Manuskripteingang: 19. 1. 1984

Kinematic Path Control of Robot Arms with Redundancy

E. Krustev, L. Lilov

I. PREFACE. The kinematic control of robot arms with redundancy has become a subject of intensified investigation in recent years. The most significant features of robot arms with redundancy that draw the attention of scientists are increased flexibility, possibility for obstacle avoidance and an admissibility for satisfaction of optimization criteria, which reflect the dynamic behavior of the mechanical system [1]. It should be noted also that research in the field of biomechanics has shown that the real antropomorphic structures are systems of the same type.

The basic problem in kinematic control is to coordinate the displacement in the arm joints, so that a desired action of the end effector be attained. With the solution of this problem joint forces and torques are computed to drive the robot arm in a specified system trajectory. Thus, a robot arm with redundancy can accomplish a variety of sophisticated movements with applications in assembling operations, welding, painting etc..

The realisation of kinematic path control of robot arms with redundancy usually is based on the solution of the following two subproblems. First, we are to determine the conditions under which a given in advance movement of the end effector is feasible for the specified robot arm [2]. As a second step, a quality criterion for the laws governing the displacement in the joints is assigned. The above subproblems, considered in their unity, formulate the general statement of the problem for kinematic path control of robot arms with redundancy, i. e. the basic problem is to guide the robot arm in such a way that the end effector follows a specified trajectory in a desired orientation and, besides, the quality criterion is to be satisfied [3] – [9].

Rate control [3] seems to be the most predominant technique that has been applied in solving the problem, here stated. However, the known applications of this approach treat implicitly and incompletely the geometric constraints imposed on the movement of the end effector. Therefore, only certain motions of the end effector along the specified trajectory are being considered. In particular, the existing methods for kinematic path control of robot arms with redundancy afford no possibility to describe such motions by which the end effector is fixed at a point of the specified path trajectory for a sufficiently short time interval, while simultaneously a continuous alteration in the arm configuration takes place. These motions prove to be of value, especially when a boundary position of an arm joint is being reached and further on, the continuous movement of the end effector is not feasible without a shift in the configuration achieved. Most of the habitual motions of the human arm (pick-and-place operations, drink test etc.) are motions of the same type. Moreover, the existing methods additionally require that the motions of the robot arm under consideration should possess such an initial configuration that the corresponding position of the end effector satisfies the geometrical constraints imposed by the choice of the path trajectory.

The work, here presented, elaborates a method for kinematic path control of robot arms with redundancy. This method employs all the continuous piecewise smooth motions of the robot arm by means of which the end effector follows a specified curve in the set of its feasible positions. The method suggests that each curve correlates to a differentiable manifold, while the laws governing the displacement in the joints are related to the integral curves of a tangent vector field. The latter is built in the above mentioned manifold in accordance with a control criterion, assigned in advance. This approach proves to be efficient in determining a continuous optimal motion of the robot arm for each prescribed path of the end effector. All this is being done by means of integration of the tangent vector field, under boundary conditions fixed in the statement of the problem. In particular, most of the methods for kinematic path control of robot arms with redundancy follow from the method here proposed.

II. PROBLEM STATEMENT. Let us consider a robot arm M, which is an open kinematic chain with rotational or translational joints. Denote with $q = (q_1, q_2, \dots, q_n)$ the ordered n-tuple of joint variables, that describe the displacement in the arm joints. The range $Q_M \subset R^n$ of these displacements is called space of configurations of the robot arm M.

The position of the end effector in the base frame $Ox_o y_o z_o$ can be determined by the coordinates r_H of the origin H of a frame Hxyz, fixed to the end effector, and the matrix G of transformation from the frame Hxyz into the frame $Ox_o y_o z_o$ (fig. 1). It is obvious, that the matrix G belongs to the set T of 3 x 3 proper orthogonal matrices, which is a three dimensional manifold [10]. Let $\Phi = (s_i, W_i)$ i = 1, a be an atlas of T, so that $\bigcup_{i=1}^{a} W_i$ forms an open covering of T and for each $i \in \{1, 2, \dots, a\}$ a diffeomorphic mapping $s_i : W_i \longrightarrow R^3$ exists. The coordinates r_H and the matrix G are smooth functions of the joint variables, i. e. $r_H : Q_M \longrightarrow R^3$ $G : Q_M \longrightarrow T$. They both determine the end effector's position in the base frame [11]. Usually, the independent parameters describing the end effector's orientation are Euler angles, Briant angles etc.. At an arbitrary $q \in Q_M$, such as $G(p) \in W_i$ for $i \in \{1, 2, \dots, a\}$ these parameters are given in the form of the composition $s_i \cdot G$. Thus, the end effector's position in the base frame is determined by a vector valued smooth function

