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A Torquefree Deformable Model Gyro

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Es ist schwierig, allgemeine Aussagen über das Verhalten von sich verformenden Kreiseln zu machen und es lohnt sich deshalb, den Einfluß bestimmter Verformungen einzeln zu betrachten. Dies ist möglich, wenn man Modellkreisel definiert, die - mit Ausnahme eines einzigen Verformungsmechanismus - als starr angenommen werden können. Die vorliegende Abhandlung befaßt sich mit dem Verhalten eines axisymmetrischen drehmomentfreien Modellkreisels, der so ausgelegt ist, daß er nur permanente Verformungen von beliebiger Größe entlang seiner Symmetrieachse zuläßt, andererseits jedoch starr ist. Damit können die Ausdrücke für Trägheitsmomente, Drall und kinetische Energie exakt angegeben werden. Auch verschiebt sich der Massenmittelpunkt dann nicht innerhalb des Kreisels. Gleichungen bleiben verhältnismäßig einfach und das Verhalten des Kreisels bleibt übersichtlich. Der Modellkreisel besteht aus zwei starren masselosen Stangen, die ihrerseits starr mit dem starren Hauptkörper verbunden sind. Entlang jeder Stange gleitet je eine Punktmasse, deren Bewegung linear gedämpft ist.

This investigation is concerned with the behaviour of an axisymmetric torque-free model gyro. The model has been devised such that it represents accurately arbitrarily large deformations. The expressions for inertia moments, angular momentum, and the kinetic energy are all exact, i.e. there are no approximations. Also, the mass center of the model gyro does not shift within the gyro. Equations remain tractable and the practicing engineer can readily get a feel for the phenomena uncovered. The model is composed of rigid massless rods connected rigidly to a rigid massive bus. Along each central rod, a point mass (bead) moves, constrained by a linear damper.

1 Introduction

The description of the behaviour of torquefree gyros which are subject to deformation is very difficult to present in general form (Magnus 1971; Hughes, 1985; Kessaris, 1992). On the other hand, it is quite feasible to obtain the behaviour of model gyros with strictly defined deformations. One such gyro is investigated in the present paper.

The principal idea underlying the present work is to describe exactly the behaviour of a specially devised dissipative model gyro.

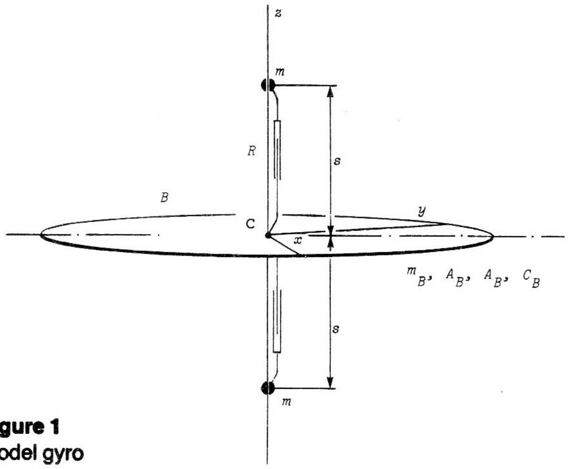

Figure 1
Model gyro

The model gyro consists of an axisymmetric bus body B, and two massless rods R along the axis of symmetry. Along each rod, a bead m can slide, constrained by a linear damper c. The point masses on each rod are further constrained such that their distances s from the gyro mass centre are always equal. This ensures that the gyro mass centre does not move within the bus body (Figure 1). The total mass of the gyro is $m_B + 2m$.

2 Kinetic Energy and Angular Momentum

The gyro's angular momentum **H** is constant, because the gyro is torquefree. Using floating *Cuvz* coordinates (Rimrott, 1988), the angular momentum is

$$\mathbf{H} = [\mathbf{e}_u \mathbf{e}_v \mathbf{e}_z] \begin{bmatrix} H_u \\ H_v \\ H_z \end{bmatrix} = [\mathbf{e}_u \mathbf{e}_v \mathbf{e}_z] \begin{bmatrix} 0 \\ H\sin\nu \\ H\cos\nu \end{bmatrix}$$

The floating *Cuvz* coordinate system has been chosen such that $H_u = 0$ (Figure 2).

