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Optimal design of elasto-plastic structures under displacement constraints

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1. Introduction

In the optimal design of bar structures presented in this paper various loading conditions and several design criteria are taken into consideration.

In elastic stage the displacements are not allowed to exceed the allowable elastic displacements due to a given arrangement of the loads acting on the structure. When the structure is submitted to a multi-parameter loading then occurance of plastic deformations might be permitted, but it must be proved, that during the entire loading history these deformations do not accumulate unrestrictedly, i. e. the structure shakes down. In some cases special extremal loads (e. g. earthquake, explosion, impact) should also be taken into account. Then again plastic deformations might be allowed but they should not exceed the values which lead to local failure or to the collapse of the entire structure.

In the procedure of optimal design the objective function is the volume of the structure and the constraints are the design criteria described above. Considering these criteria separately several independent optimal solutions can be determined which can form the basis of the design e. g. by choosing for every point of the structure the maximum cross-sectional area obtained by the separate solutions [2], [4]. A more general solution can be obtained, however, if all or several of the prescribed design criteria are simultaneously taken into consideration [8].

In the following the variational formulation of the optimal design problems described above will be presented.

2. Fundamentals

In the following linearly elastic-perfectly plastic bar structures (frames, trusses) with given shape and geometry will be considered. The structure is composed of $i = 1, 2, \ldots, n$ prismatic members with given lengths l_i and with unknown cross-sectional areas A_i as design variables. It is assumed that the specific stiffness S_i and the specific elastic and plastic strength R_i^e and R_i^p of the members can be expressed in terms of A_i in the following general forms [2], [4]:

$$S_i = \varphi E A_i^{\alpha}, \qquad R_i^e = \psi \sigma_y A_i^{\beta}, \qquad R_i^p = \rho \sigma_y A_i^{\gamma}. \tag{1}$$

Here σ, ψ, ϱ, α, β and γ are appropriately chosen constants and E and σ_y denote the Young's Modulus and the yield stress of the material. Für example in case of beams and frames S_i, R_i^e and R_i^p denote the specific bending stiffness, the maximum elastic moment M_i^e and the fully plastic moment M_i^p of the members [8].

In the following three different *loading conditions* will be considered:

a) a *one-parameter static load* $F_0(x)$ with given distribution and intensity;

b) a *multi-parameter static loading* defined by the loads $F_1(x)$, $F_2(x)$, $\ldots$, $F_p(x)$ which can act independently or simultaneously;

c) a high intensity, short-time *dynamic pressure* $F^d(x, t)$ defined by the relationships

$$\left. \begin{aligned} F^d(x, t) &= p(t)\, F_0^d(x), \\ p(t) &= p_0, \quad \text{if } 0 \le t \le t_0, \\ p(t) &= 0, \quad \text{if } t > t_0. \end{aligned} \right\} \tag{2}$$

Here x denotes the coordinate measured along the axis of the bars and t is the time.

Considering the assumptions, loading conditions and the design criteria described above the optimal design of a bar structure might be specified in the following form.

With the cross-sectional areas A_i as design variables and the volume

$$V(A_i) = \sum_{i=1}^{n} l_i A_i; \qquad (i = 1, 2, \ldots, n) \tag{3}$$

of the structure as *objective function* determine the design that minimizes V subject to the following *constraints*.

a) Under the action of the static load $F_0(x)$ the structure does not undergo plastic deformations and at given points $j = 1, 2, \ldots, m$ the *elastic displacements* w_j^e do not exceed the allowable elastic displacements w_{0j}^e, i. e.

$$Q^s_i \le R^e_i; \qquad (i = 1, 2, \ldots, n), \tag{4}$$

$$w_j^e \le w_{0j}^e; \qquad (j = 1, 2, \ldots, m). \tag{5}$$

Here Q^s_i denotes the maximum internal force caused by the load $F_0(x)$ in the i-th member of the elastic structure and R_i^e is defined by eq. (1).

