Manuskripteingang: 1. 3. 1989

Interlaminar stresses of laminated composite joints with a single cover plate [1)]

Fo-van Chan

0. Introduction

Let two identical bars of orthotropic material with rectangular cross section bh, be joined lengthwise. They are glued together by a metallic cover plate of cross section bh_1 forming a laminated composite joint with a single cover plate of length 2l (Fig. 1). The tensile force P is transmitted to the cover plate by interlaminar stresses in the adhesive surface. They being most severe near the joint's gap are responsible for delamination. E, G, μ are the elastic constants of the metal cover plate. For the orthotropic bars, one principal direction is along the length with elastic constant E_1, the others being E_2, G_1, μ_{12}.

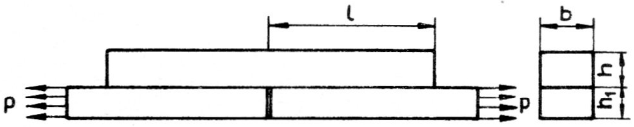

Figure 1

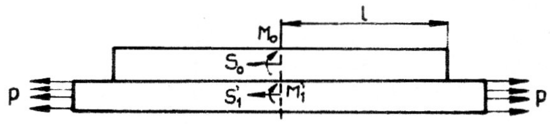

Figure 2

The interlaminar stresses of this joint can be solved by superposing that of following two parts.

(1) The two bars in Fig. 1 are continuous forming a laminated composite bar (Fig. 2), and its interlaminar stresses are the first part. In the bar's middle cross section, which is most remote from the influence of interlaminar stresses near the ends of cover plate, there will be a uniform tensile force S_1' and a bending moment M_1'. They can be determined by Mechanics of Materials.

(2) To remove the internal force components S_1' and M_1' in the middle cross section of bar (Fig. 2), apply at the two opposite faces of the gap of joint (Fig. 1) a pair of uniform compressive force $-S_1'$ and a pair of negative bending moment $-M_1'$. Fig. 3 shows the left half of the joint. To keep in equilibrium, there will be in middle cross section of the cover plate an axial tensile force S_1' and a positive bending moment $M_1' + S_1' \frac{1}{2}(h + h_1)$. The interlaminar stresses of this joint form the second part of superposition.

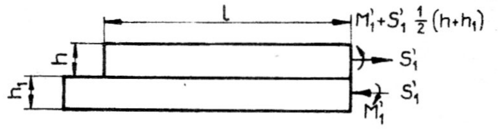

Figure 3

1. Interlaminar Stresses of Part one

1.1. Internal Force Components in Cover plate and Bar and the Relation between a_n and b_n

Fig. 4 represents the interlaminar stresses τ_o and σ_o, which are expressed by sine and cosine series with a_n and b_n to be determined.

$$\tau_o = \sum_{n=1} a_n \sin \frac{n\pi x}{l},$$

$$\sigma_o = \sum_{n=1} b_n \cos \frac{n\pi x}{l}. \qquad (1)$$

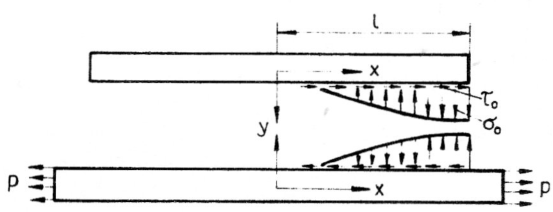

Figure 4

1) Project Supported by National Natural Science Foundation of China.

From the cover plate take an element dx (Fig. 5) upon which are acting axial force S, shearing force Q and bending moment M. Its equilibrium gives

$$\frac{dS}{dx} = -b\tau_o, \qquad \frac{dQ}{dx} = -b\sigma_o, \qquad \frac{dM}{dx} = Q - \frac{bh}{2}\tau_o. \tag{2}$$

Integrating (2) we get:

$$S = -\frac{bl}{\pi}\sum_{n=1}\frac{a_n}{n}\left(\cos n\pi - \cos\frac{n\pi x}{l}\right),$$

$$Q = -\frac{bl}{\pi}\sum_{n=1}\frac{b_n}{n}\sin\frac{n\pi x}{l}, \tag{3}$$

$$M = -\frac{blh}{\pi}\sum_{n=1}\left(\frac{l}{\pi h}\cdot\frac{b_n}{n^2} + \frac{1}{2}\frac{a_n}{n}\right)\left(\cos n\pi - \cos\frac{n\pi x}{l}\right).$$

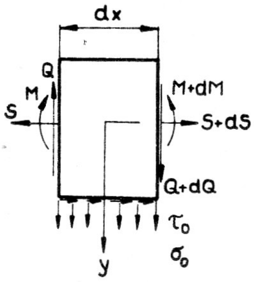

Figure 5

Similarly for the bar we have:

$$S_1 = p + \frac{bl}{\pi}\sum_{n=1}\frac{a_n}{n}\left(\cos n\pi - \cos\frac{n\pi x}{l}\right),$$

$$Q_1 = \frac{bl}{\pi}\sum_{n=1}\frac{b_n}{n}\sin\frac{n\pi x}{l},$$

$$M_1 = \frac{blh_1}{\pi}\sum_{n=1}\left(\frac{l}{\pi h_1}\cdot\frac{b_n}{n^2} - \frac{1}{2}\cdot\frac{a_n}{n}\right)\left(\cos n\pi - \cos\frac{n\pi x}{l}\right). \tag{4}$$

For cross sections outside the range of interlaminar stresses, the internal force components Q and Q_1 disappear and for remaining ones each has a constant value. They can be determined once the interlaminar stresses are known as well as simply by Mechanics of Materials. Thus we can use the two results of different approcaches to examine each other.