The angular velocity of the gyro is

$$\omega = [\mathbf{e}_u \mathbf{e}_v \mathbf{e}_z] \begin{bmatrix} 0 \\ \omega_v \\ \omega_z \end{bmatrix} \tag{2}$$

Angular velocity components (2) and angular momentum components (1) are related by

$$\omega_v = \frac{H}{A} \sin\nu \tag{3a}$$

$$\omega_z = \frac{H}{C} \cos\nu \tag{3b}$$

such that the magnitude of the angular velocity becomes

$$\omega = \sqrt{\frac{H^2 \sin^2\nu}{A^2} + \frac{H^2 \cos^2\nu}{C^2}} \tag{4}$$

The angular velocity of the *Cuvz* coordinate system can be shown to be (Rimrott, 1988)

$$\Omega = [\mathbf{e}_u \mathbf{e}_v \mathbf{e}_z] \begin{bmatrix} 0 \\ \omega_v \\ \frac{C}{A}\omega_z \end{bmatrix} \tag{5}$$

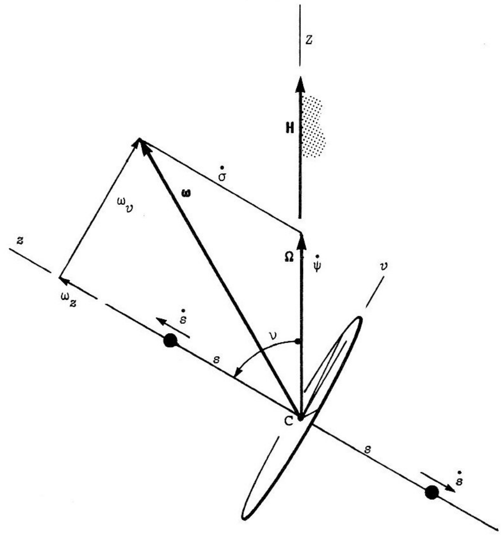

Figure 2
Gyro during motion

The magnitude of Ω is

$$\Omega = \sqrt{\omega_v^2 + \frac{C^2}{A^2}\,\omega_z^2} =$$

$$= \sqrt{\frac{H^2\sin^2\nu}{A^2} + \frac{C^2}{A^2}\,\frac{H^2\cos^2\nu}{C^2}} = \frac{H}{A} \tag{6}$$

The gyro's inertia tensor is

$$[I] = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \tag{7}$$

with

$$A = A_B + 2ms^2 \tag{8a}$$

$$B = A \tag{8b}$$

$$C = C_B \tag{8c}$$

Thus the inertia moments A and B are equal, and changing. The inertia moment C is constant. Equations (7) and (8) are valid for a gyro-fixed $Cxyz$ coordinate system, as well as the floating $Cuvz$ coordinate system.

The gyro's kinetic energy is

$$T = \frac{1}{2}\left[\frac{H^2\sin^2\nu}{A} + \frac{H^2\cos^2\nu}{C}\right] + 2\,\frac{1}{2}\,m\dot{s}^2 \tag{9}$$

with A a function (8a) of the bead position s.

When the beads m move along the rods, then the kinetic energy changes according to

$$dT = \frac{\partial T}{\partial \nu}\,d\nu + \frac{\partial T}{\partial s}\,ds + 2m\dot{s}\,d\dot{s} \tag{10}$$

Division by dt, and carrying out the differentiation, leads to

$$\dot{T} = \left[\frac{C-A}{AC}\,H^2\sin\nu\cos\nu\right]\dot{\nu} - \left[\frac{2ms}{A^2}\,H^2\sin^2\nu\right]\dot{s} + 2m\dot{s}\,\ddot{s} \tag{11}$$

3 Energy Dissipation

Extension force F and extension speed $\dot{s}$ for a linear damper are related by

$$F = c\dot{s} \tag{12}$$

The dissipation work D done by the two forces F acting on the gyro, is

$$D = 2\int F\,ds \tag{13}$$

the dissipation power

$$\frac{dD}{dt} = 2F\,\frac{ds}{dt} \tag{14}$$

or, using equations (12) and (14), and taking into consideration that the gyro is equipped with two dampers,

$$\dot{D} = 2c\,\dot{s}^2 \tag{15}$$

The quantity $\dot{D}$ is selected to be positive definite.