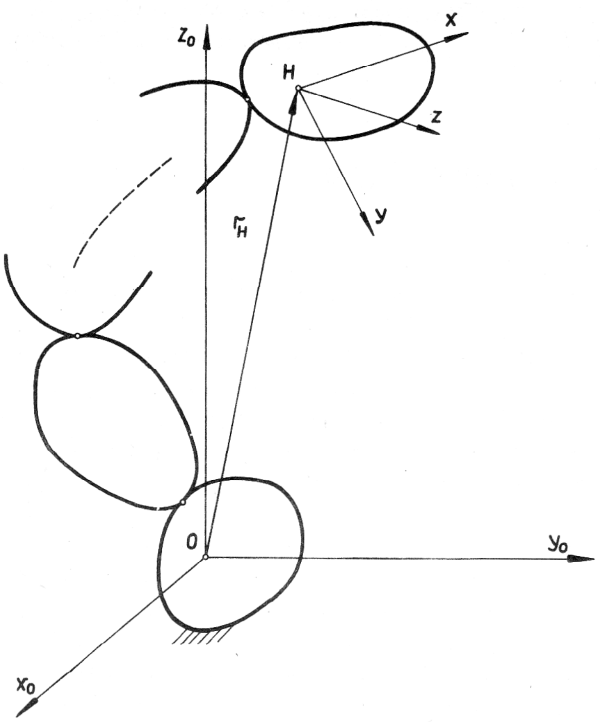


Fig. 1

$F : Q_M \to R^6$ defined as $F(q) = (r_H, s_i \cdot G)$. The function F is called a generalized function of the end effector's position and the set $F(Q_M)$ – a set of feasible positions (task-oriented space).

The functional capabilities of M and the tasks formulated, do not make it possible for the end effector to attain six degrees of freedom. Therefore, in the general case $\dim F(\text{Int } Q_M) \leqslant 6$, where Int Q_M is the interior of the space of configurations. Denote with m the maximum rank of the mapping F. The robot arm M is called [3] a robot arm with redundancy, if

$$m < \dim \text{Int } Q_M = n.$$

The existence of redundancy can intuitevely be interpreted as a capability of the robot arm to fix its end effector in a given position of the set of feasible positions with a continuum of different configurations. It follows from the inverse function theorem [12], that the boundary points of the set $F(Q_M)$ are reached only by a finite number of configurations, belonging to the boundary of the set of configurations. Thus, no redundancy exists in these points. For this reason and because of the requirement for smoothness, further on imposed, only the interior Int Q_M of the space of configurations is considered.

One of the main problems in practical applications of robot arms is to determine the joint variables as functions of time, so that the end effector follows a prescribed curve in the set of feasible positions. Let M be a robot arm with redundancy. Then M can accomplish such a task by a variety of modes. In order to include into consideration all possible modes, assume that an arbitrary parametrization of the given curve γ is introduced. Suppose γ is an arc of a smooth curve $\delta = \delta(\lambda) : R \to R^m$ and denote with $\underline{\Lambda}$ the closed interval of R that satisfies $\gamma = \delta(\underline{\Lambda})$.

Let us consider the mapping $\mathbf{F} : \text{Int } Q_M \times R \to R^m$ defined as $F(q^*) = F(q) - \delta(q_{n+1})$, where $q^* = (q, q_{n+1})$. In particular, the equality $F(q^*) = 0$ for $(q, q_{n+1}) \in Q_M \times \underline{\Lambda}$ implies that the end effector's position is $\delta(q_{n+1}) \in \gamma$. Thus, the set $B = F(0)^{-1} \cap (\text{Int } Q_M \times \underline{\Lambda})$ comprises all ordered pairs (q, λ) of configurations q for which the end effector takes a position $\gamma(\lambda)$, $\lambda \in \underline{\Lambda}$.

Suppose that the mapping F possesses maximum rank on $F(0)^{-1}$ and besides

$$\text{rank } F(q) = \text{rank } F(q) = m \quad \text{for all } q^* \in B \qquad (1)$$

Actually, such an assumption has been predominant in most methods for kinematic path control of robot arms with redundancy [1], [3] – [7], [9]. On this assumption $F(0)^{-1}$ is a differentiable submanifold of R^{n+1} with the dimension $n-m+1$. The case where $F(0)^{-1}$ is two dimensional is depicted in fig. 2. The validity of condition (1) can eventually be ensured [13] by an infinitely small variance of the arc γ admissible to the range of precision assigned by the engineering practice.