For our later use, let us find the axial forces and bending moments at middle cross sections of cover plate and bar as shown in Fig. 2. With equilibrium conditions for the right half of composite bar (Fig. 2), with equal curvature for the two laminated parts at middle cross section and with equal strain for contacting points in adhesive surface as well as in middle cross section, we have:

$$P = S_o + S_1',$$

$$M_o + M_1' - S_o\frac{1}{2}(h + h_1) = 0, \qquad \frac{M_o}{EI} = \frac{M_1'}{E_1 I_1}, \tag{5}$$

$$\frac{S_o}{AE} + \frac{M_o\frac{h}{2}}{EI} = \frac{S_1'}{A_1E_1} - \frac{M_1'\frac{h_1}{2}}{E_1I_1},$$

in which EI and E_1I_1 are flexural rigidities of cover plate and bar and A and A_1 are their respective cross sectional area. Solving the above four equations, we have:

$$S_o\left\{\frac{1}{AE} + \frac{1}{A_1E_1} + \frac{1}{4}\cdot\frac{(h+h_1)^2}{EI+E_1I_1}\right\} = \frac{P}{A_1E_1}, \qquad M_o = \frac{\frac{1}{2}(h+h_1)}{1+\frac{E_1I_1}{EI}}S_o, \tag{6}$$

$$S_1' = P - S_o \qquad M_1' = \frac{\frac{1}{2}(h+h_1)}{1+\frac{EI}{E_1I_1}}S_o,$$

Now let us find how a_n is related to b_n. As the composite bar (Fig. 2) acts as one, the cover plate and bar have same curvature, namely

$$\frac{M}{EI} = \frac{M_1}{E_1 I_1}. \tag{7}$$

Substituting M and M_1 of (3) and (4) in (7), we get

$$b_n = k\, a_n\, n, \qquad k = \frac{\pi}{2l} \cdot \frac{\dfrac{h_1}{E_1 I_1} - \dfrac{h}{EI}}{\dfrac{1}{E_1 I_1} + \dfrac{1}{EI}}. \tag{8}$$

It can be seen that when $\dfrac{h_1}{E_1 I_1} = \dfrac{h}{EI}$, no normal interlaminar stress occurs. And from (1) and (8) it follows:

$$\sigma_o = k \frac{l}{\pi} \cdot \frac{d\tau_o}{dx}. \tag{9}$$

Thus along the adhesive surface σ_o is proportional to the slope of τ_o curve at the very point. This important relation (8) can also be got by considering the equilibrium of right half of cover plate in Fig. 4.

$$\Sigma X = 0, \quad S_o - b \int_o^l \sum_{n=1} a_n \sin \frac{n\pi x}{l} dx = 0, \qquad S_o = -\frac{bl}{\pi} \sum_{n=1} \frac{a_n}{n} (\cos n\pi - 1).$$

$$\Sigma M = 0, \quad M_o - S_o \frac{h}{2} + b \int_o^l x \sum_{n=1} b_n \cos \frac{n\pi x}{l} dx = 0.$$

Using M_o in (6) and eliminating S_o, we get

$$\left\{ \frac{EI}{EI + E_1 I_1} \cdot \frac{1}{2} (h + h_1) - \frac{1}{2} h \right\} \frac{bl}{\pi} \sum_{n=1} \frac{a_n}{n} (\cos n\pi - 1) = \frac{bl^2}{\pi^2} \sum_{n=1} \frac{b_n}{n^2} (\cos n\pi - 1).$$

Or

$$\frac{bl^2}{\pi^2} \sum_{n=1} \left\{ \frac{\dfrac{h_1}{E_1 I_1} - \dfrac{h}{EI}}{\dfrac{1}{E_1 I_1} + \dfrac{1}{EI}} \cdot \frac{\pi}{2l} \cdot \frac{a_n}{n} - \frac{b_n}{n^2} \right\} (\cos n\pi - 1) = 0,$$

from which the relation between a_n and b_n can be got as given by (8). Using this relation we can express the stress components of cover plate and bar by a_n. And in the second part of superposition we shall use the latter method to obtain the relation between a_n and b_n.

1.2. To Solve a_n by the Principle of Least Work

First formulate the stress components of cover plate and bar in terms of a_n. As the cover plate is a slender bar, for σ_x we have

$$\sigma_x = \frac{S}{bh} + \frac{My}{I} = -\left[\frac{l}{\pi h} + 12 \frac{y}{h^2} \cdot \frac{l}{\pi} \left(\frac{kl}{\pi h} + \frac{1}{2}\right)\right] \sum_{n=1} \frac{a_n}{n} \left(\cos n\pi - \cos \frac{n\pi x}{l}\right), \tag{10a}$$

in which S and M in (3) as well as Fig. 4 are referred. Substituting σ_x in the integral from the equilibrium equation $\int_y^{\frac{h}{2}} \frac{\partial \sigma_x}{\partial x} dy + \int_{\tau_{xy}}^{\tau_o} \frac{\partial \tau_{xy}}{\partial y} dy = 0$, we get

$$\tau_{xy} = \left\{ \left(\frac{1}{2} + \frac{y}{h}\right) - 6 \left(\frac{1}{2} + \frac{kl}{\pi h}\right) \left(\frac{1}{4} - \frac{y^2}{h^2}\right) \right\} \sum_{n=1} a_n \sin \frac{n\pi x}{l}. \tag{10b}$$

