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Elementary investigations of the equation for a free surface

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Es wird ein freies Randwertproblem für die Navier-Stokes-Gleichungen betrachtet, das einen Vorhangbeschichtungsprozeß beschreibt. Mit Hilfe elementarer analytischer Methoden werden eine Startlösung für die Lage der unteren freien Oberfläche bestimmt und deren Eigenschaften diskutiert.

A free boundary value problem for the Navier-Stokes equations describing a curtain coating process is considered. Using elementary analytical methods a starting approach for the position of the lower free surface is determined and its properties are discussed.

1. Introduction

In the present paper we consider the following flow domain describing a curtain coating process in Chemical Engineering (cf. Fig. 1). The infinite flow domain G is bounded by the rigid walls Σ_1, Σ_2, Σ_3, and by the a priori unknown free surfaces Γ_1 and Γ_2.

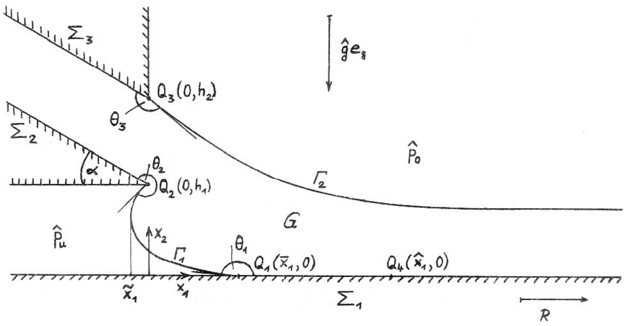

Figure 1
Flow domain of a curtain coating process

The problem is investigated in the plane $\mathbb{R}^2$ with a fixed Cartesian coordinate system $x = (x_1, x_2)$. The complete mathematical description of the flow domain and the free boundary value problem for the underlying Navier-Stokes equations are given in [1], [2].
In this paper we are especially interested in determining a starting approach for the position of the lower free surface Γ_1. In [1] the solvability of the complete problem was proved using the results given here. First of all in [2] a similar problem was solved the geometry of which was much simpler than in the case considered here.
Figure 2 shows that part of the flow domain which consists of the lower free surface Γ_1 and its two endpoints $Q_1(\bar{x}_1, 0)$, $Q_2(0, h_1)$.
Now we can formulate the equation and the boundary conditions describing the starting approach for the position of Γ_1. In order to do that we assume that $v \equiv 0$ and $p = p_c =$ const. are starting solutions for the velocity and pressure, respectively. Furthermore, we suppose that the free surface Γ_1 separates from the rigid wall Σ_2 at the static contact point Q_2 and ends at the (a priori unknown) dynamic contact point Q_1 on the moving rigid wall Σ_1. The dynamic con-

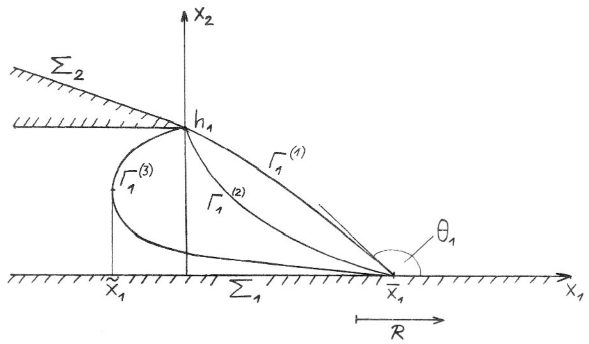

Figure 2
The lower free surface in various positions

tact angle Θ_1, i. e. the angle between the x_1 – axis and the tangent to Γ_1 at $\bar{x}_1$, is given. Finally we suppose that Γ_1 can be described as the graph of a function Ψ_1 with respect to $x_2 \in [0, h_1]$. These assumptions make physically sense.
We receive the following two-point boundary value problem (= BVP) for an ordinary differential equation of second order

$$\frac{d}{dx_2}\frac{\Psi'_1(x_2)}{[1 + (\Psi'_1(x_2))^2]^{1/2}} + \beta x_2$$
$$= W(p_c + \hat{p}_o - \hat{p}_u),\ (x_2 \in]0, h_1[) \qquad (1)$$
$$\Psi_1(h_1) = 0,\ \Psi'_1(0) = -A := \cot\Theta_1. \qquad (2.1).\ (2.2)$$

From physical point of view the restriction $\pi/2 < \Theta_1 \leqq \pi$ on Θ_1 makes sense. Thus we have $0 < A \leqq +\infty$. The constants β, W are positive and they depend only on the Weber number (i. e. surface tension) and on the acceleration of gravity. The symbols $\hat{p}_o$, $\hat{p}_u$ denote the positive (constant) athmospheric pressures outside Γ_1 and Γ_2, respectively.

