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# The Rotating Elastic-Plastic Solid Shaft with Free Ends

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*Subject of the paper is the distribution of stress, strain and displacement in a rotating solid cylinder of elastic-perfectly plastic material with free ends. The treatment is based on Tresca's yield condition and its associated flow rule.*

# Der rotierende elastisch-plastische Vollzylinder mit freien Enden

*Unter Zugrundelegung der Trescaschen Fließbedingung und der zugeordneten Fließregel wird die Verteilung von Spannungen, Dehnung und Verschiebung in einem rotierenden elastisch-idealplastischen Vollzylinder mit freien Enden untersucht.*

## 1. Introduction

The stress distribution in a rotating plastic cylinder was first studied by Nadai [1]. Since then, interest in this topic has never ceased. Due to the different aims of the authors, they investigated different aspects of the problem. While Davis and Connelly [2] and Lenard and Haddow [3] studied fully plasticized cylinders, Hodge and Balaban [4] considered the elastic-plastic case. However, the solution of Hodge and Balaban fails to meet the necessary continuity requirement of the displacement field. This was shown by Gamer and Sayir [5], who analyzed a rotating elastic-plastic solid shaft with fixed ends.

Subject of the present paper is a rotating elastic-perfectly plastic solid shaft of radius $b$ with free ends. Basis of the investigation is Tresca's yield condition and the associated flow rule. It is assumed that the shaft retains its circular symmetry throughout the loading process and is sufficiently long for the stress and strain not to vary along the shaft. Then, the principal directions of stress and strain are the radial, circumferential and axial direction.

It is well-known that plastic flow starts in the center of the shaft [1]. There, the radial stress equals the circumferential stress, and, at an angular speed $\omega = \omega_1$, two different plastic regions emerge simultaneously and spread outwards. The stress state in the core corresponds to an „edge regime" of Tresca's prism with the yield condition $\sigma_r - \sigma_z = \sigma_0$ and $\sigma_\Theta - \sigma_z = \sigma_0$. In the outer plastic region, the yield condition reads $\sigma_\Theta - \sigma_z = \sigma_0$. Next, at $\omega = \omega_2$, the elastic region adjacent to the shaft's edge disappears and a totally plastic state is attained. However, the plastic collapse state is not reached before the radial stress becomes equal to the circumferential stress throughout the shaft for $\omega = \omega_3$.

## 2. Basic equations

### 2.1. Elastic region

The equation of motion

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\Theta}{r} = -\varrho\omega^2 r, \tag{2.1.1}$$

and the geometric relations

$$\varepsilon_r = \frac{du}{dr}, \qquad \varepsilon_\Theta = \frac{u}{r} \tag{2.1.2}$$

hold in the entire shaft irrespective of the material behaviour.

In the elastic case stresses and strains are related by Hooke's law,

$$\sigma_r = 2G(\varepsilon_r + \frac{\nu}{1-2\nu}e),$$

$$\sigma_\Theta = 2G(\varepsilon_\Theta + \frac{\nu}{1-2\nu}e), \tag{2.1.3}$$

$$\sigma_z = 2G(\varepsilon_z + \frac{\nu}{1-2\nu}e),$$

where

$$e = \varepsilon_r + \varepsilon_\Theta + \varepsilon_z \tag{2.1.4}$$

means the dilatation.

Considering the condition of generalized plane strain, $\varepsilon_z = const$, one obtains the displacement

$$u = \frac{C_1}{r} + C_2 r - \frac{1-2\nu}{16(1-\nu)G}\varrho\omega^2 r^3. \tag{2.1.5}$$

The $C_i$ indicate constants of integration. Therefrom the stresses

$$\sigma_r = 2G\frac{C_1}{r^2} + \frac{2G}{1-2\nu}C_2 - \frac{1+2\nu}{8(1-\nu)}\varrho\omega^2 r^2 + \frac{2\nu G}{1-2\nu}\varepsilon_z, \tag{2.1.6}$$

$$\sigma_\Theta = 2G\frac{C_1}{r^2} + \frac{2G}{1-2\nu}C_2 - \frac{3-2\nu}{8(1-\nu)}\varrho\omega^2 r^2 + \frac{2\nu G}{1-2\nu}\varepsilon_z, \tag{2.1.7}$$

$$\sigma_z = \nu(\sigma_r + \sigma_\Theta) + 2(1+\nu)G\varepsilon_z \tag{2.1.8}$$

are arrived at.