And from the second equation of equilibrium, we have

$$\sigma_y = \left\{ k - \frac{\pi h}{l}\left[\left(\frac{3}{8} - \frac{y}{2h} - \frac{y^2}{2h^2}\right) - 6\left(\frac{1}{2} + \frac{kl}{\pi h}\right)\left(\frac{1}{12} - \frac{y}{4h} + \frac{y^3}{3h^3}\right)\right]\right\} \sum_{n=1} a_n n \cos\frac{n\pi x}{l}. \tag{10c}$$

Similarly for the bar we have:

$$\sigma'_x = \frac{S_1}{bh_1} - \frac{M_1 y}{I_1} = \frac{P}{bh_1} + \left[\frac{l}{\pi h_1} + 12\frac{yl}{h_1^2\pi}\left(\frac{1}{2} - \frac{kl}{\pi h_1}\right)\right] \sum_{n=1} \frac{a_n}{n}\left(\cos n\pi - \cos\frac{n\pi x}{l}\right),$$

$$\tau'_{xy} = -\left\{\left(\frac{1}{2} + \frac{y}{h_1}\right) - 6\left(\frac{kl}{\pi h_1} - \frac{1}{2}\right)\left(\frac{1}{4} - \frac{y^2}{h_1^2}\right)\right\} \sum_{n=1} a_n \sin\frac{n\pi x}{l}, \tag{11}$$

$$\sigma'_y = \left\{ k - \frac{\pi h_1}{l}\left[\left(\frac{3}{8} - \frac{y}{2h_1} - \frac{y^2}{2h_1^2}\right) + 6\left(\frac{kl}{\pi h_1} - \frac{1}{2}\right)\left(\frac{1}{12} - \frac{y}{4h_1} + \frac{y^3}{3h_1^3}\right)\right]\right\} \sum_{n=1} a_n n \cos\frac{n\pi x}{l}.$$

The stress components (10) and (11) for cover plate and bar satisfy both equilibrium equations and their respective boundary conditions. Now the unknown coefficients a_n have to be determined. Let U be the total strain energy of the system and the actual values of a_n should be such as to make U minimum, namely

$$\frac{\partial U}{\partial a_n} = \int_0^l \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\frac{1}{E}\left[\sigma_x\frac{\partial\sigma_x}{\partial a_n} + \sigma_y\frac{\partial\sigma_y}{\partial a_n} - \mu\left(\sigma_x\frac{\partial\sigma_y}{\partial a_n} + \sigma_y\frac{\partial\sigma_x}{\partial a_n}\right)\right] + \frac{\tau_{xy}}{G}\cdot\frac{\partial\tau_{xy}}{\partial a_n}\right\} b\,dx\,dy +$$

$$+ \int_0^l \int_{-\frac{h_1}{2}}^{\frac{h_1}{2}} \left\{\frac{\sigma'_x}{E_1}\cdot\frac{\partial\sigma'_x}{\partial a_n} + \frac{\sigma'_y}{E_2}\cdot\frac{\partial\sigma'_y}{\partial a_n} - \frac{\mu_{12}}{E_1}\left(\sigma'_x\frac{\partial\sigma'_y}{\partial a_n} + \sigma'_y\frac{\partial\sigma'_x}{\partial a_n}\right) + \frac{\tau'_{xy}}{G_1}\cdot\frac{\partial\tau'_{xy}}{\partial a_n}\right\} b\,dx\,dy = 0. \tag{12}$$

Substituting the six stress components in (12), we get:

$$a_n = -C\frac{n\cos n\pi}{n^4 + 2\eta n^2 + p^2}, \tag{13}$$

in which

$$p^2 = \frac{\frac{2}{\pi^2}\left\{\frac{1}{E}\left[1 + 3\frac{kl}{\pi h}\left(1 + \frac{kl}{\pi h}\right)\right] + \frac{1}{E_1}\frac{h}{h_1}\left[1 + 3\frac{kl}{\pi h_1}\left(\frac{kl}{\pi h_1} - 1\right)\right]\right\}}{\frac{h^4}{l^4}\cdot\frac{\pi^2}{70}\left\{\frac{1}{E}\left[\frac{1}{3} + \frac{kl}{\pi h}\left(\frac{11}{3} + 13\frac{kl}{\pi h}\right)\right] + \frac{1}{E_2}\cdot\frac{h_1^3}{h^3}\left[\frac{1}{3} + \frac{kl}{\pi h_1}\left(13\frac{kl}{\pi h_1} - \frac{11}{3}\right)\right]\right\}},$$

$$2\eta = \frac{\frac{1}{5}\left(\frac{1}{2G} - \frac{\mu}{E}\right)\left[\frac{2}{3} + \frac{kl}{\pi h}\left(1 + 6\frac{kl}{\pi h}\right)\right] + \frac{1}{5}\frac{h_1}{h}\left(\frac{1}{2G_1} - \frac{\mu_{12}}{E_1}\right)\left[\frac{2}{3} + \frac{kl}{\pi h_1}\left(b\frac{kl}{\pi h_1} - 1\right)\right] + \frac{kl}{\pi h}\left(\frac{\mu_{12}}{E_1} - \frac{\mu}{E}\right)}{\frac{h^2}{l^2}\cdot\frac{\pi^2}{70}\left\{\frac{1}{E}\left[\frac{1}{3} + \frac{kl}{\pi h}\left(\frac{11}{3} + 13\frac{kl}{\pi h}\right)\right] + \frac{1}{E_2}\cdot\frac{h_1^3}{h^3}\left[\frac{1}{3} + \frac{kl}{\pi h_1}\left(13\frac{kl}{\pi h_1} - \frac{11}{3}\right)\right]\right\}},$$

