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Effective Coefficients of Isotropic Complex Dielectric Composites in a Hexagonal Array

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Abstract: Based on the asymptotic homogenization method, the local problems related to two-phase periodic fibrous dielectric composites with isotropic and complex constituents are solved. A hexagonal periodicity distribution of the fibers is considered. Explicit formulas for the real and imaginary parts of the effective dielectric properties are derived. Such formulas can be computed for any desired precision related to a truncation order of an infinite system of algebraic linear equations. Two simple analytical expressions are specified for the first two truncation orders. Comparisons with results via other approaches show a good concordance. Hexagonal periodic lattices of acoustic scatterers are useful structures for acoustic applications.

Keywords: Effective properties, two-phase fibrous dielectric composites, complex dielectric properties, hexagonal array.

1 Introduction

The effective conductivity tensor of two-dimensional complex dielectric composites consisting of a hexagonal periodic array of circular inclusions embedded in a matrix is studied, where both matrix and inclusions have complex dielectric properties. Perfect contact conditions at the interface between the matrix and the inclusions are considered. This problem is of interest, for instance, in acoustic applications, see e.g. Guild et al. (2014).

The solution is based on the asymptotic homogenization technique combined with series expansions of elliptic functions. A similar procedure has been used in recent works. For instance, Godin (2012) solved rigorously the problem for two-dimensional real dielectric composites using series expansions of Weierstrass' function and its derivatives depending on unknown real coefficients. Then, the problem is reduced to an infinite system and found its solution as a convergent power series allowing to obtain analytical formulas of the effective conductivity tensor for different lattice of inclusions. An analogous procedure was followed in Godin (2013) for the determination of the effective complex permittivity of a similar two-dimensional composite but with complex properties of the constituents. In that case the method of undetermined coefficients was used with complex coefficients allowing the derivation of efficient formulas for the effective properties. Unlike the real case a non-monotonic behavior of the real and imaginary parts of the effective tensor as function of area fraction of the inclusions is shown. This procedure has been extended to investigate the macroscopic behavior of periodic tubular structures in Godin (2016) and the propagation of electromagnetic waves through a two-dimensional composite material containing a periodic rectangular array of circular inclusions by Godin and Vainberg (2019). These studies have been found relevance in some applications. For instance, the results of Godin (2013) have been applied in Guild et al. (2014) to acoustics showing a good agreement with experimental data and inertial enhancement. In Ren et al. (2016), the results of Godin (2013) were used for calculating eddy current losses in soft complex magnetic composites. Recently, in Bravo-Castillero et al. (2018) the study of the effective behavior of complex dielectric composites was done by the homogenization of the equivalent system of equations with real coefficients. Closed-form formulas for the effective coefficients were obtained for a square periodic distribution of the inclusions which were employed to study gain-enhancement and loss enhancement properties of the homogenized material. This procedure offers independent models to compute the real and imaginary parts of the effective complex dielectric conductivity. In this work, based on the methodology in Bravo-Castillero et al. (2018), the effective tensor of two-dimensional complex dielectric is determined for the case of a hexagonal periodic distribution of the inclusions.

The work is organized as follows. After the Introduction, section 2 is devoted to the statement of the problem. A summary of the homogenization process, and the models for the local problems and the effective coefficients is presented in section 3. In section 4, the solution of the local problems is described and the formulas for the real and imaginary part of the effective tensor are derived. In section 5, some numerical examples are discussed. Finally, some concluding remarks are given in section 6.

2 Statement of the Problem

Let $\Omega \subset \mathbb{R}^2$ be a two-dimensional domain with infinitely smooth boundary $\partial\Omega$. The components of the complex dielectric permittivity tensor of a two-phase fibrous reinforced composite (FRC) occupying Ω are $(\alpha^\varepsilon + i\beta^\varepsilon)\delta_{jl}$ $(j, l = 1, 2)$ where $i^2 = -1$, δ

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is the Kronecker's delta, ε is a small geometric parameter that characterizes the periodicity and α^ε and β^ε are the real and imaginary part, respectively. The usual global or slow coordinates $\mathbf{x} \in \Omega$ and local or fast coordinates $\mathbf{y}$ with $\mathbf{y} = \mathbf{x}/\varepsilon$ are introduced. A hexagonal array of the periodic cell Y in global coordinates is considered so that it covers the domain $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$ where $\Gamma^\varepsilon \equiv \partial\Omega_2^\varepsilon$ and $\Omega_1^\varepsilon \cap \Omega_2^\varepsilon = \emptyset$; Ω_1^ε represents the matrix or connected set, Ω_2^ε denotes the fibers or disconnected set (an ε-periodic distribution of circles of radius $R\varepsilon$) and Γ^ε is the interface between Ω_1^ε and Ω_2^ε. The boundary $\partial\Omega$ is chosen so that it does not intersect any fiber of Ω_2^ε (Fig. 1). Fig 1 also shows a blow-up of the periodic hexagonal cell cross-section $Y \subset \mathbb{R}^2$ referred as $\mathbf{y}$-coordinates with an embedded circle of radius R and boundary Γ. Therein, Y_1 denotes the matrix or connected set and Y_2 the fiber or disconnected set. The regions Ω_1^ε and Ω_2^ε are occupied with two homogeneous materials with different electric permittivity