Denote with I the time interval the robot arm needs to follow the arc γ. Let $q(t)$ and $\chi(t)$, $t \in I$ be two continuous piecewise smooth functions

$$q : I \to Q_M \ ; \ \chi : I \longrightarrow \underline{\Lambda} \qquad (2)$$

where χ maps I onto $\underline{\Lambda}$.

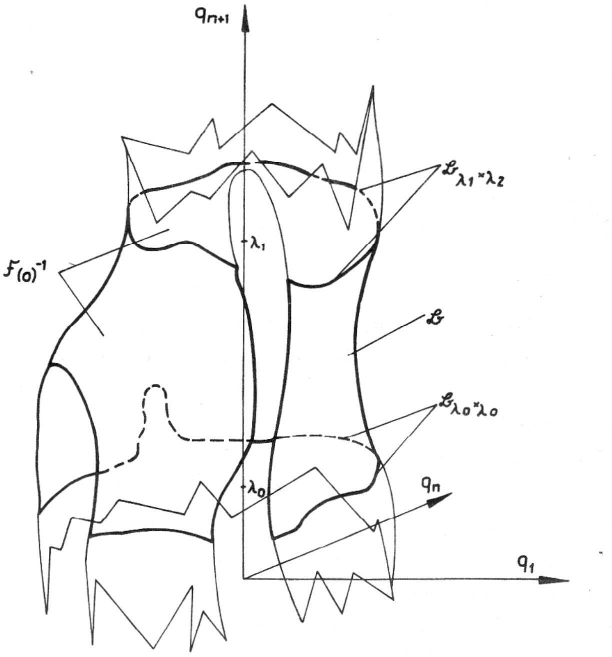


Fig. 2

Definition 1. By an absolutely defining trajectory we understand each smooth arc γ for which the set $\underline{X}_\gamma = \{(q(t), \chi(t)) : F(q(t), \chi(t)) = 0 \text{ for all } t \in I\}$ is nonempty.

Every motion of the robot arm in the time interval I such that the end effector follows a given absolutely defining trajectory γ corresponds to an element $(q(t), \chi(t))$ of the set $\underline{X}_\gamma$. Conversely, any element of $\underline{X}_\gamma$ defines a motion of the end effector along the absolutely defining trajectory. The subset of $\underline{X}_\gamma$ for which

$\dot{\chi}(t) \geqslant 0$ for all $t \in I$ is of particular importance for practical implementations. Its elements define such continuous motions of the robot arm that the end effector follows the absolutely defining trajectory by the parameter t in the same direction already specified by the parameter λ. Further on, we refer to the elements of the set $\underline{X}_\gamma$ as admissible motions of the robot arm M.

The problem of tracing a prescribed position trajectory is considered by the following formulation.

Let M be a robot arm with redundancy and an absolutely defining trajectory $\gamma : [\lambda_o, \lambda_1] \longrightarrow F(Q_M)$ with end points $\gamma_o = \gamma(\lambda_o)$; $\gamma_1 = \gamma(\lambda_1)$ be given. Find an admissible motion that minimizes an assigned functional

$$L(\dot{q}(t), \dot{\chi}(t)) \longrightarrow \min_{\underline{X}_\gamma} \quad ; \quad t \in I = [t_o, t_1] \tag{3}$$

The solution of this problem is called optimal admissible motion.

Before embarking on the solution of this problem we have to point out that the problem formulation does not require the assignment of any specific initial configuration $q_o = q(t_o)$ with the property $F(q_o) = \gamma_o$ because it can be determined as follows. Suppose a general case when the robot arm possesses an initial configuration $q_e \in \text{Int } Q_M$ for which $F(q_e) \neq \gamma_o$. Apparently the adjustment of M to an appropriate initial configuration q_o is reduced to the solution of a problem with the same formulation as stated above, where the absolutely defining trajectory originates in the computed position $F(q_e)$ and ends in the given position γ_o. This problem statement gives scope for the most expedient choice of a functional correlation between the intervals I and $\underline{\Lambda}$. Thus, the mode of motion for tracing a prescribed absolutely defining trajectory is fully determined by introducing the functional (3) in the problem formulation. These special features are of substantial importance for the application of the method for kinematic path control of robot arms with redundancy, here proposed.

III. MAIN RESULTS. The basic concept for solving the problem above formulated is that the optimal admissible motion forms a trajectory of a tangent vector field V_t, $t \in I$ of the manifold B.