### 2.2. *Plastic region I, $\sigma_r = \sigma_\theta > \sigma_z$*

Since

$$\sigma_r = \sigma_\theta, \tag{2.2.1}$$

Tresca's yield condition reads

$$\sigma_r - \sigma_z = \sigma_o, \qquad \sigma_\theta - \sigma_z = \sigma_o \tag{2.2.2}$$

where $\sigma_o$ denotes the uniaxial yield stress. Integration of the equation of motion (2.1.1) together with the yield condition leads to

$$\sigma_r = \sigma_\theta = -\frac{1}{2}\varrho\omega^2 r^2 + C_3, \tag{2.2.3}$$

$$\sigma_z = -\sigma_o - \frac{1}{2}\varrho\omega^2 r^2 + C_3. \tag{2.2.4}$$

Because of plastic incompressibility, the dilatation is governed by Hooke's law,

$$e = \frac{1-2\nu}{2(1+\nu)G}(\sigma_r + \sigma_\theta + \sigma_z). \tag{2.2.5}$$

Thus, considering the geometric relations (2.1.2), an equation in $u$ can be derived,

$$\frac{du}{dr} + \frac{u}{r} = \frac{1-2\nu}{2(1+\nu)G}\left(-\sigma_o - \frac{3}{2}\varrho\omega^2 r^2 + 3C_3\right) - \varepsilon_z, \tag{2.2.6}$$

with the solution

$$u = \frac{1-2\nu}{4(1+\nu)G}\left(-\sigma_o r - \frac{3}{4}\varrho\omega^2 r^3 + 3C_3 r\right) + \frac{C_4}{r} - \frac{1}{2}\varepsilon_z r. \tag{2.2.7}$$

The plastic parts of the strains are found as the difference of the total strains and their elastic parts, which are calculated with the help of Hooke's law,

$$\varepsilon_r^{pl} = \frac{1}{4(1+\nu)G}\left[-\sigma_o + (1-2\nu)\left(-\frac{5}{4}\varrho\omega^2 r^2 + C_3\right)\right] - \frac{C_4}{r2} - \frac{1}{2}\varepsilon_z, \tag{2.2.8}$$

$$\varepsilon_\theta^{pl} = \frac{1}{4(1+\nu)G}\left[-\sigma_o + (1-2\nu)\left(\frac{1}{4}\varrho\omega^2 r^2 + C_3\right)\right] + \frac{C_4}{r2} - \frac{1}{2}\varepsilon_z, \tag{2.2.9}$$

$$\varepsilon_z^{pl} = \frac{1}{2(1+\nu)G}\left[\sigma_o + (1-2\nu)\left(\frac{1}{2}\varrho\omega^2 r^2 - C_3\right)\right] + \varepsilon_z. \tag{2.2.10}$$

### 2.3. *Plastic region II, $\sigma_\theta > \sigma_r > \sigma_z$*

Here, the yield condition adopts the form

$$\sigma_\theta - \sigma_z = \sigma_o. \tag{2.3.1}$$

As a consequence of the flow rule

$$\varepsilon_\theta^{pl} = -\varepsilon_z^{pl}, \qquad \varepsilon_r^{pl} = 0. \tag{2.3.2}$$

Hence

$$\varepsilon_r = \varepsilon_r^{el} \tag{2.3.3}$$

and

$$\varepsilon_\theta = \varepsilon_\theta^{el} + \varepsilon_\theta^{pl} = \varepsilon_\theta^{el} + \varepsilon_z^{el} - \varepsilon_z. \tag{2.3.4}$$

Next, these strains are inserted into the compatibility condition

$$\varepsilon_r = \frac{d}{dr}(\varepsilon_\theta r). \tag{2.3.5}$$

Their elastic parts can be expressed in terms of the stresses via Hooke's law, so that

$$\sigma_r - \nu(2\sigma_\theta - \sigma_o) = (1-\nu)\left[2\frac{d}{dr}(\sigma_\theta r) - \sigma_o\right] - 2\nu\frac{d}{dr}(\sigma_r r) - 2(1+\nu)G\varepsilon_z \tag{2.3.6}$$