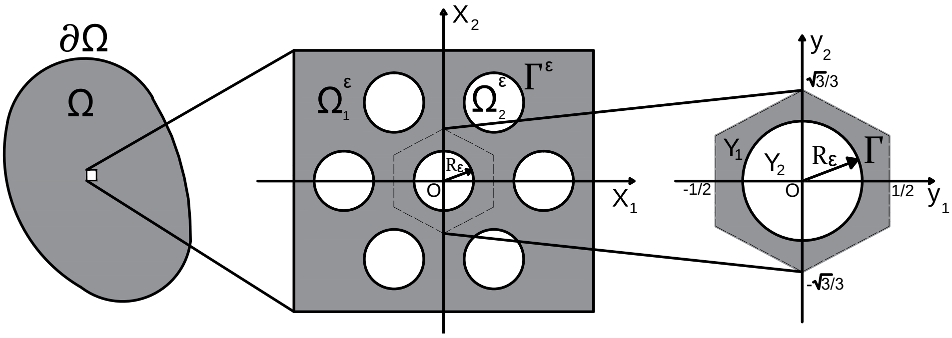

Fig. 1: (Left) Domain Ω with boundary $\partial\Omega$. (Centre) A blow-up domain contained in Ω showing a FRC type of geometry in global coordinates. (Right) Hexagonal cell in y-coordinates.

properties and the jl-components of the electric permittivity tensor are given by

$$\alpha^\varepsilon + i\beta^\varepsilon = \begin{cases} \alpha^{(1)} + i\beta^{(1)} & \text{in } \Omega_1^\varepsilon, \\ \alpha^{(2)} + i\beta^{(2)} & \text{in } \Omega_2^\varepsilon. \end{cases} \tag{1}$$

The complex electric potential $u^\varepsilon = \varphi^\varepsilon + i\psi^\varepsilon$ in Ω is sought as ε tends to zero so that Maxwell's equation in the quasi-static approximation in absence of free conduction currents are satisfied in Ω_1^ε and Ω_2^ε together with continuity of electric potential and normal component of electric displacement field across the interface Γ^ε. The Dirichlet condition "$u^\varepsilon = \tilde{u}_1 + i\tilde{u}_2$" is given on $\partial\Omega$. The related boundary value problem with complex coefficients is equivalent to the following system of two-coupled real partial differential equations

$$\frac{\partial}{\partial x_j}\left(\mathcal{A}_{jl}^\varepsilon \frac{\partial \mathbf{U}^\varepsilon}{\partial x_l}\right) = \mathbf{0} \quad \text{in } \Omega \setminus \Gamma^\varepsilon, \tag{2a}$$

$$[\![\mathbf{U}^\varepsilon]\!] = \mathbf{0} \quad \text{on } \Gamma^\varepsilon, \tag{2b}$$

$$\left[\!\!\left[\left(\mathcal{A}_{jl}^\varepsilon \frac{\partial \mathbf{U}^\varepsilon}{\partial x_l}\right) n_j\right]\!\!\right] = \mathbf{0} \quad \text{on } \Gamma^\varepsilon, \tag{2c}$$

$$\mathbf{U}^\varepsilon = \tilde{\mathbf{U}} \quad \text{on } \partial\Omega, \tag{2d}$$

where $\mathbf{U}^\varepsilon = (\varphi^\varepsilon, \psi^\varepsilon)^T$, $\tilde{\mathbf{U}} = (\tilde{u}_1, \tilde{u}_2)^T$ and $\mathbf{0} = (0,0)^T$ is the null vector of $\mathbb{R}^2$. The superscript T means transposition and the components of the 2×2 symmetric matrix-valued $\mathcal{A}^\varepsilon$ are given by

$$\mathcal{A}_{11}^\varepsilon = \alpha^\varepsilon, \quad \mathcal{A}_{12}^\varepsilon = \mathcal{A}_{21}^\varepsilon = -\beta^\varepsilon, \quad \mathcal{A}_{22}^\varepsilon = -\alpha^\varepsilon, \tag{3}$$

where Einstein repeated indexes summation convention is adopted. The j-th component of unit normal vector to Γ^ε, denoted with n_j, is taken in the direction from Ω_1^ε to Ω_2^ε. The notation $[\![.]\!]$ is used to denote the jump of the enclosed function across the interface Γ^ε in the direction of the normal $\mathbf{n}$.

3 Homogenization, Effective Coefficients and Local Problems

Following Bravo-Castillero et al. (2018) a formal asymptotic solution of (2a)–(2d) can be constructed up to $O(\varepsilon^2)$ as follows

$$\mathbf{U}^\varepsilon(\mathbf{x}) = \mathbf{U}^{(0)}(\mathbf{x}) + \varepsilon \mathbf{N}_k(\mathbf{y}) \frac{\partial \mathbf{U}^{(0)}(\mathbf{x})}{\partial x_k}, \tag{4}$$

with

$$\mathbf{U}^{(0)}(\mathbf{x}) = \left(\varphi^{(0)}(\mathbf{x}), \psi^{(0)}(\mathbf{x})\right)^T \quad \text{and} \quad \mathbf{N}_k(\mathbf{y}) = \begin{pmatrix} w^k(\mathbf{y}) & g^k(\mathbf{y}) \\ \zeta^k(\mathbf{y}) & \xi^k(\mathbf{y}) \end{pmatrix},$$

where the 2×2 matrices $\mathbf{N}_k$ are Y-periodic solutions of the local problems