$$C = \frac{\frac{1}{E}\frac{h}{l}\frac{\sigma^*}{\pi} + \frac{4}{\pi^2}\left\{\frac{1}{E}\left[1 + \frac{3kl}{\pi h}\left(1 + \frac{kl}{\pi h}\right)\right] + \frac{1}{E_1}\frac{h}{h_1}\left[1 + 3\frac{kl}{\pi h_1}\left(\frac{kl}{\pi h_1} - 1\right)\right]\right\}\sum_{n=1}\frac{a_n}{n}\cos n\pi}{\frac{h^4}{l^4}\cdot\frac{\pi^2}{70}\left\{\frac{1}{E}\left[\frac{1}{3} + \frac{kl}{\pi h}\left(\frac{11}{3} + 13\frac{kl}{\pi h}\right)\right] + \frac{1}{E_2}\frac{h_1^3}{h^3}\left[\frac{1}{3} + \frac{kl}{\pi h_1}\left(13\frac{kl}{\pi h_1} - \frac{11}{3}\right)\right]\right\}}, \tag{14}$$

and $\sigma^* = \dfrac{P}{bh_1}$.

It can be seen that for cover plate and bar to be of same isotropic material, p^2, 2η and C are independent of elastic constants. And interlaminar stresses do not depend upon them. When the cover plate and bar are of same orthotropic material as happens to wooden joint, then in (14) in numertors E, G, μ have to be changed to E_1, G_1, μ_{12}; and in denominators E changed to E_2.

1.3. Interlaminar Stresses

From (1) and (13) we get

$$\tau_o = -C \sum_{n=1} \frac{n \cos n\pi \sin \dfrac{n\pi x}{l}}{n^4 + 2\eta n^2 + p^2} . \tag{15}$$

This series is of same form as the deflection curve expressed by sine series of a simply supported tie rod on elastic foundation under tension S and bent by an end couple (Fig. 6). Thus the two problems are analogous. The deflections correspond to τ_o and its slopes to σ_o.

As the series can be summed, our solutions are in closed forms.

For $\eta > p$ and using (9), we get

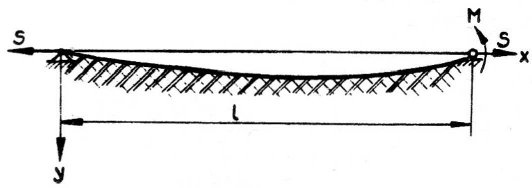

Figure 6

$$\tau_o = \frac{C\pi}{4\sqrt{\eta^2 - p^2}} \left\{ \frac{\sinh \dfrac{\gamma\pi}{l} x}{\sinh \gamma\pi} - \frac{\sinh \dfrac{\beta\pi}{l} x}{\sinh \beta\pi} \right\} , \tag{16}$$

$$\sigma_o = k \frac{l}{\pi} \cdot \frac{d\tau_o}{dx} = \frac{kC\pi}{4\sqrt{\eta^2 - p^2}} \left\{ \gamma \frac{\cosh \dfrac{\gamma\pi}{l} x}{\sinh \gamma\pi} - \beta \frac{\cosh \dfrac{\beta\pi}{l} x}{\sinh \beta\pi} \right\}, \tag{17}$$

in which $\beta = \sqrt{\eta + \sqrt{\eta^2 - p^2}}$, $\gamma = \sqrt{\eta - \sqrt{\eta^2 - p^2}}$. The shearing force transmitted by cover plate is

$$\int_0^l b\tau_o \, dx = \frac{Cbl}{4\sqrt{\eta^2 - p^2}} \left(\frac{1}{\gamma} - \frac{1}{\beta}\right) . \tag{18}$$

The value C can be found by the third equation of (14), and the series in it can be summed by using (13).

$$\sum_{n=1} \frac{a_n}{n} \cos n\pi = -C \sum_{n=1} \frac{1}{n^4 + 2\eta n^2 + p^2} = -C \left\{ \frac{\pi}{4\sqrt{\eta^2 - p^2}} \left(\frac{1}{\gamma} - \frac{1}{\beta}\right) - \frac{1}{2p^2} \right\} . \tag{19}$$

For $\eta < p$ and with $\beta\pi$ pretty large such that $\sinh \beta\pi = \cosh \beta\pi$, we have

$$\tau_o = \frac{C\pi}{4\beta\gamma \sinh \beta\pi} \left\{ \sin \gamma\pi \sinh \frac{\beta\pi}{l} x \cos \frac{\gamma\pi}{l} x - \cos \gamma\pi \cosh \frac{\beta\pi}{l} x \sin \frac{\gamma\pi}{l} x \right\} , \tag{20}$$

$$\sigma_o = \frac{kC\pi}{4\beta\gamma \sinh \beta\pi} \left\{ (\beta \sin \gamma\pi - \gamma \cos \gamma\pi) \cosh \frac{\beta\pi}{l} x \cos \frac{\gamma\pi}{l} x - (\gamma \sin \gamma\pi + \beta \cos \gamma\pi) \sinh \frac{\beta\pi}{l} x \sin \frac{\gamma\pi}{l} x \right\}, \tag{21}$$

in which $\beta = \sqrt{\frac{1}{2}(p + \eta)}$, $\gamma = \sqrt{\frac{1}{2}(p - \eta)}$.