b) The plastic deformations caused by the multi-parameter loading do not accumulate unrestrictedly, i. e. the structure *shakes down*.

c) At given points $j = 1, 2, \ldots, m$, in the structure the plastic displacements w_j^p caused by the dynamic pressure $F^d(x, t)$ do not exceed the allowable plastic displacements w_{0j}^p, i. e.

$$w_j^p \le w_{0j}^p; \quad (j = 1, 2, \ldots, m). \tag{6}$$

Using eq. (1) the constraint defined by eq. (4) can be written in the form

$$\left. \begin{array}{l} A_i \le A_{ij0}, \\ \text{where} \\ A_{i0} = \left[\dfrac{Q_i^s}{\psi \, \sigma_y} \right]^{1/\beta}. \end{array} \right\} \tag{7}$$

Here Q_i^s denotes the maximum elastic internal force caused by the load $F_0(x)$ in the i-th member of the structure. Using the above relationships and introducing the independent „slack variables" a_i, e_j, g_j the inequalities (4) – (6) can be converted into equality constraints which have the following forms

$$(A_i - A_{i0}) - a_i^2 = 0, \tag{4a}$$

$$(w_j^e - w_{0j}^e) + e_j^2 = 0, \tag{5a}$$

$$(w_j^p - w_{0j}^p) + g_j^2 = 0. \tag{6a}$$

Next, we discuss the above design constraints in detail and present separately the variational formulation of each optimal design problem.

3. Static analysis of the elastic structure

3.1. General relationships

Under the action of the static load $F_0(x)$ the structure under consideration is in elastic state and the corresponding internal force distribution is denoted by $Q^s(x, S_i)$. Then the elastic displacement w_j^e at the point j in the structure can be obtained from the following relationship

$$w_j^e = \sum_{i=1}^{n} \int_{l_1} \frac{Q^s(x, S_i)\, Q_j^D(x)}{S_i} dx. \tag{8}$$

Here $Q_j^D(x)$ denotes any statically admissible internal force distribution equilibrating a „dummy unit force" acting at the point j in the direction of w_j^e. Note that $Q^s(x, S_i)$ is function of the design variable A_i, $Q_j^D(x)$ is, however, independent of it. Introducing the flexibility coefficient

$$f_{ij}(S_i) = \int_{l_i} Q^s(x, S_i)\, Q_j^D(x)\, dx \tag{9}$$

elastic displacement constraint (5a) can be expressed in the form

$$C_j^e = \left| \sum_{i=1}^{n} \frac{f_{ij}(S_i)}{S_i} \right| - w_{0j}^e + e_j^2 = 0; \tag{10}$$

$(j = 1, 2, \ldots, m)$.

3.2. Variational formulation

Using variational formulation the optimal design A_i satisfying the geometric and design constraints (4a) and (10) is identified with the stationarity of the functional

$$J_e = \sum_{i=1}^{n} A_i l_i + \sum_{j=1}^{m} \lambda_j \left[\left| \sum_{i=1}^{n} \frac{f_{ij}(S_i)}{S_i} \right| - w_{0j}^e + e_j^2 \right] + \sum_{i=1}^{n} \kappa_i (A_i - A_{i0} - a_i^2). \tag{11}$$

Here λ_j and κ_i denote Langrangian multipliers. The variation of the functional J_e with respect to the variables A_i, e_j and a_i yields to the following equations:

$$\frac{\partial J_e}{\partial A_i} = l_i + \sum_{j=1}^{m} \frac{\lambda_i}{(S_i)^2} \left| \left[\left(\frac{\partial f_{ij}}{\partial S_i} S_i - f_{ij} \right) \frac{\partial S_i}{\partial A_i} \right] \right| + \kappa_i = 0;$$

$$(i = 1, 2, \ldots n), \tag{12}$$

$$\frac{\partial J_e}{\partial e_j} = \lambda_j e_j = 0; \quad (j = 1, 2, \ldots, m), \tag{13}$$

$$\frac{\partial J_e}{\partial a_i} = \kappa_i a_i = 0; \quad (i = 1, 2, \ldots, n). \tag{14}$$