To begin with, we introduce a generalized representation of the tangent vector fields of B. Denote with TB the tangent vector bundle of B and with $J(q) = \dfrac{D(F)}{D(q)}$ the Jacobian matrix of F(q) at $q \in \text{Int } Q_M$. For each element $v^* = (v^T, v_{n+1})^T \in TB$, $v = (v_1, v_2, \ldots, v_n)^T$ such a point $q^* \in B$ exists, that the vector v^* belongs to the tangent space $T_{q^*}B$ of B at q^* i. e. $J(q)v - \dfrac{\partial \gamma}{\partial q_{n+1}}(q_{n+1})\, v_{n+1} = 0$. According to (1) the system obtained can be considered as a system of m linear equations with respect to the components of the vector v

$$Jv = \frac{\partial \gamma}{\partial q_{n+1}} v_{n+1} \tag{4}$$

Hence [12], the set of solutions of this system forms a linear variety $L_{v_{n+1}}$ which may be written as $L_{v_{n+1}} = v^{(o)} + N(J)$ where $v^{(o)}$ is an arbitrary solution of (4) and the set $N(J) = \{ x \in R^n : Jx = 0 \}$ is the null space of J(q). We find it more convenient to consider the following representation of $L_{v_{n+1}}$

$$L_{v_{n+1}} = v^{(o)} + Pu \tag{5}$$

where u is an arbitrary vector of dimension n, $P = E - J^+(q) J(q)$ $J^+(q) = J^T(q)(J(q)J^T(q))^{-1}$ is the pseudoinverse of J(q) and E is the identity n x n matrix. The linear space R^n is projected onto N(J) by the linear operator $P = P(q)$ (fig. 3).

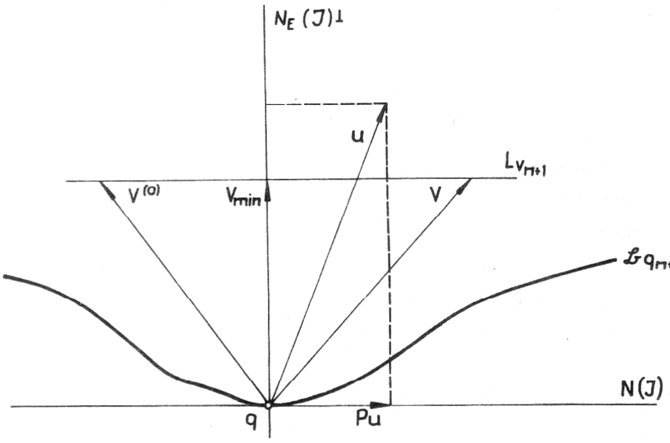


Fig. 3

The null space of J(q) is defined on the smooth manifold

$$B_{q_{n+1}} = \{ q \in \text{Int } Q_M : F(q) - \gamma(q_{n+1}) = 0 \} \tag{6}$$

that consists of configurations q, for which the end effector's position coincides with a given position $\gamma(q_{n+1})$, $q_{n+1} \in \underline{\Lambda}$ and besides, N(j) is identical with the tangent space of $B_{q_{n+1}}$ at q (fig. 2, 3).

We are going to prove that an arbitrary element $v^{(o)}$ of $L_{v_{n+1}}$ can be represented in the form

$$v^{(o)} = A^{-1} J^T (JA^{-1} J^T)^{-1} \frac{\partial \gamma}{\partial q_{n+1}} v_{n+1} \tag{7}$$

where A is an appropriate regular n x n matrix. Indeed, if $v^{(o)} \in L_{v_{n+1}}$ is the element v_{min} with the minimum Euclidean norm, then [12]

$$v_{min} = J^T (JJ^T)^{-1} \frac{\partial \gamma}{\partial q_{n+1}} v_{n+1} \tag{8}$$

and thus, it satisfies the assertion with A = E. Assume that $v^{(o)} \neq v_{min}$. Then such a proper orthogonal n x n matrix A_1, exists [14], that

$$v^{(o)} = \sqrt{(v^{(o)T} v^{(o)} / v^T_{min} v_{min})}\; A_1 v_{min} \tag{9}$$

Let $A^{-1} = \sqrt{(v^{(o)T} v^{(o)} / v_{min}^T v_{min})}\; A_1$. We verify that the matrix A represents $v^{(o)}$ in the form desired. Obviously, the matrix A is regular and relation (1) implies that $JA^{-1}J$ is regular as well. According to (8) and (4) we obtain

$$(JJ^T)^{-1} \frac{\partial \gamma}{\partial q_{n+1}} v_{n+1} = (JA^{-1} J^T)^{-1} \frac{\partial \gamma}{\partial q_{n+1}} v_{n+1} .$$

Multiply this equation leftwise by $A^{-1} J^T$. Now the desired representation (7) of $v^{(o)}$ follows from (8) and (9).