In the derivation, use has been made of the yield condition (2.3.1) and the condition of generalized plane strain, $\varepsilon_z = const.$ With the help of the equation of motion (2.1.1) one obtains

$$r^2\frac{d^2\sigma_\theta}{dr^2} + 3r\frac{d\sigma_\theta}{dr} + (1-R^2)\sigma_\theta = \frac{1-R^2}{1-2\nu}\left[\sigma_o - (1+6\nu)\varrho\omega^2 r^2 + 2(1+\nu)G\varepsilon_z\right] \tag{2.3.7}$$

where

$$R^2 = \frac{1}{2(1-\nu)}. \tag{2.3.8}$$

The solution of (2.3.7) is

$$\sigma_\theta = C_5 r^{-(1-R)} + C_6 r^{-(1+R)} + \frac{1}{1-2\nu}\sigma_o - \frac{1+6\nu}{17-18\nu}\varrho\omega^2 r^2 + \frac{2(1+\nu)G}{1-2\nu}\varepsilon_z. \tag{2.3.9}$$

From (2.1.1) there follows

$$\sigma_r = \frac{C_5}{R}r^{-(1-R)} - \frac{C_6}{R}r^{-(1+R)} + \frac{1}{1-2\nu}\sigma_o - \frac{2(3-2\nu)}{17-18\nu}\varrho\omega^2 r^2 + \frac{2(1+\nu)G}{1-2\nu}\varepsilon_z \tag{2.3.10}$$

and from the yield condition (2.3.1)

$$\sigma_z = C_5 r^{-(1-R)} + C_6 r^{-(1+R)} + \frac{2\nu}{1-2\nu}\sigma_o - \frac{1+6\nu}{17-18\nu}\varrho\omega^2 r^2 + \frac{2(1+\nu)G}{1-2\nu}\varepsilon_z. \tag{2.3.11}$$

Now, (2.3.4) and (2.3.9) – (2.3.11) yield the displacement

$$u = \frac{1}{(1+\nu)G}\left[\left(1-\nu-\frac{\nu}{R}\right)C_5 r^R + \left(1-\nu+\frac{\nu}{R}\right)C_6 r^{-R}\right] + \frac{1}{2G}\sigma_o r - \frac{1-2\nu}{(17-18\nu)G}\varrho\omega^2 r^3 + \varepsilon_z r. \tag{2.3.12}$$

Finally, since $\varepsilon_\theta^{pl} = -\varepsilon_z^{pl} = \varepsilon_z^{el} - \varepsilon_z$, Hooke's law gives the plastic strains

$$\varepsilon_\theta^{pl} = -\varepsilon_z^{pl} = \frac{1}{2(1+\nu)G}\Big[\Big(1-\nu-\frac{\nu}{R}\Big)C_5 r^{-(1-R)} \tag{2.3.13}$$

$$+\Big(1-\nu+\frac{\nu}{R}\Big)C_6 r^{-(1+R)} - \frac{(1+\nu)(1-2\nu)}{17-18\nu}\varrho\,\omega^2 r^2\Big].$$

## 3. Stress distributions

### 3.1. *In the elastic range $\omega \leqslant \omega_1$*

For the determination of the three unknowns $C_1$, $C_2$ and $\varepsilon_z$ in the equations for the stresses and the displacement, three conditions have to be found. Two of them read:

$$r = 0: \quad u = 0, \tag{3.1}$$

$$r = b: \quad \sigma_r = 0. \tag{3.2}$$

The third one follows from the presupposition of free ends, which implies that the total axial force on any section is equal to zero,

$$2\pi \int_0^b \sigma_z r\,dr = 0. \tag{3.3}$$

In the Appendix, these integrals are listed for the different elastic and plastic regions.