$$\frac{\partial}{\partial y_j}\left(\mathcal{A}_{jl}(\mathbf{y})\frac{\partial \mathbf{N}_k(\mathbf{y})}{\partial y_l} + \mathcal{A}_{jk}(\mathbf{y})\right) = \mathbf{O} \quad \text{in } Y \setminus \Gamma, \tag{5a}$$

$$[\![\mathbf{N}_k(\mathbf{y})]\!] = \mathbf{O} \quad \text{on } \Gamma, \tag{5b}$$

$$\left[\!\!\left[\left(\mathcal{A}_{jl}(\mathbf{y})\frac{\partial \mathbf{N}_k(\mathbf{y})}{\partial y_l} + \mathcal{A}_{jk}(\mathbf{y})\right) n_j\right]\!\!\right] = \mathbf{O} \quad \text{on } \Gamma, \tag{5c}$$

with $\langle \mathbf{N}_k(y)\rangle = \mathbf{O}$. In (5a)–(5c), $\mathbf{O}$ denotes the 2×2 null matrix.
The term $\mathbf{U}^{(0)}$ in (4) is the solution of the homogenized problem

$$\widehat{\mathcal{A}}_{jk}\frac{\partial^2 \mathbf{U}^{(0)}(\mathbf{x})}{\partial x_j \partial x_k} = \mathbf{0} \quad \text{in } \Omega, \tag{6a}$$

$$\mathbf{U}^{(0)} = \tilde{\mathbf{U}} \quad \text{in } \partial\Omega, \tag{6b}$$

where the effective coefficients $\widehat{\mathcal{A}}_{jl}$ are constants and given by

$$\widehat{\mathcal{A}}_{jk} = \left\langle \mathcal{A}_{jk}(\mathbf{y}) + \mathcal{A}_{jl}(\mathbf{y})\frac{\partial N_k(\mathbf{y})}{\partial y_l}\right\rangle. \tag{7}$$

The angular brackets represent the volume average per unit length over the unit periodic cell, i.e. $\langle f(\mathbf{y})\rangle \equiv \int_Y f(y)dy$. The components of the effective coefficient $\widehat{\mathcal{A}}$ are

$$\widehat{\mathcal{A}}_{11} = \widehat{\boldsymbol{\alpha}}, \quad \widehat{\mathcal{A}}_{12} = \widehat{\mathcal{A}}_{21} = -\widehat{\boldsymbol{\beta}}, \quad \widehat{\mathcal{A}}_{22} = -\widehat{\boldsymbol{\alpha}}. \tag{8}$$

The effective coefficients $\widehat{\boldsymbol{\alpha}}$ and $\widehat{\boldsymbol{\beta}}$ can be found by using the following formulas

$$\widehat{\alpha} = \begin{cases} \langle\alpha\rangle - \frac{[\![\alpha]\!]}{|Y|}\int_\Gamma g^1 \mathrm{d}y_2 - \frac{[\![\beta]\!]}{|Y|}\int_\Gamma \xi^1 \mathrm{d}y_2, & \text{for } k = 1 \\ \langle\alpha\rangle + \frac{[\![\alpha]\!]}{|Y|}\int_\Gamma g^2 \mathrm{d}y_1 + \frac{[\![\beta]\!]}{|Y|}\int_\Gamma \xi^2 \mathrm{d}y_1, & \text{for } k = 2 \end{cases} \tag{9a}$$

$$\widehat{\beta} = \begin{cases} \langle\beta\rangle - \frac{[\![\beta]\!]}{|Y|}\int_\Gamma g^1 \mathrm{d}y_2 + \frac{[\![\alpha]\!]}{|Y|}\int_\Gamma \xi^1 \mathrm{d}y_2, & \text{for } k = 1 \\ \langle\beta\rangle + \frac{[\![\beta]\!]}{|Y|}\int_\Gamma g^2 \mathrm{d}y_1 - \frac{[\![\alpha]\!]}{|Y|}\int_\Gamma \xi^2 \mathrm{d}y_1, & \text{for } k = 2 \end{cases} \tag{9b}$$

where $\langle f\rangle = f_1 |Y_1| + f_2 |Y_2|$, with $|Y| = |Y_1| + |Y_2|$. The local functions g^k and ξ^k are solutions of the local problems defined as follows

Problem $\mathfrak{I}^k$: Find the Y-periodic functions g^k, ξ^k, such that:

$$\Delta g^k = 0, \quad \Delta \xi^k = 0, \quad \text{in } Y \setminus \Gamma, \tag{10a}$$

$$[\![g^k]\!] = 0, \quad [\![\xi^k]\!] = 0, \quad \text{on } \Gamma, \tag{10b}$$

$$\left[\!\!\left[\left(\alpha\frac{\partial \xi^k}{\partial y_l} + \beta\frac{\partial g^k}{\partial y_l}\right) n_j\right]\!\!\right] = -[\![\alpha]\!]\, n_k \quad \text{on } \Gamma, \tag{10c}$$

$$\left[\!\!\left[\left(\beta\frac{\partial \xi^k}{\partial y_l} - \alpha\frac{\partial g^k}{\partial y_l}\right) n_j\right]\!\!\right] = -[\![\beta]\!]\, n_k \quad \text{on } \Gamma, \tag{10d}$$

with $\langle g^k\rangle = 0$ and $\langle \xi^k\rangle = 0$. In (10a), $\Delta \equiv \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$ is the two-dimensional Laplace operator in a Cartesian coordinate system.