2. The solvability of BVP (1), (2)

We define $c_1 := W(p_c + \hat{p}_o - \hat{p}_u)/\beta$. From Eq. (1) we then receive

$$\frac{d}{dx_2}\frac{\Psi'_1(x_2)}{[1 + (\Psi'_1(x_2))^2]^{1/2}}$$
$$= -\beta(x_2 - c_1)\cdot(x_2 \in]0, h_1[) \qquad (3)$$

Integrating Eq. (3) with respect to x_2 we obtain

$$\frac{\Psi_1(x_2)}{[1+(\Psi_1'(x_2))^2]^{1/2}} = -\frac{\beta}{2}(x_2-c_1)^2+c_2$$

and after taking into account condition (2.2)

$$c_2 = \frac{\beta}{2}c_1^2 - \frac{A}{\sqrt{1+A^2}}, \quad \frac{\Psi_1'(x_2)}{[1+(\Psi_1')_2]^{1/2}} = -\left(\frac{\beta}{2}x_2^2-\beta c_1 x_2+\tilde{A}\right), \tag{4}$$

where $\tilde{A}:=A(1+A^2)^{-1/2}$ was set. Obviously, $0<\tilde{A}\leqq 1$ holds. Eq. (4) yields the following necessary conditions on the solution

$$-1<F(x_2):=\frac{\beta}{2}x_2^2-\beta c_1 x_2+\tilde{A},$$
$$F(x_2)\leqq 1.\ (x_2\in]0,h_1[) \tag{5.1), (5.2}$$

In the sequel we have to distinguish some cases for the parameter c_1.

a) The case $c_1 \leqq 0$

The expression on the left-hand side of Eq. (3) is equivalent to the curvature of Γ_1 at x_2. Thus for $c_1\leqq 0$ the function Ψ_1 is concave on the whole interval $J_1:=]0,h_1[$. From the definition of F it follows that $F(x_2)>0$ holds on J_1 and hence the function Ψ_1 is a strongly decreasing one (cf. Eq. (4)). To fulfill the inequality (5.2) it is sufficient to require

$$\frac{\beta}{2}h_1^2-\beta c_1 h_1+\tilde{A}\leqq 1.$$

This condition is equivalent to the inequality

$$c_1 \geqq \frac{h_1}{2}-\frac{1-\tilde{A}}{\beta h_1}. \tag{6}$$

Inequality (6) can be fulfilled only, if the right-hand side of (6) is negative, i. e. for $h_1\leqq\sqrt{\dfrac{2(1-\tilde{A})}{\beta}}$.

b) The case $0<c_1<h_1$

From Eq. (3) it follows that the function Ψ_1 is convex on the interval $x_2\in[0,c_2[$ and concave on the interval $x_2\in]c_1, h_1]$. Due to $F'(x_2)=\beta(x_2-c_1)$ we have

$$\min_{x_2\in J_1} F(x_2)=F(c_1)=\tilde{A}-\frac{\beta}{2}c_1^2.$$

If $F(c_2)>-1$, *i. e.* if

$$c_1<\sqrt{\frac{2}{\beta}\cdot(\tilde{A}+1)} \tag{7}$$

holds, then the inequality (5. 1) is fulfilled. Furthermore, note that

$$\max_{x_2\in J_1} F(x_2)=\max[F(0),F(h_1)] = \max\left[\tilde{A},\frac{\beta}{2}h_1^2-\beta c_1 h_1+\tilde{A}\right].$$