With the help of the above conditions, one obtains

$$C_1 = 0, \tag{3.4}$$

$$C_2 = \frac{(1-2\nu)(3-2\nu)}{16(1-\nu)G}\varrho\,\omega^2 b^2 - \nu\varepsilon_z, \tag{3.5}$$

where

$$\varepsilon_z = -\frac{\nu}{4(1+\nu)G}\varrho\,\omega^2 b^2. \tag{3.6}$$

Note, that $\sigma_r = \sigma_\Theta > \sigma_z$ for $r = 0$ and $\sigma_\Theta > \sigma_r > \sigma_z$ for $0 < r \leqslant b$!

### 3.2. *In the elastic-plastic range $\omega_1 \leqslant \omega \leqslant \omega_2$*

At the angular speed

$$\omega = \omega_1 = \sqrt{\frac{8(1-\nu)\sigma_o}{(3-4\nu)\varrho b^2}}, \tag{3.7}$$

yielding starts in the center of the shaft and the plastic regions I and II emerge. Subsequently, the shaft is composed of the plastic region I for $0 \leqslant r < r_1$, the plastic region II for $r_1 < r < r_2$ and the elastic region for $r_2 < r \leqslant b$. Besides the border radii $r_1$ and $r_2$ there are seven more unknowns: $C_1$, $C_2$, $C_3$, $C_4$, $C_5$, $C_6$ and $\varepsilon_z$. Of course, the conditions (3.1) – (3.3) are needed here again and also in the following stages of plastic flow. They are completed by

$$r = r_1: \quad \sigma_r^{(I)} = \sigma_r^{(II)}, \tag{3.8}$$

$$u^{(I)} = u^{(II)}, \tag{3.9}$$

$$\sigma_r^{(II)} - \sigma_z^{(II)} = \sigma_o, \tag{3.10}$$

$$r = r_2: \quad \sigma_r^{(II)} = \sigma_r^{(el)}, \tag{3.11}$$

$$u^{(II)} = u^{(el)}, \tag{3.12}$$

$$\sigma_\Theta^{(el)} - \sigma_z^{(el)} = \sigma_o. \tag{3.13}$$

In these equations the superscript denotes the region. From the above conditions, beginning with those for the elastic outer shell, the constants of integration can be expressed in terms of the radii $r_1$, $r_2$ and the axial strain $\varepsilon_z$:

$$C_1 = \frac{r_2^2 b^2}{2[(1-2\nu)r_2^2+b^2]G}\Big\{\sigma_o + \frac{1-2\nu}{8(1-\nu)}\varrho\,\omega^2[r_2^2-(3-2\nu)b^2] \tag{3.14}$$

$$+ \; 2(1+\nu)G\varepsilon_z\Big\},$$

$$C_2 = \frac{1-2\nu}{2[(1-2\nu)r_2^2+b^2]G}\Big\{\sigma_o r_2^2 + \frac{1}{8(1-\nu)}\varrho\,\omega^2[(1-2\nu)r_2^4 \tag{3.15}$$

$$+ (3-2\nu)b^4] + 2\Big(r_2^2 - \frac{\nu}{1-2\nu}b^2\Big)G\varepsilon_z\Big\},$$

$$C_3 = \frac{C_5}{R}r_1^{-(1-R)} - \frac{C_6}{R}r_1^{-(1+R)} + \frac{\sigma_o}{1-2\nu}$$

$$+ \frac{5(1-2\nu)}{2(17-18\nu)}\varrho\,\omega^2 r_1^2 + \frac{2(1+\nu)G}{1-2\nu}\varepsilon_z, \tag{3.18}$$

$$C_4 = 0. \tag{3.19}$$

$$C_5 = \frac{1-\nu+\frac{\nu}{R}}{2(1-\nu)}R r_2^{1-R}\Big\{\frac{-1}{1-2\nu}\sigma_o - (3-2\nu)\Big[\frac{1}{8(1-\nu)} - \frac{2}{17-18\nu}\Big]\varrho\,\omega^2 r_2^2 -$$

$$- \frac{2}{1-2\nu}G\varepsilon_z - 2G\Big(\frac{C_1}{r_2^2} - \frac{C_2}{1-2\nu}\Big)\Big\} +$$