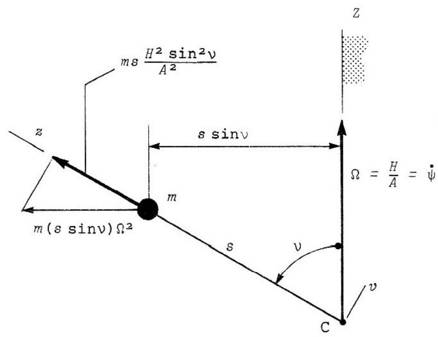

Figure 3
Inertial force acting on bead

An inspection of Figure 3 shows that the force F acting on the damper is the z component of the centrifugal force minus the linear acceleration force

$$F = ms\,\frac{H^2\sin^2\nu}{A^2} - m\ddot{s} \tag{16}$$

According to equation (12)

$$\dot{s} = \frac{ms}{c}\,\frac{H^2\sin^2\nu}{A^2} - \frac{m}{c}\,\ddot{s} \tag{17}$$

The energy dissipation rate (15) can consequently be written

$$\dot{D} = \left[\frac{2ms}{A^2}\,H^2\sin^2\nu - 2m\ddot{s}\right]\dot{s} \tag{18}$$

Equation (17) represents a nonlinear differential equation of second order.

$$m\ddot{s} + c\dot{s} - \frac{mH^2\sin^2\nu}{(A_B+2ms^2)^2}\,s = 0 \tag{19}$$

When the damping coefficient c is very large, the acceleration term becomes so insignificant, that the term $m\ddot{s}$ can be neglected. Then

$$\dot{s} = \frac{mH^2\sin^2\nu}{c(A_B+2ms^2)^2}\,s \tag{20}$$

4 Power Balance

The present system has only two energies, one is the kinetic energy (9), the other the dissipation energy (13). The time derivatives of the two are related by

$$\dot{T} = -\dot{D} \tag{21}$$

Using equation (11) for $\dot{T}$ and equation (18) for $\dot{D}$, we obtain

$$\left[\frac{C-A}{AC} H^2 \sin\nu \cos\nu\right]\dot{\nu} - \left[\frac{2ms}{A^2} H^2 \sin^2\nu\right]\dot{s} + 2m\dot{s}\ddot{s} = -\left[\frac{2ms}{A^2} H^2 \sin^2\nu - 2m\ddot{s}\right]\dot{s} \tag{22}$$

which leads to the conclusion that

$$\dot{\nu} = 0 \tag{23}$$

or that the attitude

$$\nu = \text{constant} \tag{24}$$

throughout the energy dissipation process.

Figure 4 shows an attitude diagram. It is seen that the final kinetic energy is, with $s = \infty$.

