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On Failure of Determinism in Classical Mechanics

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Newtonian mechanics is generally considered to be deterministic: Once the initial conditions are known, all the future behaviour of a system can be predicted by solving the equations of motion. (That is the idea of Laplace's demon.) But a simple example will reveal that the solution of the initial value problem need not be unique. A prediction thus becomes impossible. An effect can happen without a cause, so that causality is annulled.

1 Introduction

Classical mechanics is ruled by differential equations. If the initial values of position and velocity of a system are given, the future values can, in principle, be calculated by integrating these equations. Laplace (1814) applied this idea to the whole universe and concluded that its future is fully determined by the presence. (The intelligent being who should know all the initial conditions and solve the equations was later on named Laplace's demon.) So Newtonian mechanics seems to be a fully deterministic theory. But this conviction is based on the tacit belief that the solution of the differential equations is unique.
We will present the following simple example that allows an infinite number of solutions and thus disproves the idea of determinism.

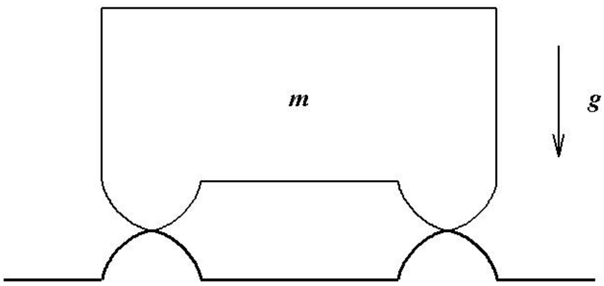

Figure 1: The state of rest of a plate under the influence of gravity

A rigid plate is supported by a rigid basis. The mutual contact occurs at the vertices of four geometrically identical cams. The situation depicted in Fig. 1 is obviously a state of equilibrium. So the plate can remain in this position for all times. Our question is whether it is also possible that the plate begins to move and leaves this position without noticeable cause.

2 The Geometry of our Example

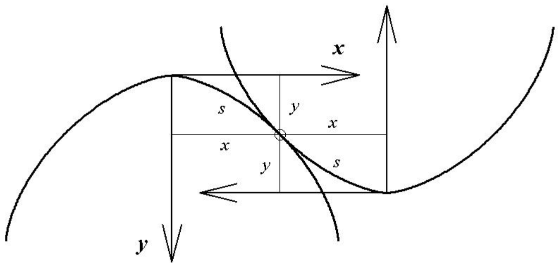

Figure 2: Contact of the cams during the motion

We describe the progress of the motion by the arc length s of the boundary of a cam from the vertex to the actual point of contact as shown in Fig. 2. The coordinates of the point of contact shall be given as

$$x = x(s)\,, \qquad y = y(s) \tag{1}$$

and the coordinates of the center of gravity of the plate are then

$$x_C = x_{C0} + 2x\,, \qquad y_C = y_{C0} + 2y \tag{2}$$

where x_{C0} and y_{C0} denote the position of the center of gravity in the state of rest. The square of the velocity of the center of gravity is

$$v_C^2 = \dot{x}_C^2 + \dot{y}_C^2 = 4(\dot{x}^2 + \dot{y}^2) = 4\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right)\left(\frac{ds}{dt}\right)^2 = 4\frac{dx^2 + dy^2}{ds^2}\left(\frac{ds}{dt}\right)^2 = 4\dot{s}^2 \tag{3}$$

3 Frictionless Motion

The sum of the potential and kinetic energy remains constant during the motion.

$$(E+U)_0 = E + U = \frac{1}{2}mv_C^2 - mg(y_C - y_{C0}) = 2m\dot{s}^2 - 2mgy(s) \tag{4}$$