Near the end of cover plate when $\sinh \frac{\beta\pi}{l} x = \cosh \frac{\beta\pi}{l} x$, the above two equations can further be simplified.

$$\tau_o = \frac{C\pi}{4\beta\gamma} \cdot \frac{\sinh \dfrac{\beta\pi}{l} x}{\sinh \beta\pi} \sin \gamma\pi \left(1 - \frac{x}{l}\right). \tag{22}$$

$$\sigma_o = \frac{C\pi^2}{4\beta\gamma} \cdot \frac{h}{2l} \cdot \frac{\sinh \frac{\beta\pi}{l} x}{\sinh \beta\pi} \left\{ \beta \sin \gamma\pi \left(1 - \frac{x}{l}\right) - \gamma \cos \gamma\pi \left(1 - \frac{x}{l}\right) \right\}. \tag{23}$$

The shearing force transmitted by cover plate and the sum of series are

$$\int_o^l b\,\tau_o\,dx = \frac{C h_1 b}{4 p \beta \frac{h_1}{l}}. \tag{24}$$

$$\sum_{n=1} \frac{a_n}{n} \cos n\pi = -C\left(\frac{\pi}{4p\beta} - \frac{1}{2p^2}\right). \tag{25}$$

2. Interlaminar stresses of Part two

Fig. 7 represents the interlaminar stresses of the second part of superposition (Fig. 3). For facilitating our analysis, S_1' and M_1' are

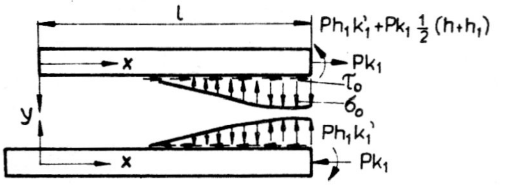

Figure 7

changed to Pk_1 and $Ph_1 k_1'$. As $S_1' = Pk_1$ and $M_1' = Ph_1 k_1'$, we have

$$k_1 = \frac{\frac{1}{AE} + \frac{1}{4} \cdot \frac{(h+h_1)^2}{EI + E_1 I_1}}{\frac{1}{AE} + \frac{1}{A_1 E_1} + \frac{1}{4} \cdot \frac{(h+h_1)^2}{EI + E_1 I_1}} \tag{26a}$$

$$k_1' = \frac{\frac{1}{2}\left(1 + \frac{h}{h_1}\right)}{1 + \frac{EI}{E_1 I_1}} \cdot \frac{\frac{1}{A_1 E_1}}{\frac{1}{AE} + \frac{1}{A_1 E_1} + \frac{1}{4} \cdot \frac{(h+h_1)^2}{EI + E_1 I_1}} \tag{26b}$$

Still use the two series in (1) to express the interlaminar stresses. From equilibrium of cover plate in Fig. 7, we have

$$\Sigma X = 0, \quad Pk_1 + b \int_o^l \sum_{n=1} a_n \sin \frac{n\pi x}{l}\,dx = 0, \qquad Pk_1 = \frac{bl}{\pi} \sum_{n=1} \frac{a_n}{n} (\cos n\pi - 1).$$

$$\Sigma M = 0, \quad Pk_1 \frac{h}{2} + b \int_o^l x \sum_{n=1} b_n \cos \frac{n\pi x}{l}\,dx = Pk_1 \left[\frac{1}{2}(h+h_1) + h_1 \frac{k_1'}{k_1}\right].$$

By eliminating Pk_1, the second equation gives

$$b\,\frac{l^2}{\pi^2} \sum_{n=1} \frac{b_n}{n^2} (\cos n\pi - 1) = b\,\frac{l}{\pi} \cdot \frac{h_1}{2} \left(1 + 2\frac{k_1'}{k_1}\right) \sum_{n=1} \frac{a_n}{n} (\cos n\pi - 1).$$

Hence we get the relation between a_n and b_n as given by

$$b_n = k\,a_n\,n, \qquad k = \frac{h_1 \pi}{2l}\left(1 + 2\frac{k_1'}{k_1}\right), \tag{27}$$

$$\text{and} \quad \sigma_o = \frac{h_1}{2}\left(1 + 2\frac{k_1'}{k_1}\right) \frac{d\tau_o}{dx}. \tag{28}$$

An element taken from the cover plate is the same as given by Fig. 5, and its three equilibrium equations are also the same as (2). The internal force components of cover plate are

$$S = Pk_1 - \frac{bl}{\pi} \sum_{n=1} \frac{a_n}{n} \left(\cos n\pi - \cos \frac{n\pi x}{l}\right), \qquad Q = -\frac{bl}{\pi} \sum_{n=1} \frac{b_n}{n} \sin \frac{n\pi x}{l},$$

$$M = P h_1 k'_1 + P k_1 \frac{1}{2}(h+h_1) - \frac{b h l}{\pi} \sum_{n=1} \left(\frac{l}{\pi h} \cdot \frac{b_n}{n^2} + \frac{1}{2} \cdot \frac{a_n}{n} \right) (\cos n\pi - \cos \frac{n \pi x}{l}).$$