$$T_\infty = \frac{1}{2} \frac{H^2 \cos\nu}{C_B} \tag{25}$$

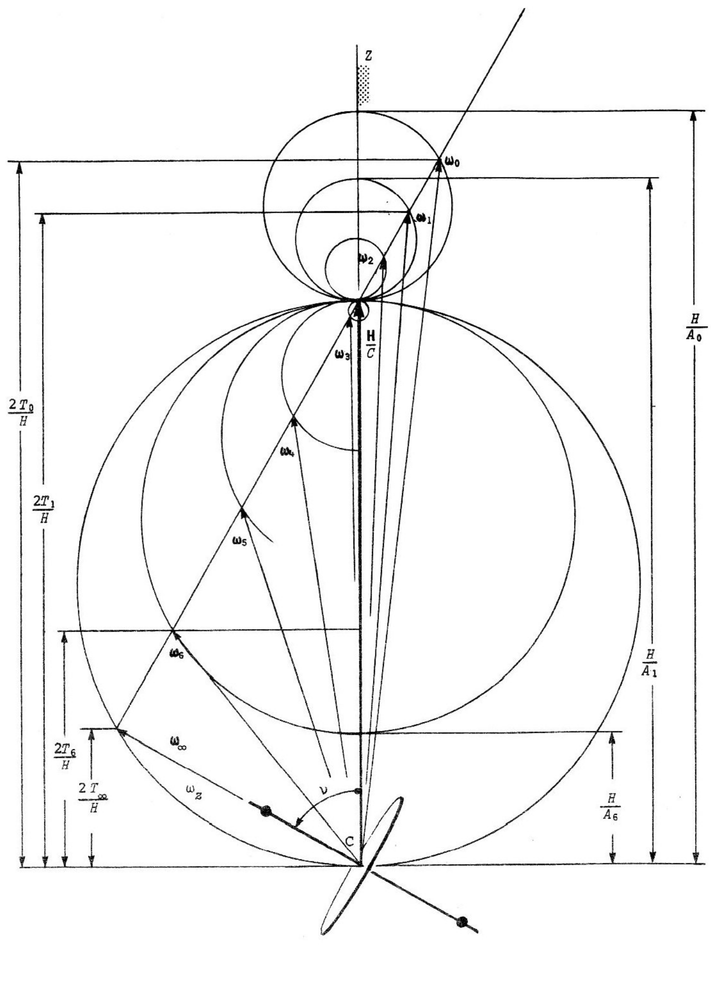

Figure 4
Attitude diagram

while the initial kinetic energy is

$$T_0 = \frac{1}{2}\left[\frac{H^2 \sin^2\nu}{A_B + 2ms_0^2} + \frac{H^2 \cos^2\nu}{C_B}\right] + m\dot{s}^2 \tag{26}$$

The kinetic energy lost is equal to the energy dissipated, from equation (21)

$$D_\infty = -\Delta T = T_0 - T_\infty = \frac{1}{2} \frac{H^2 \sin^2\nu}{A_B + 2ms_0^2} + m\dot{s}_0^2 \tag{27}$$

5 Bead Position and Time

Equation (17), with $\ddot{s} = 0$, can be rearranged into

$$\int_0^t dt = \frac{c}{mH^2 \sin^2\nu} \int_{s_0}^{s} \frac{(A_B + 2ms^2)^2}{s} ds \tag{28}$$

and integrated to give the time t as function of the bead position s.

$$t = \frac{cA_B^2}{mH^2 \sin^2\nu}\left[\ln\frac{s}{s_0} + 2\frac{m}{A_B}(s^2 - s_0^2) + \frac{m^2}{A_B^2}(s^4 - s_0^4)\right] \tag{29}$$

6 Attitude Diagram

In Figure 4 an attitude diagram is shown for an initially oblate gyro $(C_0 > A_0)$. The original angular velocity is ω_0. In the course of time, the angular velocity assumes smaller and smaller values, until ω_∞ is reaches when $t = \infty$ and $s = \infty$. The initial angular velocity has a magnitude of

$$\omega_0 = \sqrt{\frac{H^2 \sin^2\nu}{A_0^2} + \frac{H^2 \cos^2\nu}{C_B^2}} \tag{30}$$

with

$$A_0 = A_B + 2ms_0^2 \tag{31}$$

The final angular velocity, with $A_\infty = \infty$, has a magnitude of

$$\omega_\infty = \frac{H \cos\nu}{C_B} \tag{32}$$

The relationship between kinetic energy (9) and bead position s for the same gyro is plotted in Figure 5. Figures 4 and 5 are for en example with $\nu =$ 60°, $A_B = 60$ m^2 kg, $C_B = 10^2$m^2 kg, m = 3 kg, $s_0 = 0{,}5$ m, $H = 150$ Ws^2, and $c = 12$ MNs/m.