From eqs. (13) and (14) follows that along the structure either e_i or λ_i and either κ or a_i must vanish. Considering these „switching conditions" the structure can be subbivided into different regions.
In the region where $e_j = 0$ and $\kappa = 0$ eqs. (10) and (12) provide n+m equations for the determination of A_i and λ_i. Where, on the other hand, $e_j = 0$ and $a_i = 0$ eq. (4a) yields to the solution $A_i = A_{io}$.
In the region where $\lambda_j = 0$ according to eq. (12) $\kappa \neq 0$ therefore, independently from the value of e_j, a_i must vanish. Hence for this region eq. (4a) provides again the solution $A_j = A_{io}$.
We can conclude that the above variational formulation uniquely defines the optimal solution of our problem.

4. Shakedown analysis of the elastic-plastic structure

4.1. General relationships

The condition of shakedown of a linearly elastic – perfectly plastic, statically q times indeterminate structure is defined by the following relationsships [6], [7]

$$\left.\begin{aligned} Q_k^{max}(S_i) + Q_k^R &\leqslant R_k^p, \\ Q_k^{min}(S_i) + Q_k^R &\geqslant -R_k^p. \end{aligned}\right\} \quad (k = 1, 2, \ldots, s). \tag{15}$$

Here $Q_k^{max}(S_i)$ and $Q_k^{min}(S_i)$ denote the maximum and minimum values of the internal forces of the linearly elastic structure calculated from all the possible combinations of the multi-parameter loading $F_1(x)$, $F_2(x)$ at the critical cross-sections $k = 1,2,\ldots,s$ and R_k^P and Q_k^R are the plastic strengths (e.g. plastic moments) and the self-equilibrating internal residual forces of the critical cross-sections, respectively. The latter can be expressed in terms of the unknown statically indeterminate forces X_l $(l = 1,2,\ldots,q)$ in linear forms

$$Q_k^R = \sum_{l=1}^{q} a_{kl} X_l; \quad (k = 1, 2, \ldots, s), \tag{16}$$

where a_{kl} are constant coefficients. Note that Q_k^{max}, Q_k^{min} and R_k^P are functions of the design variables A_i, Q_k^R but X_l are independent of them.
Substituting eq. (16) in eqs. (15) and introducing the independent "slack variables" d_k and f_k the condition of shakedown can be defined as

$$\left.\begin{aligned} Q_k^{max}(S_i) + \sum_{l=1}^{q} a_{kl} X_l - R_k^p + d_k^2 &= 0, \\ Q_k^{min}(S_i) + \sum_{l=1}^{q} a_{kl} X_l + R_k^p - f_k^2 &= 0. \\ (k = 1, 2, \ldots, s) \end{aligned}\right\} \quad (17a/b)$$

In addition, to fulfil some constructional requirements, it might be necessary to prescribe a minimum value A_0 for the cross-sectional area. This geometrical constraint is expressed as

$$(A_i - A_0) - a_i^2 = 0; \quad (i = 1,2,\ldots n), \tag{18}$$

where a_i is a slack variable.

4.2. Variatonal formulation

The variational formulation of the optimal design A_i staisfying the constraints (17a-b) and (18) is identified with stationarity of the functional

$$\begin{aligned} J_s = \sum_{i=1}^{n} l_i A_i + \sum_{k=1}^{n} \mu_k \Big[Q_k^{max}(S_i) + \sum_{l=1}^{q} a_{kl} X_l - R_k^p + d_p^2\Big] \\ + \sum_{k=1}^{n} \nu_k \Big[Q_k^{min}(S_i) + \sum_{l=1}^{q} a_{kl} X_l + R_k^p - f_k^2\Big] \\ + \sum_{i=1}^{n} \kappa_i (A_i - A_0 - a_i^2). \end{aligned} \tag{19}$$