$$+ \frac{1+\nu}{2(1-\nu)}r_2^{1-R}\Big\{-\frac{1}{2}\sigma_o - \frac{1-2\nu}{2}\Big[\frac{1}{8(1-\nu)} - \frac{2}{17-18\nu}\Big]\varrho\,\omega^2 r_2^2 - G\varepsilon_z +$$

$$+ G\Big(\frac{C_1}{r_2^2} + C_2\Big)\Big\}, \tag{3.16}$$

$$C_6 = R r_2^{1+R}\Big\{\frac{C_5}{R}r_2^{-(1-R)} + \frac{1}{1-2\nu}\sigma_o + (3-2\nu)\Big[\frac{1}{8(1-\nu)} - \frac{2}{17-18\nu}\Big]\varrho\,\omega^2 r_2^2 +$$

$$+ \frac{2G}{1-2\nu}\varepsilon_z + 2G\Big(\frac{C_1}{r_2^2} - \frac{C_2}{1-2\nu}\Big)\Big\}, \tag{3.17}$$

In the derivation of these relations, no use has been made of the conditions (3.3), (3.9) and (3.10). They form a system of three equations in the three unknowns $r_1$, $r_2$ and $\varepsilon_z$. Although a lengthy expression for $\varepsilon_z(r_1, r_2)$ could be found, it is more convenient to solve the whole system numerically.

### 3.3. *In the totally plastic range* $\omega_2 \leqslant \omega \leqslant \omega_3$

At $\omega = \omega_2$, the elastic region disappears and the entire shaft behaves plastically. The angular speed $\omega_2$ is determined by the condition

$$r_2(\omega_2) = b. \tag{3.20}$$

However, this is not the plastic collapse speed. A further increase of the angular speed causes the border $r_1$ to migrate to the edge, and the collapse state is reached when the plastic region II has vanished. During this flow phase, the conditions (3.1)–(3.3) and (3.8)–(3.10) are still valid. They suffice to determine the six unknowns $C_3$, $C_4$, $C_5$, $C_6$, $r_1$ and $\varepsilon_z$. While $C_3$ and $C_4$ remain unchanged, one obtains

$$C_5 = \frac{-(1+R)b^{1+R}r_1^{-(1+R)}[\dfrac{1}{1-2\nu}\sigma_o - \dfrac{2(3-2\nu)}{17-18\nu}\varrho\omega^2 b^2 + \dfrac{2(1+\nu)G}{1-2\nu}\varepsilon_z] - \dfrac{5(1-2\nu)}{17-18\nu}\varrho\omega^2 r_1^2}{(1-\dfrac{1}{R})r_1^{-(1-R)} + (1+\dfrac{1}{R})b^{2R}r_1^{-(1+R)}} \tag{3.21}$$

$$C_6 = Rb^{1+R}[\frac{C_5}{R}b^{-(1-R)} + \frac{1}{1-2\nu}\sigma_o - \frac{2(3-2\nu)}{17-18\nu}\varrho\omega^2 b^2 + \frac{2(1+\nu)G}{1-2\nu}\varepsilon_z]. \tag{3.22}$$

The equations (3.3) and (3.9 )are used to calculate $r_1$ and $\varepsilon_z$. *Here,* $\varepsilon_z$ can be expressed in terms of $r_1$ in principle.

At

$$\omega = \omega_3 = \frac{2}{b}\sqrt{\frac{\sigma_o}{\varrho}}, \tag{3.23}$$

the border $r_1$ coincides with the outer boundary $b$, that is, the plastic region I occupies the whole shaft (compare [3]). Then, the stresses take the forms

$$\sigma_r = \sigma_\theta = \frac{1}{2}(b^2 - r^2)\varrho\,\omega_3^2, \tag{3.24}$$

$$\sigma_z = -\sigma_o + \frac{1}{2}(b^2 - r^2)\varrho\,\omega_3^2. \tag{3.25}$$

It is not possible, for a shaft of perfectly plastic material, to exceed $\omega_3$.

## 4. Numerical results

For the numerical treatment of the problem, the following non-dimensional quantities are introduced:

$$\bar{u} = uE/(b\sigma_o), \quad x = r/b, \tag{4.1}$$

$$\bar{\varepsilon}_i = \varepsilon_i E/\sigma_o, \quad \bar{\sigma}_i = \sigma_i/\sigma_o, \quad \Omega = \varrho\omega^2 b^2/\sigma_o.$$

In the following, Poisson's number, $\nu$, equals 0.3.