4 Solution of the Local Problem $\mathfrak{I}^k$ for Hexagonal Array

In order to solve the problem (10a)–(10d), let us consider a hexagonal lattice of inclusions of radius R (see Fig. 1). Particularly, doubly-periodic harmonic functions that satisfy the given interface conditions and the null average condition over the hexagonal cell are sought. Following Guinovart-Díaz et al. (2001), for $k = 1, 2$, the solutions of the local problems are sought in the form

$$\begin{aligned} g_1^1 &= \mathcal{R}e\left\{\sum_{q=1}^{\infty\, o}\left(a_q^1 z^{-q} - A_q^1 z^q\right)\right\}, \quad & g_2^1 &= \mathcal{R}e\left\{\sum_{q=1}^{\infty\, o} c_q^1 z^q\right\} \\ g_1^2 &= \mathcal{I}m\left\{\sum_{q=1}^{\infty\, o}\left(a_q^2 z^{-q} - A_q^2 z^q\right)\right\}, \quad & g_2^2 &= \mathcal{I}m\left\{\sum_{q=1}^{\infty\, o} c_q^2 z^q\right\} \end{aligned} \tag{11}$$

and

$$\begin{aligned} \xi_1^1 &= \mathcal{R}e\left\{\sum_{q=1}^{\infty\ o}\left(b_q^1 z^{-q} - B_q^1 z^q\right)\right\}, \quad \xi_2^1 = \mathcal{R}e\left\{\sum_{q=1}^{\infty\ o} d_q^1 z^q\right\} \\ \xi_1^2 &= \mathcal{I}m\left\{\sum_{q=1}^{\infty\ o}\left(b_q^2 z^{-q} - B_q^2 z^q\right)\right\}, \quad \xi_2^2 = \mathcal{I}m\left\{\sum_{q=1}^{\infty\ o} d_q^2 z^q\right\} \end{aligned} \tag{12}$$

where $\mathcal{R}e$ and $\mathcal{I}m$ indicate the real and imaginary parts, respectively. The superscript o specifies that the sum is carried out over odd indices, the unknown coefficients a_q^k, b_q^k, c_q^k and d_q^k are real and

$$A_q^k = \sum_{p=1}^{\infty\ o} p a_p^k \eta_{pq}^k, \quad B_q^k = \sum_{p=1}^{\infty\ o} p b_p^k \eta_{pq}^k, \tag{13}$$

with

$$(k=1)\ \ \eta_{pq}^1 = \begin{cases} \frac{2\pi}{\sqrt{3}}, & p+q=2 \\ \dfrac{(p+q-1)!}{p!q!} S_{p+q}, & p+q>2 \end{cases} \qquad (k=2)\ \ \eta_{pq}^2 = \begin{cases} -\pi, & p+q=2 \\ \dfrac{(p+q-1)!}{p!q!} S_{p+q}, & p+q>2 \end{cases}$$

and S_j are the reticulate sums given by

$$S_{p+q} = \sum_{n^2+m^2\neq 0} \frac{1}{(m\omega_1 + n\omega_2)^{p+q}},$$

where $\omega_1 = 1$ and $\omega_2 = e^{\frac{\pi}{3}i}$ are the periods. As the cross-section of the inclusion is described by a circle of radius R, the interface in the unit cell is defined by $\Gamma = Re^{i\theta}$ with $0 \leq \theta < 2\pi$, then substituting (11)–(12) into the interface conditions (10b)–(10d), one obtain the following infinite system of algebraic equations

$$\begin{pmatrix} \mathbf{I} + (-1)^{k+1}\chi_\alpha \mathbf{W}^k & \chi_{\beta\alpha}^+ \mathbf{I} + (-1)^{k+1}\chi_{\beta\alpha}^- \mathbf{W}^k \\ \chi_{\beta\alpha}^+ \mathbf{I} + (-1)^{k+1}\chi_{\beta\alpha}^- \mathbf{W}^k & -\left(\mathbf{I} + (-1)^{k+1}\chi_\alpha \mathbf{W}^k\right) \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{A}}^k \\ \tilde{\mathbf{B}}^k \end{pmatrix} = (-1)^{k+1} \begin{pmatrix} \mathbf{V}^1 \\ \mathbf{V}^2 \end{pmatrix}, \tag{14}$$

where $\mathbf{I}$ is the infinite identity matrix, $\tilde{\mathbf{A}}^k = (\tilde{a}_1^k, \tilde{a}_3^k, \ldots)^T$, $\tilde{\mathbf{B}}^k = (\tilde{b}_1^k, \tilde{b}_3^k, \ldots)^T$, $a_q^k = \tilde{a}_q^k R^q/\sqrt{q}$, $b_q^k = \tilde{b}_q^k R^q/\sqrt{q}$, $\mathbf{V}^1 = (\chi_\alpha R, 0, \ldots)^T$, $\mathbf{V}^2 = (\chi_{\beta\alpha}^- R, 0, \ldots)^T$, and

$$(k=1)\ \ \mathbf{W}^1 = \begin{cases} \frac{2\pi}{\sqrt{3}} R^2, & p+q=2 \\ \sum_{p=1}^{\infty\ o} \sqrt{pq}\,\eta_{pq}^1 R^{p+q}, & p+q>2. \end{cases} \qquad (k=2)\ \ \mathbf{W}^2 = \begin{cases} -\pi R^2, & p+q=2 \\ \sum_{p=1}^{\infty\ o} \sqrt{pq}\,\eta_{pq}^2 R^{p+q}, & p+q>2. \end{cases}$$