Using (27) to replace b_n by a_n, we get the stress component σ_x:

$$\sigma_x = k_1 \frac{P}{b h} + P h_1 \frac{y}{l} [k'_1 + \frac{k_1}{2}(1 + \frac{h}{h_1})] - [\frac{l}{\pi h} + 12 \frac{y l}{h^2 \pi} (\frac{k l}{\pi h} + \frac{1}{2})] \sum_{n=1} \frac{a_n}{n} (\cos n\pi - \cos \frac{n \pi x}{l}).$$

It differs from the σ_x given by (10a) of first part only in the first two terms, which are idependent of x. Therefore, the other two stress components τ_{xy} and σ_y remain the same as those of first part, k being different. And for the bar the stress component σ'_x is:

$$\sigma'_x = -k_1 \frac{P}{b h_1} + P h_1 k'_1 \frac{y}{l_1} + [\frac{l}{\pi h_1} + 12 \frac{y l}{h_1^2 \pi} (\frac{1}{2} - \frac{k l}{\pi h_1})] \sum_{n=1} \frac{a_n}{n} (\cos n\pi - \cos \frac{n \pi x}{l}).$$

Again, it differs from the σ'_x of (11) in the terms independent of x, so that τ'_{xy} and σ'_y remain the same as that of first part, k being different. This simplifies very much our work.

Substituting the six stress components in (12), we again get a_n as given by (13) in which p^2 and 2η remain the same as that in (14) except with a different value of k. For they depend upon $\tau_{xy}, \sigma_y, \tau'_{xy}, \sigma'_y$ and σ_x, σ'_x involving variable x. Now the third equation becomes

$$C = \frac{-\frac{P}{b h_1} \cdot \frac{h}{l \pi} \left\{ \frac{1}{E} \frac{h_1}{h} [k_1 + 12 \frac{h_1}{h} (\frac{1}{2} + \frac{k l}{\pi h}) [k'_1 + \frac{1}{2} k_1 (1 + \frac{h}{h_1})]] + \frac{1}{E_1} [k_1 + 12 k'_1 (\frac{k l}{\pi h_1} - \frac{1}{2})] \right\}}{\frac{h^4}{l^4} \cdot \frac{\pi^2}{70} \left\{ \frac{1}{E} [\frac{1}{3} + \frac{k l}{\pi h} (\frac{11}{3} + 13 \frac{k l}{\pi h})] + \right.} \cdots$$

$$\cdots \frac{\frac{-4}{\pi^2} \left\{ \frac{1}{E} [1 + \frac{3 k l}{\pi h} (1 + \frac{k l}{\pi h})] + \frac{1}{E_1} \cdot \frac{h}{h_1} [1 + \frac{3 k l}{\pi h_1} (\frac{k l}{\pi h_1} - 1)] \right\} \sum_{n=1} \frac{a_n}{n} \cos n\pi}{\left. + \frac{1}{E_2} \frac{h_1^3}{h^3} [\frac{1}{3} + \frac{k l}{\pi h_1} (13 \frac{k l}{\pi h_1} - \frac{11}{3})] \right\}} \quad (29)$$

When the cover plate and bar are of same orthotropic material, the elastic constants in above equation have to be changed to suit our case as we did in the first part of superposition.

Now τ_o is still given by (16) and (20). As for σ_o, using (28) for $\eta > p$ we have

$$\sigma_o = \frac{h_1}{8 l} (1 + 2 \frac{k'_1}{k_1}) \frac{C \pi^2}{\sqrt{\eta^2 - p^2}} \left\{ \gamma \frac{\cosh \frac{\gamma \pi}{l} x}{\sinh \gamma \pi} - \beta \frac{\cosh \frac{\beta \pi}{l} x}{\sinh \beta \pi} \right\}, \quad (30)$$

in which

$$\beta = \sqrt{\eta + \sqrt{\eta^2 - p^2}}, \quad \gamma = \sqrt{\eta - \sqrt{\eta^2 - p^2}}.$$

For $\eta < p$, we have

$$\sigma_o = \frac{h_1}{8 l} (1 + 2 \frac{k'_1}{k_1}) \frac{C \pi^2}{\beta \gamma \sinh \beta \pi} \left\{ (\beta \sin \gamma \pi - \gamma \cos \gamma \pi) \cosh \frac{\beta \pi}{l} x \cos \frac{\gamma \pi}{l} x - \right. \quad (31)$$

$$\left. - (\gamma \sin \gamma \pi + \beta \cos \gamma \pi) \sinh \frac{\beta \pi}{l} x \sin \frac{\gamma \pi}{l} x \right\},$$

together with

$$\sigma_o = \frac{h_1}{8l}\left(1+2\frac{k_1'}{k_1}\right)\frac{C\pi^2}{\beta\gamma}\cdot\frac{\sinh\frac{\beta\pi}{l}x}{\sinh\beta\pi}\left\{\beta\sin\gamma\pi\left(1-\frac{x}{l}\right)-\gamma\cos\gamma\pi\left(1-\frac{x}{l}\right)\right\}, \qquad (32)$$

in which $\beta = \sqrt{\frac{1}{2}(p+\eta)}$, $\gamma = \sqrt{\frac{1}{2}(p-\eta)}$. Equations (18) and (24) for shearing force transmitted by cover plate together with series sum (19) and (25) remain unchanged for the second part of superposition.