First, Fig. 1 shows the evolution of the border radii with increasing angular speed.

The stress distribution for four different angular speeds is depicted in Fig. 2. It is interesting to watch the development of the plastic regions and the growth of the plastic strains presented in Fig. 3 simultaneously. Note, that at the collapse speed $\Omega_3$ the radial and the circumferential stresses are equal throughout the shaft, but not the corresponding plastic strains! Figure 4 exhibits the evolution of the displacement. In contrast to a rotating tube with free ends (Mack [6]), the displacements remain small even beyond $\Omega_2$, that is, when the shaft has reached the totally plastic state.

Finally, Fig. 5 shows the stresses remaining after the stand-still for three different maximum angular speeds. As soon as the angular speed decreases, the whole shaft behaves elastically again and the stresses after stand-still are found by subtraction of the stresses occurring in an unlimited elastic shaft from those in the actual one at the same maximum angular speed. However, this holds true only because the residual stresses reach nowhere the yield limit, that is, secondary plastic flow does not occur.

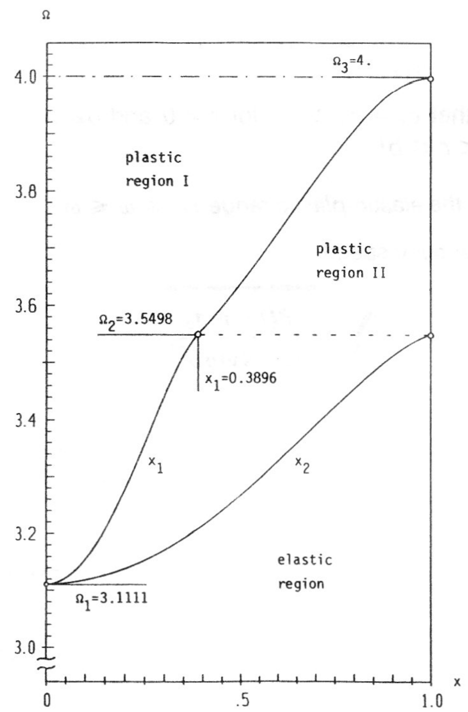