Furthermore,

$$\chi_\alpha = \frac{[\![\alpha]\!]}{\alpha^{(1)} + \alpha^{(2)}}, \quad \chi_{\beta\alpha}^+ = \frac{\beta^{(1)} + \beta^{(2)}}{\alpha^{(1)} + \alpha^{(2)}} \quad \text{and} \quad \chi_{\beta\alpha}^- = \frac{[\![\beta]\!]}{\alpha^{(1)} + \alpha^{(2)}}. \tag{15}$$

The matrix $\mathbf{W}^k$, $k = 1, 2$ is real, symmetric and bounded, and consequently the classical results from the theory of infinite systems Kantorovich and Krylov can be used to solve (14). In this sense, the infinite linear system can be truncated into an appropriate order $p = q = 2n_o - 1$, with $n_o \in \mathbb{N}$. In this way, (14) is transformed into a linear system of order $2n_o$. Now, the use of (11) and (12) into (9a) and (9b) leads to,

$$\widehat{\alpha} = \alpha^{(1)} - (-1)^{k+1}\frac{2\pi}{|Y|}\left(\alpha^{(1)} a_1^k + \beta^{(1)} b_1^k\right), \tag{16a}$$

$$\widehat{\beta} = \beta^{(1)} - (-1)^{k+1}\frac{2\pi}{|Y|}\left(\beta^{(1)} a_1^k - \alpha^{(1)} b_1^k\right). \tag{16b}$$

For the particular case of real dielectric composites with isotropic constituents (i.e., for $\beta^{(1)}, \beta^{(2)} = 0$), the formulas for the effective coefficients (16a)–(16b) reduce to formulas (3.15)–(3.16), p . 228 in Guinovart-Díaz et al. (2001).

5 Analytical Formulas, Numerical Examples and Some Comparisons

5.1 Analytical Formulas

Following Bravo-Castillero et al. (2018), the system (14) can also be written as follows

$$\left(\boldsymbol{\eta}\mathcal{I} + (-1)^{k+1}\boldsymbol{\lambda}\mathcal{W}^k\right)\begin{pmatrix}\tilde{\mathbf{A}}^k\\ \tilde{\mathbf{B}}^k\end{pmatrix} = (-1)^{k+1}\begin{pmatrix}\mathbf{V}^1\\ \mathbf{V}^2\end{pmatrix}, \tag{17}$$

where

$$\boldsymbol{\eta} = \begin{pmatrix}1 & \chi^+_{\beta\alpha}\\ \chi^+_{\beta\alpha} & -1\end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix}\chi_\alpha & \chi^-_{\beta\alpha}\\ \chi^-_{\beta\alpha} & -\chi_\alpha\end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix}\mathbf{I} & \Theta\\ \Theta & \mathbf{I}\end{pmatrix} \quad \text{and} \quad \mathcal{W}^k = \begin{pmatrix}\mathbf{W}^k & \Theta\\ \Theta & \mathbf{W}^k\end{pmatrix},$$

with Θ denoting the infinite null matrix. After multiplication of (11) by $\boldsymbol{\lambda}^{-1}$ and noticing that $\boldsymbol{\lambda}^{-1}(\mathbf{V}^1, \mathbf{V}^2)^T = R\mathbf{e}_1$, where $\mathbf{e}_1$ is the infinite vector $(1, 0, 0, \dots)^T$. Then, equation (17) becomes

$$\begin{pmatrix}\tilde{\mathbf{A}}^k_{n_o}\\ \tilde{\mathbf{B}}^k_{n_o}\end{pmatrix} = (-1)^{k+1}(\boldsymbol{\theta}\mathcal{I}_{n_o} + (-1)^{k+1}\mathcal{W}^k_{n_o})^{-1}R\mathbf{e}^T_{2n_o}, \tag{18}$$

or equivalently

$$\begin{pmatrix}\tilde{\mathbf{A}}^k_{n_o}\\ \tilde{\mathbf{B}}^k_{n_o}\end{pmatrix} = (-1)^{k+1}\begin{pmatrix}\theta_{11}\mathbf{I}_{n_o} + (-1)^{k+1}\mathbf{W}^k_{n_o} & \theta_{12}\mathbf{I}_{n_o}\\ -\theta_{12}\mathbf{I}_{n_o} & \theta_{11}\mathbf{I}_{n_o} + (-1)^{k+1}\mathbf{W}^k_{n_o}\end{pmatrix}^{-1} R\mathbf{e}^T_{2n_o}, \tag{19}$$

where the sub-index n_o represents the truncation order of the vectors $\tilde{\mathbf{A}}^k$, $\tilde{\mathbf{B}}^k$, $\mathbf{e}_1$, and the matrices $\mathcal{I}$ and $\mathcal{W}^k$. The matrix $\boldsymbol{\theta} = \boldsymbol{\lambda}^{-1}\boldsymbol{\eta}$ has the form

$$\boldsymbol{\theta} = \begin{pmatrix}\theta_{11} & \theta_{12}\\ -\theta_{12} & \theta_{11}\end{pmatrix}, \tag{20}$$

and its components are

$$\theta_{11} = \frac{\chi_\alpha + \chi^-_{\beta\alpha}\chi^+_{\beta\alpha}}{(\chi_\alpha)^2 + (\chi^-_{\beta\alpha})^2} \quad \text{and} \quad \theta_{12} = \frac{\chi_\alpha\chi^+_{\beta\alpha} - \chi^-_{\beta\alpha}}{(\chi_\alpha)^2 + (\chi^-_{\beta\alpha})^2}. \tag{21}$$

Using the finite system (19), we find the unknowns $\tilde{\mathbf{A}}^k_{n_o}$ and $\tilde{\mathbf{B}}^k_{n_o}$ for different orders of truncation, which are then substituted into the effective coefficients expressions (16a)–(16b). In this way, formula (19) is helpful in finding closed-forms for the effective coefficients.