Here μ_k, ν_k and κ_i denote Lagrangian multipliers.
The variation of the functional J_s with respect to the variables A_i, X_j, d_k and a_i equations:

$$\frac{\partial J_s}{\partial A_i} = l_i + \sum_{k=1}^{n} \mu_k \frac{\partial Q_k^{max}(S_i)}{\partial S_i} \frac{\partial S_i}{\partial A_i} - \frac{\partial R_k^p}{\partial A_i} \tag{20}$$

$$+ \sum_{k=1}^{s} \nu_k \Big[\frac{\partial Q_k^{min}(S_i)}{\partial S_i} \frac{\partial S_i}{\partial A_i} + \frac{\partial R_k^p}{\partial A_i}\Big] + \kappa_i = 0;$$

$$(i = 1, 2, \ldots n),$$

$$\frac{\partial J}{\partial X_l} = \sum_{k=1}^{s} (\mu_k + \nu_k) a_{kl} = 0; \quad (l = 1, 2, \ldots, q), \tag{21}$$

$$\frac{\partial J_s}{\partial d_k} = \mu_k d_k = 0; \quad (k = 1, 2, \ldots, s), \tag{22}$$

$$\frac{\partial J_s}{\partial f_k} = \nu_k f_k = 0; \quad (k = 1, 2, \ldots, s). \tag{23}$$

Considering the „switching conditions" (21)–(23) different regions can be distinguished in the structure.
In the region where $d_k = 0$, $f_k \neq 0$ and $\kappa = 0$ according to eq. (23) ν_k must vanish. Then, eqs. (17a), (20) and (21) provide (s+n+ q) equations for the determination of ν_k, X_i and A_j.
On the other hand, in the region where $d_k \neq 0$, $f_k = 0$ and $\kappa_i =$ according to eq. (22) μ_k must vanish. Hence, 'eqs. (17b), (20) and (21) provide again (s+n+q) equations for the determination of ν, X_i and A_j.
In the region where $d_k = 0$, $f_k = 0$ and $\nu_i = 0$ both μ_k and ν_k can be different from zero. Now eqs. (17a–b), (20) and (21) provide (2s+n+q) equations from which μ_k, ν_k, X_i and A_i can be calculated.
It can be easily seen, that in all the remaining parts of the structure the above switching conditions yield to a constrained solution i.e. in these regions $A_i = A_o$. Hence, we can conclude that the above variational formulation uniquely defines the optimum solution of the problem.

5. Dynamic analysis of the rigid – perfectly plastic structure

5.1. *General relationsship*

The maximum permanent displacements of a rigid – perfectly plastic structure subjected to a high intensity short-time dynamic pressure given by eq. (2) can be determined among others by the kinematic approximation [3], [5]. The basic ideas of this approximation is that during the dynamic response the structure has stationary motion wich is described by a function expressed in product form

$$w_P(x, t) = W(t)\, w^k(x). \tag{24}$$

Here $w^k(x)$ denotes any arbitrary kinematically admissible displacement field (yield mechanism) and W(t) is an unknown displacement parameter function W(t) is determined by the differential equation of motion of the structure and reaches its maximum value W^P when the structure comes to standstill. Omitting the details for W^P the following expression can be obtained [5], [6]

$$W^P = \frac{1}{2} K\, p_0\, t_0^2 \left[\frac{p_0}{p^k} - 1 \right] \tag{25}$$

Here

$$K = \frac{\int_L F_0^d(x)\, w^k(x)\, dx}{\rho \sum_{i=1}^{n} A_i \int_{l_i} [w^k(x)]^2\, dx}, \tag{26}$$

ϱ is the density per unite volume of the material and p^k denotes the kinematically admissible multiplier associated with the load $F_0^d(x)$ and displacement field $w_k(x)$ and is defined by the expression

$$p^k = \frac{\sum_{i=1}^{n} R_i^p\, \bar{q}_i^k}{\int_L F_0^d(x)\, w^k(x)\, dx}. \tag{27}$$