**Figure 1**
Evolution of the plastic regions with increasing angular speed

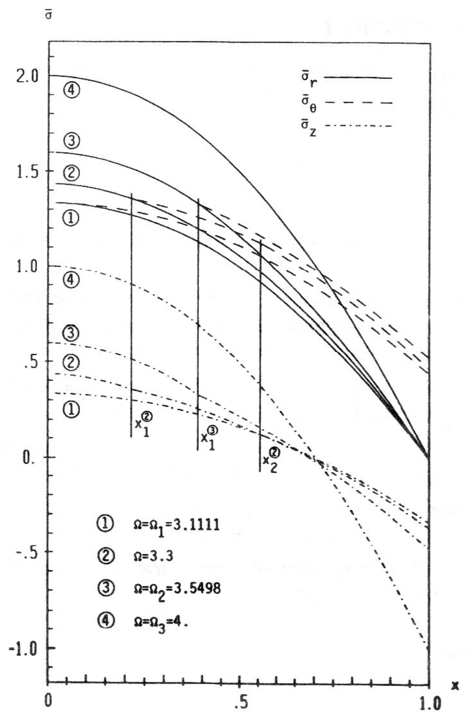


**Figure 2**
Stresses in the shaft

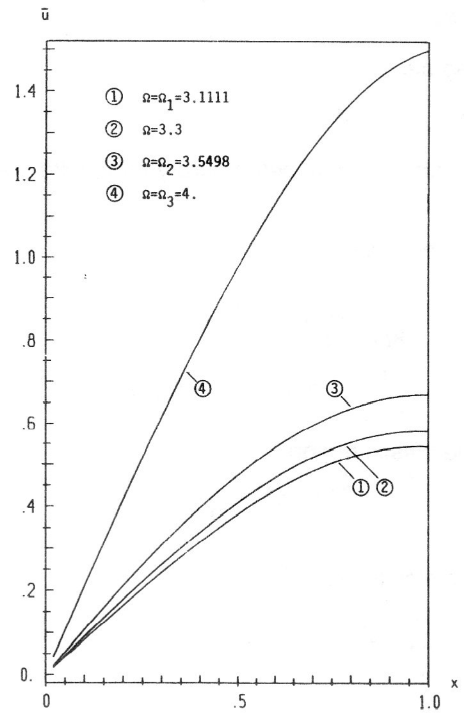


**Figure 4**
Displacements in the shaft

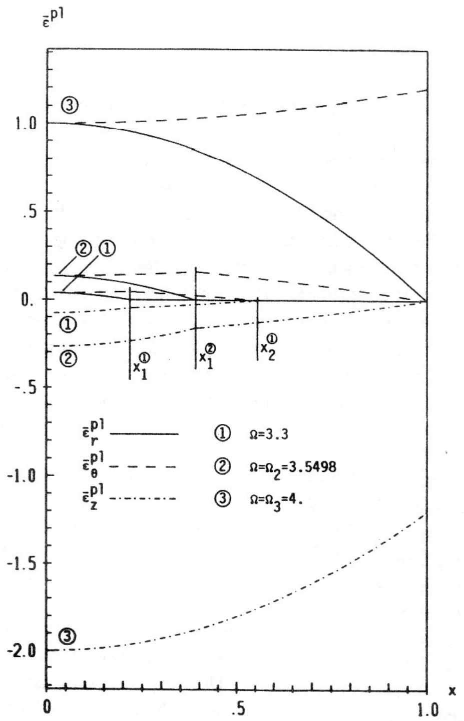


**Figure 3**
Evolution of the plastic strains

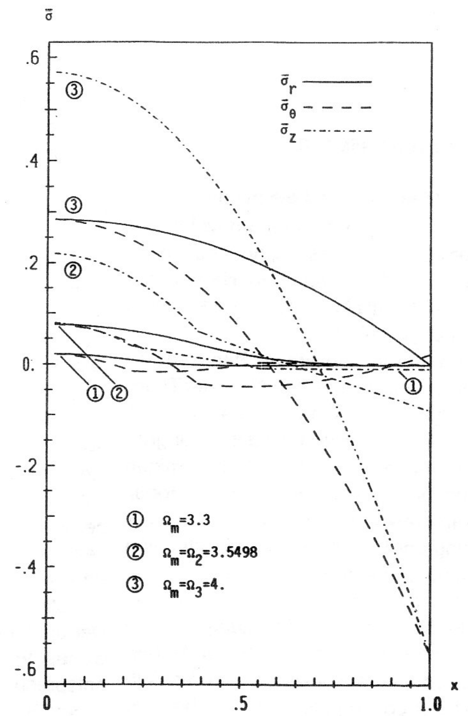


**Figure 5**
Residual stresses for three different maximum angular speeds $\Omega_m$

## Appendix

The integrals

$$I(s,t) = \int_s^t \sigma_z r \, dr, \qquad (A1)$$

where s and t denote two arbitrary radii, take the following forms:

a) in the elastic region

$$I(s,t) = -\frac{\nu}{8(1-\nu)} \varrho \omega^2 (t^4 - s^4) + \frac{G}{1-2\nu} [2\nu C_2 + (1-\nu)\varepsilon_z](t^2 - s^2) \qquad (A2)$$

b) in the plastic region I

$$I(s,t) = -\frac{1}{8} \varrho \omega^2 (t^4 - s^4) + \frac{1}{2} (C_3 - \sigma_o)(t^2 - s^2) \qquad (A3)$$

c) in the plastic region II

$$I(s,t) = \frac{C_5}{1+R}(t^{1+R} - s^{1+R}) + \frac{C_6}{1-R}(t^{1-R} - s^{1-R}) + \frac{\nu}{1-2\nu}\sigma_o(t^2 - s^2) - \frac{1+6\nu}{4(17-18\nu)} \varrho \omega^2 (t^4 - s^4) + \frac{(1+\nu)G}{1-2\nu}\varepsilon_z(t^2 - s^2) \qquad (A4)$$

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