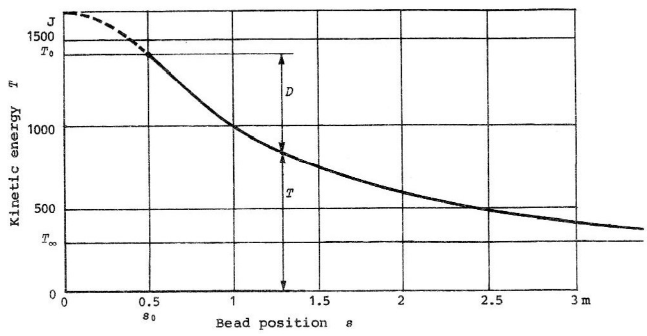

Figure 5
Kinetic energy versus bead position

7 Euler Angles as Generalized Coordinates

If the kinetic energy expression, in terms of generalized coordinates, is known, i. e. for the present gyro,

$$T = \frac{1}{2}\left[\frac{1}{A\sin^2\nu}p_\psi^2 + \frac{1}{A}p_\nu^2 + \left[\frac{1}{C} + \frac{1}{A\tan^2\nu}\right]p_\sigma^2 - \frac{2}{2\sin\nu\tan\nu}p_\psi p_\sigma\right] + \frac{p_s^2}{4m} \tag{33}$$

then the generalized velocities are obtained by partial differentiation, i. e. $\dot{q}_i = \frac{\partial T}{\partial p_i}$, resulting in

$$\dot{\Psi} = \frac{1}{A\sin^2\nu}p_\Psi - \frac{1}{A\sin\nu\tan\nu}p_\sigma \tag{34a}$$

$$\dot{\nu} = \frac{1}{A}p_\nu \tag{34b}$$

$$\dot{\sigma} = -\frac{1}{A\sin\nu\tan\nu}p_\psi + \left[\frac{1}{C} + \frac{1}{A\tan^2\nu}\right]p_\sigma \tag{34c}$$

$$\dot{s} = \frac{1}{2m}p_s \tag{34d}$$

The constant angular momentum H has the covariant components

$$p_\psi = H \tag{35a}$$

$$p_\nu = 0 \tag{35b}$$

$$p_\sigma = H\cos\nu \tag{35c}$$

Consequently we have for the Euler rates

$$\dot{\Psi} = \frac{H}{A} \tag{36a}$$

$$\dot{\nu} = 0 \tag{36b}$$

$$\dot{\sigma} = \frac{A-C}{AC}H\cos\nu \tag{36c}$$

In order to obtain a relationship for the velocity $\dot{s}$, we need the kinetic coenergy

$$T^* = \frac{1}{2}\left[A(\dot{\Psi}^2\sin^2\nu + \dot{\nu}^2) + C(\dot{\Psi}\cos\nu + \dot{\sigma})^2 + 2m\dot{s}^2\right] \tag{37}$$

The Lagrange equation for s requires that

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{s}} - \frac{\partial T^*}{\partial s} = Q_s \tag{38}$$

Recalling equation (8a) for the changing inertia moment A, and writing $Q_s = -2c\dot{s}$, and using equations (36) we obtain

$$m\ddot{s} - \frac{mH^2\sin^2\nu}{A^2}s = -c\dot{s} \tag{39}$$

which is the same as equation (19).

8 Conclusions

The torquefree model gyro used in the present paper has been devised such that it dissipates energy while deforming. The deformation is chosen such that only two principal inertia moments are affected, and that the gyro remains axisymmetric during deformation. The model gyro adopted is consequently of a type which ensures that all equations are exact. It is then shown that the attitude angle remains constant during energy dissipation, and that the latter continues until the time becomes infinite and the beads reach an infinite displacement.

Nomenclature

A, A, C	Principal inertia moments, m^2 kg
B	Bus body
C	Mass centre
D	Damping energy, J
F	Force, N
H	Angular momentum, Ws^2
I	Inertia moment, m^2 kg
R	Massless rod
T	Kinetic energy, J
X, Y, Z	Space-fixed coordinates, m
c	Damping coefficient, Ns/m
m	Bead mass, kg
m_B	Bus mass, kg
s	Bead position, m
t	Time, s
u, v, z	Floating principal coordinates, m
x, y, z	Gyro-fixed coordinates, m
$\boldsymbol{\Omega}$	Angular velocity, rad/s, of floating $Cuvz$
ν	Nutation angle („Attitude")
$\dot{\sigma}$	Spin, rad/s
$\dot{\Psi}$	Precession, rad/s
$\boldsymbol{\omega}$	Angular velocity, rad/s, of gyro

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