We are interested in motions that start from the state of rest with $s = 0,\ y = 0,\ \dot{s} = 0$, so that $(E+U)_0 = 0$ holds. The equation of motion then becomes

$$\dot{s} = \ +\sqrt{gy(s)} \equiv r(s) \tag{5}$$

We choose the positive square root to describe motions to the right.
An obvious solution is $s(t) \equiv 0,\ y(t) \equiv 0,\ \dot{s}(t) \equiv 0$, so that the plate remains in the state of rest for an arbitrarily long time. We want to know whether there are other solutions of the differential equation. Uniqueness requires the fulfilment of Lipschitz' condition. But that condition is surely violated if the derivative of the right-hand side $r(s)$ is not finite, *i.e.*

$$\left|\frac{dr(s)}{ds}(s=0)\right| = \infty \tag{6}$$

The last condition allows a geometric interpretation. Let α denote the angle of the tangent and κ the curvature of the boundary curve. Then

$$\sin\alpha = \frac{dy}{ds}\,, \qquad \kappa = \frac{d\alpha}{ds} = \frac{d\alpha}{d\sin\alpha}\frac{d\sin\alpha}{ds} = \frac{1}{\cos\alpha}\frac{d^2y}{ds^2} = \frac{y''(s)}{\sqrt{1 - y'(s)^2}} \tag{7}$$

Now

$$y(s) = \frac{1}{g}r(s)^2\,, \qquad y'(s) = \frac{2}{g}r(s)r'(s)\,, \qquad y''(s) = \frac{2}{g}\Big(r'(s)^2 + r(s)r''(s)\Big) \tag{8}$$

At the vertex, we have $s = 0,\ \alpha = 0,\ \kappa(0) = y''(0) \propto r'(0)^2 = \infty$. So an infinite curvature at the vertex is sufficient to allow the spontaneous deviation of the plate from the state of rest. If this happens at some instant $t = t_D$, then the solution of our differential equation can be found by separation of the variables.

$$\int_{\hat{s}=0}^{s} \frac{d\hat{s}}{r(\hat{s})} = \int_{\hat{t}=t_D}^{t} d\hat{t} = t - t_D \tag{9}$$

4 A Special Geometry

We study the following class of boundary curves, depending on a positive constant a and a real parameter θ.

$$y(s) \equiv a s^{2\theta}\,, \quad y'(s) \equiv 2\theta a s^{2\theta-1}\,, \quad y''(s) \equiv 2\theta(2\theta-1) a s^{2(\theta-1)}\,, \quad r(s) \equiv \sqrt{ga}\, s^{\theta}\,, \quad r'(s) \equiv \theta\sqrt{ga}\, s^{\theta-1} \tag{10}$$

The condition $y'(0) = 0$ requires $\theta > 1/2$ and $r'(0)$ is infinite if $\theta < 1$. The curvature at the vertex $y''(0)$ is then infinite, too, as we already know. We are therefore only interested in values of θ satisfying $1/2 < \theta < 1$. Eq. (9) then gives

$$\frac{1}{\sqrt{ga}}\frac{s^{1-\theta}}{1-\theta} = t - t_D \qquad \Longrightarrow \qquad s(t) = \Big((1-\theta)\sqrt{ga}(t-t_D)\Big)^{\frac{1}{1-\theta}} \qquad t \geq t_D \tag{11}$$

Now let the initial conditions be $s(t_I) = 0$, $\dot{s}(t_I) = 0$ at some initial time $t_I < t_D$. A possible solution of this initial value problem is the remaining in the state of rest from t_I to the time t_D of deviation

$$s(t) \equiv 0 \qquad t_I \leq t \leq t_D \tag{12}$$

followed by a deviation off the state of rest according to eq. (11). Choosing the special value $\theta = 3/4$, we find

$$y(s) \equiv a s^{\frac{3}{2}} \qquad \Longrightarrow \qquad s(t) \equiv \begin{cases} 0 & \text{if} \quad t_I \leq t \leq t_D \\ \left(\frac{ga}{16}\right)^2 (t-t_D)^4 & \text{if} \quad t \geq t_D \end{cases} \tag{13}$$

So an infinite set of solutions of the initial value problem exists, depending on the parameter t_D.