Since $\tilde{A}\leqq 1$ we obtain the necessary condition

$$\frac{\beta}{2}h_1^2-\beta c_1 h_1+\tilde{A}\leqq 1$$

in order to satisfy inequality (5.2). The last condition leads to

$$c_1\geqq\frac{h_1}{2}-\frac{1-\tilde{A}}{\beta h_1}. \tag{8}$$

Since the right-hand side of (8) is less than h_1 we have to verify only the condition

$$\frac{h_1}{2}-\frac{1-\tilde{A}}{\beta h_1}<\sqrt{\frac{2}{\beta}(\tilde{A}+1)},$$

which is fulfilled iff

$$h_1<\sqrt{\frac{2}{\beta}(\tilde{A}+1)}+\frac{2}{\sqrt{\beta}} \tag{9}$$

hold. Next, we study the monotonicity of the function Ψ_1. Note that ψ_1 is decreasing at $x_2=0$ (cf. Eq. (4)). The condition

$$F(x_2)>0 \qquad (x_2\in[0,h_1]) \tag{10}$$

is necessary and sufficient for the monotonicity of Ψ_1 on the whole interval J_1. Inequality (10) implies

$$\min_{x_2\in J_1} F(x_2)=F(c_1)=\tilde{A}-\frac{\beta}{2}c_1^2>0,$$

$$\text{i. e. } c_1<\sqrt{\frac{2\tilde{A}}{\beta}}. \tag{11}$$

The condition (11) can be fulfilled together with (8) iff

$$\frac{h_1}{2}-\frac{1-\tilde{A}}{\beta h_1}>\sqrt{\frac{2\tilde{A}}{\beta}}, \text{ i. e. } h_1<\sqrt{\frac{2\tilde{A}}{\beta}}+\sqrt{\frac{2}{\beta}}, \tag{12}$$

holds. If condition (11) or (12) is not fulfilled then the function Ψ_1 always possesses a local minimum at $\check{x}_2=c_1-\sqrt{c_1^2-2\tilde{A}/\beta}$. If, additionally, the inequality $c_1<\min\left[h_1,\dfrac{h_1}{2}+\dfrac{\tilde{A}}{\beta h_1}\right]$ holds then Ψ_1 has also a local maximum at $\hat{x}_2=c_1+\sqrt{c_1^2-2\tilde{A}/\beta}$.

c) The case $h_1\leqq c_1$

If $h_1\leqq c_1$ holds then the function Ψ_1 is convex on the whole interval J_1. This follows immediately from Eq. (3).
Due to $F'(x_2)=\beta(x_2-c_1)\leqq 0$ on $\bar{J}_1$ we get

$$\min_{x_2\in\bar{J}_1} F(x_2)=F(h_1)=\frac{\beta}{2}h_1^2-\beta c_1 h_1+\tilde{A},$$
$$\max_{x_2\in\bar{J}_1} F(x_2)=F(0)=\tilde{A}\leqq 1.$$

To fulfill condition (5. 1), i. e. $F(x_2)>-1$. on $\bar{J}_1$, we require $F(h_1)>-1$. This inequality is equivalent to

$$c_1\leqq\frac{h_1}{2}+\frac{\tilde{A}+1}{\beta h_1}. \tag{13}$$

Because of $c_1\geqq h_1$ the right-hand side of (13) must be greater than h_1, i. e.

$$h_1<\sqrt{\frac{2(\tilde{A}+1)}{\beta}}. \tag{14}$$

If, additionally, $F(h_1)\geqq 0$ holds then the solution Ψ_1 is strongly decreasing. The last condition is fulfilled for

interval of h_1	feasible solution of c_1	convexity	monotonicity	minimum at $\tilde{x}_2 =$
$]0, (2\tilde{A}/\beta)^{1/2}[$	$[\frac{h_1}{2} - \frac{1-\tilde{A}}{\beta h_1}, h_1[$	concave, as $c_1 < 0$	decreasing	-
	$[h_1, \frac{h_1}{2} + \frac{\tilde{A}}{\beta h_1}[$	convex	decreasing	-
$[(2\tilde{A}/\beta)^{1/2}, (2(\tilde{A}+1)/\beta)^{1/2}[$	$[\frac{h_1}{2} - \frac{1-\tilde{A}}{\beta h_1}, (2\tilde{A}/\beta)^{1/2}[$	concave, as $c_1 < 0$	decreasing	-
	$[(2\tilde{A}/\beta)^{1/2}, h_1[$	-	-	$c_1 - (c_1^2 - 2\tilde{A}/\beta)^{1/2}$
	$[h_1, \frac{h_1}{2} + \frac{\tilde{A}+1}{\beta h_1}[$	convex	-	$c_1 - (c_1^2 - 2\tilde{A}/\beta)^{1/2}$
$[(2(\tilde{A}+1)/\beta)^{1/2}, (2\tilde{A}/\beta)^{1/2} + (2/\beta)^{1/2}[$	$[\frac{h_1}{2} - \frac{1-\tilde{A}}{\beta h_1}, (2\tilde{A}/\beta)^{1/2}[$	-	decreasing	-
$[(2\tilde{A}/\beta)^{1/2} + (2/\beta)^{1/2}, (2(\tilde{A}+1)/\beta)^{1/2} + 2/\beta^{1/2}[$	$[\frac{h_1}{2} - \frac{1-\tilde{A}}{\beta h_1}, 2(\tilde{A}+1)/\beta[$	-	-	$c_1 - (c_1^2 - 2\tilde{A}/\beta)^{1/2}$
$[(2(\tilde{A}+1)/\beta)^{1/2} + 2/\beta^{1/2}, +\infty[$	no solution	-	-	-