1. If $n_o = 1$, equation (19) takes the form

$$\begin{pmatrix}\tilde{a}^k_1\\ \tilde{b}^k_1\end{pmatrix} = (-1)^{k+1}\begin{pmatrix}\theta_{11} + (-1)^{k+1}W^k_{11} & \theta_{12}\\ -\theta_{12} & \theta_{11} + (-1)^{k+1}W^k_{11}\end{pmatrix}^{-1}\begin{pmatrix}R\\ 0\end{pmatrix}, \tag{22}$$

where W^k_{pq} denote the elements of $\mathbf{W}^k$. Then,

$$\tilde{a}^k_1 = (-1)^{k+1}\frac{(\theta_{11} + (-1)^{k+1}W^k_{11})R}{(\theta_{11} + (-1)^{k+1}W^k_{11})^2 + \theta^2_{12}}, \tag{23a}$$

$$\tilde{b}^k_1 = (-1)^{k+1}\frac{\theta_{12}R}{(\theta_{11} + (-1)^{k+1}W^k_{11})^2 + \theta^2_{12}}. \tag{23b}$$

Substitution of equations (23a)–(23b) in the expressions for the effective coefficients (16a)–(16b) yields

$$\widehat{\alpha} = \alpha^{(1)} - \frac{2(\alpha^{(1)}\theta_{11} + \beta^{(1)}\theta_{12})Y_2 + 2\alpha^{(1)}Y^2_2}{(\theta_{11} + Y_2)^2 + \theta^2_{12}}, \tag{24a}$$

$$\widehat{\beta} = \beta^{(1)} - \frac{2(\beta^{(1)}\theta_{11} - \alpha^{(1)}\theta_{12})Y_2 + 2\beta^{(1)}Y^2_2}{(\theta_{11} + Y_2)^2 + \theta^2_{12}}, \tag{24b}$$

where V_γ is the volume fraction of the phase γ. Particularly, $Y_1 + Y_2 = \frac{\sqrt{3}}{2}$ with $Y_2 = \pi R^2$.

2. If $n_o = 2$, equation (19) takes the form

$$\begin{pmatrix} \tilde{a}_1^k \\ \tilde{a}_3^k \\ \tilde{b}_1^k \\ \tilde{b}_3^k \end{pmatrix} = (-1)^{k+1} \begin{pmatrix} \theta_{11} + (-1)^{k+1} w_{11} & 0 & \theta_{12} & 0 \\ 0 & \theta_{11} + (-1)^{k+1} w_{33} & 0 & \theta_{12} \\ -\theta_{12} & 0 & \theta_{11} + (-1)^{k+1} w_{11} & 0 \\ 0 & -\theta_{12} & 0 & \theta_{11} + (-1)^{k+1} w_{33} \end{pmatrix}^{-1} \begin{pmatrix} R \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{25}$$

To find $\tilde{a}_1^1$, $\tilde{a}_3^1$, $\tilde{b}_1^1$ and $\tilde{b}_3^1$, the above linear system (25) must be solved. Then, the coefficients $\tilde{a}_1^1$ and $\tilde{b}_1^1$ are substituted into the effective coefficients expressions (16a)–(16b).

5.2 Numerical Examples

Now, we compare the effective coefficients (16a)–(16b) for successive truncation orders $n_o = 1, 2, 3, 4$. In particular, we fix

$$\kappa^{(1)} = 1 - 5i \qquad \kappa^{(2)} = 30 - 0.3i,$$

and denote by $V_p = \frac{\pi}{4}$ the percolation limit where the cylinders are in contact. Fig 2 and 3 displays the real and imaginary parts of the effective complex dielectric coefficient $\hat{\kappa}$ as a function of the inclusion volume fraction V_o. It is observed that the first approximation is a very good estimation of the complex effective dielectric coefficient for $V_o < 0.7$. Besides, the effective coefficients for a truncation order at $n_o = 3$ and 4 are quite similar. This agreement shows that the second order approximation is good enough for higher orders of approximations. Therefore, in what follows, we restrict our analysis to first, second and third approximation orders of the effective coefficients.

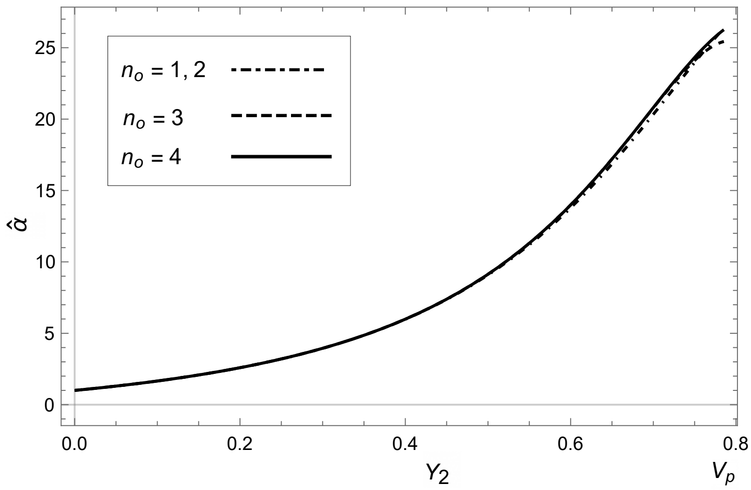

Fig. 2: The real part of the complex effective dielectric coefficient $\widehat{\kappa}$ as a function of the volume fraction Y_2 shown for successive truncation orders $n_o = 1, 2, 3, 4$.