5 The Role of Friction

It would be erroneous to assume that our phenomenon of indeterminism depends on the crude idealization of a frictionless motion. Let $F(s)$ be the work of friction exerted during the motion at each of the two contacts. Then the balance of work (4) has to be modified as follows.

$$0 = E + U + 2F = 2m\dot{s}^2 - 2mgy(s) + 2F(s) \tag{14}$$

and the equation of motion (5) has to be replaced by

$$\dot{s} = +\sqrt{gy(s) - \frac{1}{m}F(s)} \equiv r(s) \tag{15}$$

A non-trivial solution can only exist if

$$F(s) < mgy(s) \tag{16}$$

The power of friction is the product of the frictional force f and the relative velocity $v_R = v_C = 2\dot{s}$ at the points of contact.

$$\dot{F} = F'(s)\dot{s} = 2f(s)\dot{s} \tag{17}$$

In the case of dry friction, $F'(0) = 2f(0) = 2f_0 > 0$ is finite but $y'(0)$ is zero. So the inequality (16) cannot be satisfied near $s = 0$ and a deviation from the state of rest is impossible. The same happens if sticking friction is present.

However, viscous damping can be allowed. To demonstrate this, we study the rather special case

$$F(s) = \lambda m g y(s) \qquad \text{with} \qquad 0 < \lambda < 1 \tag{18}$$

The equation of motion becomes

$$\dot{s} = +\sqrt{(1-\lambda)gy(s)} \tag{19}$$

The solutions of the frictionless case remain valid if g is replaced by $(1-\lambda)g$. The appertaining nonlinear viscous law is obtained as follows

$$f = \frac{1}{2}F'(s) = \lambda\theta m g a s^{2\theta-1} = \lambda\theta m g a \left(\frac{v_R}{2\sqrt{(1-\lambda)ga}}\right)^{2-\frac{1}{\theta}} \equiv f(v_R) \tag{20}$$

6 Conclusions

Let us interpret our result (13). Newtonian mechanics reveals the following possibility: The plate remains in a state of rest for a certain time interval and then, at some instant t_D, suddenly starts a motion and leaves the state of rest.

- It is disturbing that the point t_D of deviation remains totally uncertain. Not even a statement of probability like a half-value time can be given.
- It cannot be known, too, whether the motion will ocur to the right-hand or the left-hand side.
- Our solution is an example of an indetermined motion. Note that the begin of the motion is not triggered by any external disturbance. No cause of this effect can be found. On the other hand, Laplace (1814), guided by his investigation on celestial mechanics, wrote : ”Les événemens actuels ont avec les précédens, une liaison fondée sur le principe évident, qu’une chose ne peut pas commencer d’être, sans une cause qui la produise. Cet axiome connu sous le nom de *principe de la raison suffisante,* s’étend aux actions même les plus indifférentes. (The connexion of the actual events with the preceding ones is based on the evident principle that nothing can begin to exist without a reason by which it is produced. This axiom, known under the name of principle of sufficient reason, even applies to actions of utmost irrelevance.)” But our finding indicates that this principle of sufficient reason is perhaps not so evident and even invalid in special situations.
- It is surprising that the plate can start its motion although, at the beginning, it has no information whether the curvature at the vertex is infinite and the friction small enough to allow the motion at all.

7 Delimitation

- The phenomenon of indeterminacy may be considered as a heightened stage of instability. Whenever there is a positive curvature at the vertex then the state of rest of the plate is unstable. An arbitrarily small disturbance is sufficient to cause a permanent deviation from that state. In a case like $y = as^2$, such a disturbance is also necessary. Otherwise the plate remains at rest in a deterministic way. In our indeterminate case, however, the state of rest is of course unstable, but no disturbance at all is necessary to start the deviation.
- Indeterminate behaviour must not be confused with chaotic behaviour. The latter is deterministic and characterized by a sensitive dependence on initial conditions. So all the intermediate states of the orbit are unstable. In our indeterminate case, we do not discuss various initial conditions but only one, the state of rest. Only this state is unstable but not the following ones during the motion.
- Indeterminism is not a problem for engineers but one of natural philosophy. The plate of our example cannot be manufactured with sufficient accuracy to test its behaviour by an experiment.

Remark: The indeterminate behaviour of eq. (13) was already discussed by the author in a text book (Krawietz (1997), p. 262). It was inferred there from the motion of a point mass, which is a cruder idealization than our plate. The same example was afterwards presented by Norton (2003) in a critical philosophical treatment on the principle of causation.

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