Table 1
Survey of all solutions to BVP (1), (2)

$$c_1 \leqq \frac{h_1}{2} + \frac{\tilde{A}}{\beta h_1}. \tag{15}$$

Inequality (15) can be fulfilled only, if

$$h_1 < \sqrt{\frac{2\tilde{A}}{\beta}} \tag{16}$$

holds. If Ψ_1 is not monotonous then Ψ_1 possesses a global minimum at $\tilde{x}_2 = c_1 - \sqrt{c_1^2 - 2\tilde{A}/\beta}$. If h_1 is greater than the right-hand side of (9) then a parameter c_1 satisfying conditions (5.1) and (5.2) does not exist. Thus in that case there is no solution Ψ_1 to BVP (1), (2).

Finally in this section, we give a survey of the solvability of BVP (1), (2) and the features of the solution.

3. The solution to BVP (1), (2)

As shown in Table 1 for $h_1 < \sqrt{\frac{2(\tilde{A}+1)}{\beta}} + \frac{2}{\sqrt{\beta}}$ a feasible parameter c_1 exists. Now we want to write the solution Ψ_1 to BVP (1), (2). From Eq. (4) we get $\psi_1'(x_2) = -F(x_2)[1 - F^2(x_2)]^{-1/2}$. Integrating and taking into account boundary condition (2.1) we obtain the formula

$$x_1 = \Psi_1(x_2) = \int_{x_2}^{h_1} \frac{F(t)}{\sqrt{1-F^2(t)}} dt$$

$$= \int_{x_2}^{h_1} \frac{\frac{\beta}{2}t^2 - \beta c_1 t + \tilde{A}}{\sqrt{1 - (\frac{\beta}{2}t^2 - \beta c_1 t + \tilde{A})^2}} dt. \tag{17}$$

As a starting approach $\tilde{x}_1^0$, of the position of the dynamic contact point $Q_1(\tilde{x}_1 0)$ *we finally obtain*

$$\tilde{x}_1{}^0 = \int_0^{h_1} \frac{\frac{\beta}{2}t^2 - \beta c_1 t + \tilde{A}}{\sqrt{1 - (\frac{\beta}{2}t^2 - \beta c_1 t + \tilde{A})^2}} dt. \tag{18}$$

If a local minimum of Ψ_1 exists at $\tilde{x}_2 = c_1 - \sqrt{c_1{}^2 - 2\tilde{A}/\beta}$ then this minimum can be calculated by the formula

$$\tilde{x}_1 = \Psi_1(\tilde{x}_2) = \int_{\tilde{x}_2}^{h_1} \frac{\frac{\beta}{2}t^2 - \beta c_1 t + \tilde{A}}{\sqrt{1 - (\frac{\beta}{2}t^2 - \beta c_1 t + \tilde{A})^2}} dt.$$

From Eq. (17) one can conclude that Ψ_1 is infinitely differentiable on $]0, h_1[$. For $\tilde{A} < 1$, i. e. as $A < +\infty$, the function Ψ_1 is infinitely differentiable even in the closed interval $[0, h_1]$. Thus we have proved the following lemma.

Lemma 1. For any h_1 and c_1 satisfying the conditions of an arbitrary row of Table 1 the BVP (1), (2) possesses a unique inifinitely differentiable solution Ψ_1.

Similar analytical studies of free surfaces with capillary contact angles were given by Finn and Shinbrodt (cf. [3, 4]).

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