For the second part of superposition, we have used the left half of the joint, while the right half is used for the first part. Owing to symmetry of σ_o and antisymmetry of τ_o, it is easy to convert the solution of first part to the left half of the joint and superpose with that of second part.

3. Numerical illustrative Examples

(1) The cover plate is of hard aluminium: $E = 7 \times 10^5$ kg/cm^2, $G = 2.69 \times 10^5$ kg/cm^2, $\mu = 0.3$. The bars are of pine: $E_1 = 10^5$ kg/cm^2, $E_2 = .042 \times 10^5$ kg/cm^2, $G_1 = .075 \times 10^5$ kg/cm^2, $\mu_{12} = .238$. $h = h_1 = {}^l/6$.

First part of superposition:

From (8) and (14), $k = .19635$, $p = 7.9997$, $\eta = 10.535$. Using equations for $\eta > p$ we get β and γ from (16) $\beta = 4.1700$, $\gamma = 1.9183$, and from (19) $\sum_{n=1} \frac{a_n}{n} \cos n\pi = -.024437\,C$. And the third equation of (14) gives $C = 6.1432\,\sigma^*$.
As a check use (18) to find the shearing force taken by the cover plate.

$$\int_o^l b\,\tau_o\,dx = \frac{6.1432 \times 6\,P}{4 \times 6.8550}\left(\frac{1}{1.9183} - \frac{1}{4.170}\right) = .37838\,P.$$

The axial force in middle cross section of cover plate given by S_o in (6) is

$$\frac{S_o}{bh \times 10^5}\left\{\frac{1}{7} + 1 + \frac{3}{2}\right\} = \frac{P}{bh \times 10^5}, \qquad S_o = \frac{14}{37}\,P = .37838\,P.$$

The identity of two values confirms our analysis. Now by (16) and (17) τ_o and σ_o are found for the right half in Fig. 4 and are tabulated as follows.

Table 1

X	l	.98	.96	.94	.92	.9	.85	
τ_o	0	.0823 σ^*	.1363	.1695	.1878	.1954	.1864	
σ_o	–.3112	–.2085	–.1329	–.0779	–.0384	–.0104	.0266	
X	.8	.75	.7	.6	.5	.4	.3	.2
τ_o	.1596	.1294	.1016	.0595	.0335	.0185	.0100	.0051
σ_o	.0375	.0370	.0322	.0207	.0122	.0070	.0040	.0023

Second part of superposition:

From (26) and (27), $k_1 = .62162$, $k_1' = .047297$ and $k = .30164$. Using (14) we obtain $p = 4.9272$, $\eta = 5.9983$. As $\eta > p$, β, γ in (16) are 3.0691 and 1.6054; (19) and (29) give $\sum_{n=1} \frac{a_n}{n} \cos n\pi = -.047608\,C$ and $C = -4.7722\,\sigma^*$.
By using (18) the shearing force transmitted by the cover plate is

$$\int_o^l b\,\tau_o\,dx = \frac{-4.7722 \times 6\,P}{4 \times 3.4209}\left(\frac{1}{1.6054} - \frac{1}{3.0691}\right) = -.62162\,P.$$

The negative sign means direction opposite to that shown in Fig. 7. Summing up the shearing forces by the two parts of superposition, we get: $.37838\,P + .62162\,P = P$.

This is exactly what the cover plate is required to carry. Now τ_o and σ_o as given by (16) and (30) are tabulated as follows.

Table 2

X	l	.98	.96	.94	.92	.9	.85	.8
τ_o	0	–.0870	–.1505	–.1952	–.2253	–.2439	–.2562	–.2403
σ_o	.4837 σ^*	.3567	.2561	.1767	.1146	.0663	–.0097	–.0460
X	.75	.7	.6	.5	.4	.3	.2	.1
τ_o	–.2122	–.1806	–.1226	–.0792	–.0488	–.0293	–.0163	–.0073
σ_o	–.0593	–.0606	–.0493	–.0347	–.0284	–.0151	–.0102	–.0075

in two tables, positive τ_o and σ_o are shown in Fig. 5. By superposing results of two parts, the interlaminar stresses of the joint are got in Fig. 8 for the left half of it.

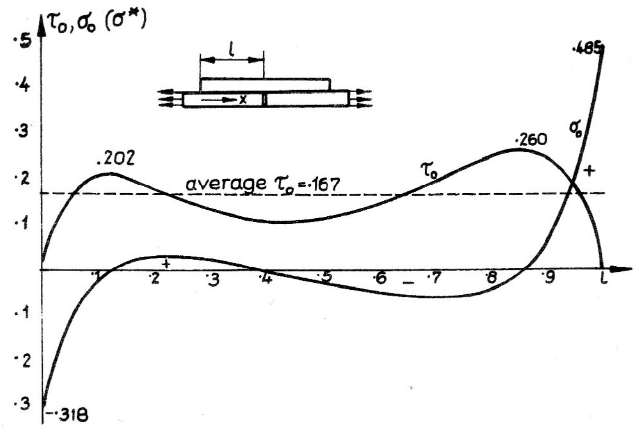

Figure 8

(2) The joint is made of pine cover plate and bars; $E = 10^5$ kg/cm^2, $E_2 = .042 \times 10^5$ kg/cm^2, $G_1 = .075 \times 10^5$ kg/cm^2, $\mu_{12} = .238$. $h = 1.2h_1$, $h/l = 1/5$.