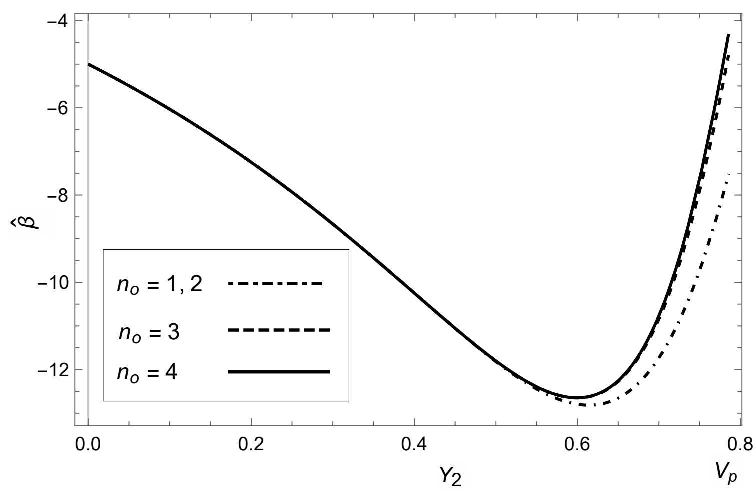

Fig. 3: The imaginary part of the complex effective dielectric coefficient $\widehat{\kappa}$ as a function of the volume fraction Y_2 shown for successive truncation orders $n_o = 1, 2, 3, 4$.

5.3 Comparisons

We compare our results with those obtained in Godin (2013) for

$$\kappa^{(1)} = 2 - 0.3i \quad \text{and} \quad \kappa^{(2)} = 1 - 8i.$$

The approximation to the real (imaginary) part of the complex effective coefficient reported in Godin (2013), are determined by

$$\widehat{\alpha} = Re\left(\varepsilon^*\right) \qquad \text{and} \qquad \widehat{\beta} = Im\left(\varepsilon^*\right), \tag{26}$$

where

$$\varepsilon^* = \kappa^{(1)} \frac{1+\alpha\lambda f}{1-\alpha\lambda f},$$

and

$$\alpha = \frac{\kappa^{(2)} - \kappa^{(1)}}{\kappa^{(2)} + \kappa^{(1)}}, \qquad f = \frac{2}{\sqrt{3}}\pi R^2 = \frac{2}{\sqrt{3}}Y_2, \qquad \lambda = 1 + 5\alpha^2 S_3^2 R^{12} + \alpha^2\left(25\alpha^2 S_3^4 + 11 S_6^2\right) R^{24} + O\left(R^{36}\right)$$

and while the only non-zero real lattice sums are S_{3k}, $k = 1, 2, \ldots$, here

$$S_3 = \sum_{n^2+m^2\neq 0} \frac{1}{\left(m + ne^{\frac{\pi}{3}i}\right)^6} \approx 5.86303 \qquad S_6 = \sum_{n^2+m^2\neq 0} \frac{1}{\left(m + ne^{\frac{\pi}{3}i}\right)^{12}} \approx 6.00964.$$

As it was pointed out in the previous section, it is sufficient to work up to a truncation order of $n_o = 3$. Fig 4 and 5 show the comparison between the results using the present approach and those from Godin (2013). In particular, we note that the second order approximation of the effective coefficients agrees with the result in Godin (2013), whereas the results using a first order truncation is close to the data reported in Godin (2013). These comparisons assure the use of the obtained short formulas arising from (19) to investigate the complex effective dielectric coefficient.

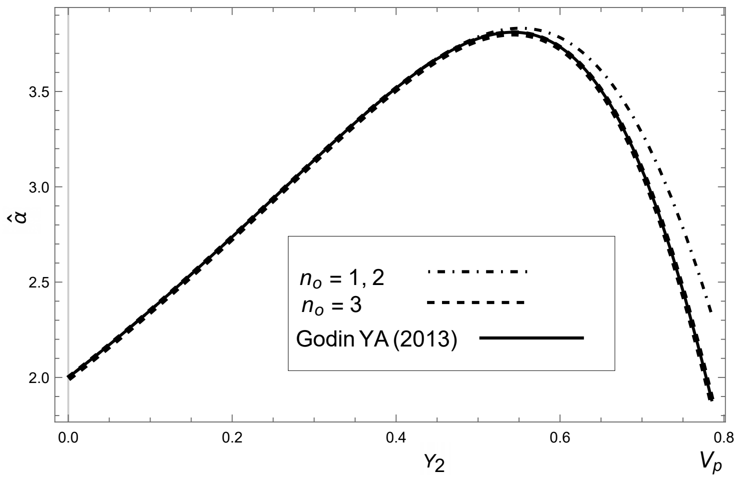

Fig. 4: Comparison of the real part of the complex effective dielectric coefficient $\widehat{\kappa}$ depending on the volume fraction Y_2 calculated using (16a)–(16b) truncated at $n_o = 1, 2, 3$. Also plotted the results from Godin (2013).

