r\right) &= D\left[\beta_{r,rr}+\frac{1}{r}\beta_{r,r}-\frac{\beta_r}{r^2}+\frac{1-\nu}{2}\frac{1}{r^2}\beta_{r,\theta\theta}\right]+D\left[\frac{1+\nu}{2}\frac{1}{r}\beta_{\theta,r\theta}-\frac{3-\nu}{2}\frac{1}{r^2}\beta_{\theta,\theta}\right] \\ Gh\left(\frac{1}{r}w_{,\theta}+\beta_\theta\right) &= D\left[\frac{1+\nu}{2}\frac{1}{r}\beta_{r,r\theta}+\frac{3-\nu}{2}\frac{1}{r^2}\beta_{r,\theta}\right]+D\left[\frac{1-\nu}{2}\left(\beta_{\theta,rr}+\frac{1}{r}\beta_{\theta,r}-\frac{\beta_\theta}{r^2}\right)+\frac{1}{r^2}\beta_{\theta,\theta\theta}\right] \end{aligned} \tag{47}$$

$$D\left[L_{\beta_r}\left(\beta_r\right)+L_{\beta_\theta}\left(\beta_\theta\right)\right]+L\left(F,w+w_0\right)+q_n=0 \qquad \frac{1}{A}\nabla^4F=\frac{1}{2}L\left(w,w+w_0\right) \tag{48}$$

in which r is the radius of a circular plate, and

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{49}$$

$$L(A,B) = \left(\frac{1}{r}A_{,r}+\frac{1}{r^2}A_{,\theta\theta}\right)B_{,rr} + A_{,rr}\left(\frac{1}{r}B_{,r}+\frac{1}{r^2}B_{,\theta\theta}\right) - 2\frac{1}{r^2}\left(\frac{1}{r}A_{,r\theta}-A_{,\theta}\right)\left(\frac{1}{r}B_{,r\theta}-B_{,\theta}\right)$$

(d) *Sandwich cylinder* $\left(s \to x, s\cos\varphi \to R, r\theta = y\right)$

$$Gh\left(w_{,x}+\beta_x\right) = D\left[\beta_{x,xx}+\frac{1-\nu}{2}\beta_{x,yy}+\frac{1+\nu}{2}\beta_{y,xy}\right] \qquad Gh\left(w_{,y}+\beta_y\right) = D\left[\frac{1+\nu}{2}\beta_{x,xy}+\frac{1-\nu}{2}\beta_{y,xx}+\beta_{y,yy}\right]$$

$$D\nabla^2\left(\beta_{x,x}+\beta_{y,y}\right) - \frac{1}{R}F_{,xx} + F_{,yy}\left(w+w_0\right)_{,xx} + F_{,xx}\left(w+w_0\right)_{,yy} - 2F_{,xy}\left(w+w_0\right)_{,xy} + q_n = 0 \tag{50}$$

$$\frac{1}{A}\nabla^4F = \frac{1}{2}w_{,yy}\left(w+2w_0\right)_{,xx} + \frac{1}{2}w_{,xx}\left(w+2w_0\right)_{,yy} - w_{,xy}\left(w+2w_0\right)_{,xy} + \frac{1}{R}w_{,xx}$$

where

$$\nabla^2 = \partial^2(:)/\partial x^2+\partial^2(:)/\partial y^2 \tag{51}$$

The above equation are exactly the same as those of Tennyson and Chan (1990) except for the +/- of some notations.

Compared with the governing equations of a sandwich circular plate, cylinder and cone, we can see that the governing equations of plate and cone are non-linear partial differential equations with variable coefficients, the sandwich cylinder excepted.

- To be continued -

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