First part of superposition:

From (8), $k = .05067$. Changing the relevant elastic constants in (14) to suit our case, we get $p = 10.245$, $\eta = 8.1158$, As $p > \eta$, β and γ in (20) are 3.0299 and 1.0318. And (25) gives $\sum\limits_{n=1} \frac{a_n}{n} \cos n\pi = -.020538C$ and from the third equation of (14) we get $C = 2.8919\ \sigma^*$. By using (24) the shearing force transmitted by cover plate is

$$\int_0^l b\,\tau_o\,dx = \frac{2.8919 \times 6\,P}{4 \times 3.0299 \times 10.245} = .13974\,P.$$

To check our calculation, find the axial force S_o in the middle cross section of cover plate from Mechanics of Materials. From (6) we have

$$S_o \left\{ \frac{1}{1.2} + 1 + \frac{1}{4} \cdot \frac{2.2^2}{\frac{1}{12}(1.2^3 + 1)} \right\} = P, \qquad S_o = \frac{P}{7.1559} = .13974\,P.$$

This confirms our analysis. Now τ_o and σ_o obtained from (20) to (23) are tabulated as follows.

Table 3

X	l	.98	.96	.94	.92	.9	.85	.8
τ_o	0	.0389 σ^*	.0642	.0793	.0870	.0893	.0814	.0654
σ_o	–.0380	–.0254	–.0159	–.0101	–.0038	–.0002	.0049	.0055

X	.75	.7	.65	.6	.55	.5	.45	.4
τ_o	.0487	.0345	.0235	.0155	.0100	.0062	.0046	.0022
σ_o	.0054	.0041	.0031	.0022	.0015	.0010	.0006	.0004

Second part of superposition:

From (26) and (27), k_1 = .86026, k_1' = .05635 and k = .29610. With equation (14), the elastic constants of which have been converted to suit our case, we obtain p = 4.4925 and η = 3.1331. As $p > \eta$, β and γ in (20) are 1.9526 and .82444. Then (25) gives $\sum\limits_{n=1} \frac{a_n}{n} \cos n\pi = -.06476C$ and from (29) we get $C = -5.0309\,\sigma^*$. By using (24), the shearing force transmitted by the cover plate is

$$\int_o^l b\,\tau_o\,dx = \frac{-5.0309 \times 6\ P}{4 \times 4.4925 \times 1.9526} = -.86027\ P.$$

Summing up the shearing forces transmitted by cover plate for the two parts of superposition, we get

.13974P + .86027P = 1.00001P.

It differs so little from the load the joint is required to carry and again confirms our results. Now τ_o and σ_o are obtained from (20) and (31) and tabulated as follows.

Table 4

X	l	.98	.96	.94	.92	.9	.85	.8
τ_o	0	–.1124	–1986	–.2629	–.3091	–.3403	–.3705	–.3564
σ_o	.5992	.4643	.3515	.2577	.1802	.1168	.0068	–.0534
X	.75	.7	.65	.6	.5	.4	.3	.2
τ_o	–.3194	–.2732	–.2264	–.1816	–.1099	–.0621	–.0327	–.0159
σ_o	–.0816	–.0901	–.0874	–.0787	–.0559	–.0355	–.0209	–.0117

Superposing the results of two parts, we get the interlaminar stresses in Fig. 9 for the left half of the joint.

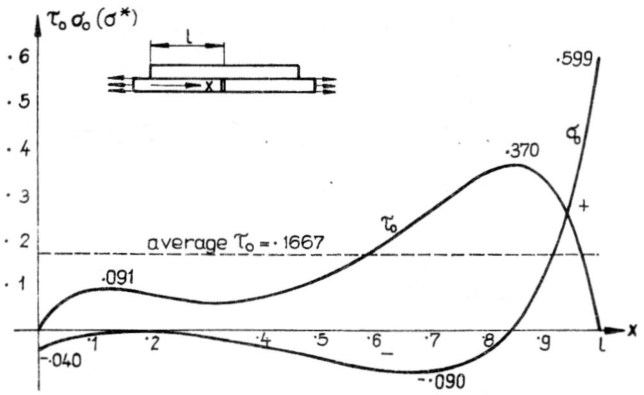

Figure 9

4. Conclusions

The second part of superposition involves severe loading as in Fig. 7. The two end moments tend to tear open the cover plate from the bar over gap, while the two axial forces equal and opposite tend to shear off the two parts at the very place. This accounts for the predominant interlaminar stresses near the gap of joint and is liable to cause delamination. At the same time, τ_o and σ_o at the ends of cover plate due to first part of superposition are not important to the strength of joint.

REFERENCES

[1] Zhang Fufan: Interlaminar Stresses of a Laminated Composite Lap Joint. Scientia Sinica (Series A), Vol 31 No. L (1988), 69 – 78.

[2] Zhang Fufan: Interlaminar Stresses of Laminated Composite Bars Under Bending. I, II, Appl. Maths. and Mech. (published in China), 7, 8 (1985) 583 – 594, 699 – 710.

[3] S. Timoshenko: Theory of Elasticity. McGraw-Hill (1934).

[4] N. J. Hoff: The Analysis of Structures. John Wiley & Sons, Inc. (1956).

Address of the author:

Fo-van Chang
Department of Engineering Mechanics
Tsinghua University
Peking
China