Here $\bar{q}_i^k$ denotes the sum of the absolute values of the generalized strains (e.g. rotations) occuring in the perfectly plastic cross-sections (e.g. in the plastic hinges) of the bar i. Making use of eq. (25) the approximate values of the maximum plastic displacements can be expressed in the form

$$w_j^p = W^p\, w_j^k = \frac{1}{2} K\, p_0\, t_0^2 \left[\frac{p_0}{p^k} - 1 \right] w_j^k. \tag{28}$$

Note that the accuracy of the approximation might be improved by introducing several kinematically admissible displacement fields [5], [6]. Then, the maximum values of the permanent displacements obtained by the use of these displacement fields are competent in the design.

Substituting eqs. (26) and (27) in eq. (28) and introducing the notations

$$G = \int_L F_0^d(x)\, w^k(x)\, dx, \quad D_i = \int_{l_i} [w^k(x)]^2\, dx \tag{29}$$

for the design constraint (11a) we get the expression

$$\frac{p_0\, t_0^2\, G\, |w_j^k|}{2\rho \sum_{i=1}^{n} D_i A_i} \left[\frac{p_0\, G}{\sum_{i=1}^{n} R_i^p\, \bar{q}_i^k} - 1 \right] - w_{0j}^p + g_j^2 = 0; \tag{30}$$

$(j = 1, 2, \ldots, m)$

and the geometrical constraint (18) has the form

$$(A_i - A_0) - a_2^i = 0 \qquad (i = 1, 2, \ldots, n). \tag{31}$$

5.2. *Variational formulation*

The variational formulation of optimal design A_i satisfying the design and geometric constraints (30) and (31) is identified with the stationarity of the functional

$$J_p = \sum_{i=1}^{n} l_i A_i + \sum_{j=1}^{n} \psi_j \left[\frac{p_0 t_0^2 G\, |w_j^k|}{2\rho \sum_{i=1}^{n} D_i A_i} \left[\frac{p_0\, G}{\sum_{i=1}^{n} R_i^p\, \bar{q}_i^k} - 1 \right] - w_{0j}^p + g_j^2 \right] + \sum_{i=1}^{n} \kappa_i (A_i - A_0 - a_i^2). \tag{32}$$

Here ψ_i and κ_i denote Lagrangian multipliers. The variation of the functional J_P with respect to the variables A_i, g_j and a_i yields to the following equations:

$$\frac{\partial J_p}{\partial A_i} = l_i + \frac{p_0\, t_0^2\, G}{2\rho} \cdot \frac{\partial}{\partial A_i} \left[\frac{1}{\sum_{i=1}^{n} D_i A} \left(\frac{p_0\, G}{\sum_{i=1}^{n} R_i^p\, \bar{q}_i^k} - 1 \right) \right] \sum_{j=1}^{m} \psi_j |w_j^k| + \kappa_i = 0, \quad (i = 1, 2, \ldots, n) \tag{33}$$

$$\frac{\partial J_p}{\partial g_j} = \psi_j g_j = 0, \qquad (j = 1, 2, \ldots, m) \tag{34}$$

$$\frac{\partial J_p}{\partial a_i} = \kappa_i a_i = 0, \qquad (i = 1, 2, \ldots, n) \tag{35}$$

Similarly to the former problems we can see that for the region where $g_j = 0$ and $\kappa_i = 0$ an unconstrained solution can be obtained for the determination of A_i and ψ_j. In the other parts of the structure we get a constrained solution, i.e. $A_i = A_0$. Hence, the above variational formulation uniquely defines the optimal solution of the problem under consideration.

6. Concluding remarks

The optimal design problems presented above can also be described in form of mathematical programming. Since the equations are highly nonlinear the solution can be obtained by the application of iterative procedure. The details of this numerical solution and the applications are published elsewhere [8].

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