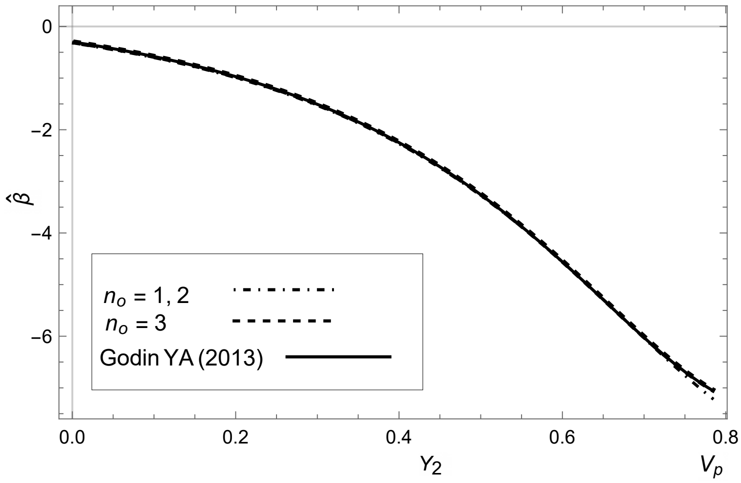

Fig. 5: Comparison of the imaginary part of the complex effective dielectric coefficient $\widehat{\kappa}$ depending on the volume fraction Y_2 calculated using (16a)–(16b) truncated at $n_o = 1, 2, 3$. Also plotted the results from Godin (2013).

6 Concluding Remarks

A system of two equations with real periodic and rapidly oscillating coefficients (2a)–(2d) is studied via asymptotic homogenization for predict the macroscopic behavior of two-phase fibrous dielectric composites wherein the constituents exhibit complex dielectric isotropy. Series expansions of complex potentials with unknown real coefficients (11)–(12) are used to solve the local problems (10a)–(10d) for a hexagonal periodic distribution of the fibers. The unknown coefficients are solutions of an infinity system of linear algebraic equations (14). An explicit solution (19) of the infinite system was derived for any truncation order. Formula (19) is useful in finding closed-forms expressions for the effective coefficients. Therefore, two simple analytical formulas (24a)–(24b) and (25) are specified for the two first truncation order. Numerical examples illustrated a very good concordance of such formulas with those reported in Godin (2013). These results could be useful for acoustic applications wherein hexagonal periodic lattices of acoustic scatterers structures are present Guild et al. (2014). Besides, these formulas can be used for estimating gain and loss enhancement properties of active and passive composites in certain volume fraction intervals as in Bravo-Castillero et al. (2018). Besides it is interesting to mention that results display for either the real part of the effective dielectric coefficient a monotonic behavior and the imaginary part a non-monotonic one, or the opposite. Some examples show gain- or loss-enhancement properties.

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Bibliography

J. Bravo-Castillero, A. Ramírez-Torres, F.J. Sabina, C. García-Reimbert, R. Guinovart-Díaz, and R. Rodríguez-Ramos. Analytical formulas for complex permittivity of periodic composites. estimation of gain and loss enhancement in active and passive composites. *Waves in Random and Complex Media*, pages 1–21, 2018. doi: 10.1080/17455030.2018.1546063.

Y.A. Godin. Effective complex permittivity tensor of a periodic array of cylinders. *Journal of Mathematical Physics*, 53(6): 053505, 2012. doi: 10.1063/1.4726213.

Y.A. Godin. Effective complex permittivity tensor of a periodic array of cylinders. *Journal of Mathematical Physics*, 54(5): 053505, 2013. doi: doi: 10.1063/1.4803490.

Y.A. Godin. Effective properties of periodic tubular structures. *The Quarterly Journal of Mechanics and Applied Mathematics*, 69 (2):181–193, 2016. doi: 10.1093/qjmam/hbw003.

Y.A. Godin and B. Vainberg. Dispersive and effective properties of two-dimensional periodic media. *Proceedings of the Royal Society A*, 475(2221):20180298, 2019. doi: 10.1098/rspa.2018.0298.

M.D. Guild, V.M. García-Chocano, W. Kan, and J. Sánchez-Dehesa. Enhanced inertia from lossy effective fluids using multi-scale sonic crystals. *AIP Advances*, 4(12):124302, 2014. doi: 10.1063/1.4901880.

R. Guinovart-Díaz, J. Bravo-Castillero, R. Rodríguez-Ramos, and F. J Sabina. Closed-form expressions for the effective coefficients of fibre-reinforced composite with transversely isotropic constituents. i: Elastic and hexagonal symmetry. *Journal of the Mechanics and Physics of Solids*, 49(7):1445–1462, 2001. doi: 10.1016/S0022-5096(01)00005-9.

L.V. Kantorovich and V.I. Krylov. Approximate methods of higher analysis. *translated by C. D. Benster. XII + 681 S. m. 68 Abb. Groningen, 1958. P. Noordhoff Ltd*.

X. Ren, R. Corcolle, and L. Daniel. A homogenization technique to calculate eddy current losses in soft magnetic composites using a complex magnetic permeability. *IEEE Transactions on Magnetics*, 52(12):1–9, 2016. doi: 10.1109/TMAG.2016.2594048.