Conversely, it is easy to verify that any $v^{(o)}$ of the form (7) satisfies $v^{(o)} \in L_{v_{n+1}}$

Finally, from (5) and (7) we obtain

$$L_{v_{n+1}} = \xi v_{n+1} + Pu \tag{10}$$

where $\xi = A^{-1} J^T (JA^{-1} J^T)^{-1} \frac{\partial \gamma}{\partial q_{n+1}}$ and A is a regular matrix.

We employ the matrix A as follows. The generalized function of position maps Int Q_M into a m-dimensional smooth manifold in the linear space R^6. Suppose that in R^6 some sort of metric is defined, so that the distance between the generalized positions of the end effector can be determined. This metric defines a Riemann metric on Int Q_M i. e. the inner product of every two n-dimensional vectors u, v is defined at an arbitrary configuration q Int Q_M as $\langle u, v\rangle = u^T A(q) v$ where $A(q)$ is a n x n positive definite smooth matrix. We choose the matrix A in (7) as the matrix A (q) that defines the Riemann metric on Int Q_M. In this case, one can establish that $(Pu)^T A(q) v^{(o)} = 0$ if $v^{(o)}$ is given by (7). Therefore, the metric introduced, implies that the vector $v^{(o)}$ belongs to the orthogonal complement $N_A(J)^{\perp} = \{ v \in R^n : v^T A(q) x = 0, Jx = 0 \}$ of N(J) and $Pu \in N(J)$ for an arbitrary $u \in R^n$, i. e. in (5) each element of $L_{v_{n+1}}$ is defined as the sum of two orthogonal vectors $v^{(o)}$ and Pu. When the functional (3) is minimized, this property proves to be of great utility.

A Riemann metric is introduced in the Cartesian product Int Q_M x R by means of the matrix $A^* = \mathrm{diag}(A(q), 1)$, i. e. $\langle u^*, v^* \rangle = u^{*T} A^* v^* = u^T A v + u_{n+1} v_{n+1}$. Thus, the inner geometry of Int Q_M remains invariant during the end effector's motion.

Most of the existing methods for kinematic path control of robot arms with redundancy, for example [1], [3]–[8] presuppose that the interval $\underline{\Lambda}$ coincides with the time interval I and besides $q_{n+1} \equiv t$. Hence, these methods consider this, and only this proper subset of $\underline{X}_\gamma$ that consists of elements $(q(t), \chi(t))$ with $\dot{\chi}(t) \equiv v_{n+1} = 1$ for all $t \in I$. Apparently, such an approach makes it impossible to consider simultaneously all admissible motions of the robot arm M. Besides, it artificialy sets limits to the capabilities of the kinematic control.

We solve the problem stated in the set $\underline{X}_\gamma$, that defines all possible motions of the end effector along the absolutely defining trajectory γ. The method proposed is based on the specific differential structure of B.

Let $v^* \in T_{q*} B$ be an arbitrary vector, tangent to B. Hence, this vector satisfies (4) and $v \in L_{v_{n+1}}$. If $u_{n+1} = v_{n+1}$, then it follows from (10)

$$v^* = Ku^* \quad \text{for all } q^* \in B \tag{11}$$

where $K(q^*) = \left(\frac{P(p)}{0} \mid \alpha(q^*) \right)$, $\alpha(q^*) = (\xi(q^*)^T, 1)^T$ and $u^* = (u^T, u_{n+1})^T \in R^{n+1}$. Thus, the matrix K defines a linear mapping $K : R^{n+1} \longrightarrow T_{q*} B$ for all $q^* \in B$. Moreover, it follows from (11) that v^* can be written as $v^* = w^{(1)} + w^{(2)}$, where $w^{(1)} = [(Pu)^T, 0]^T$ and $w^{(2)} = (\xi^T, 1)^T u_{n+1}$ are two linearly independent vectors with the property $w^{(1)T} A^* w^{(2)} = 0$. In this representation the vector v^* is unique. It can easily be established that $w^{(1)}$ and $w^{(2)}$ belong to T_{q*} B. The following relations are straightforward:

$$w^{(1)\,T} A^* e_{n+1} = 0 \tag{12}$$

where e_{n+1} is the unit vector along the n+1 coordinate direction

$$w^{(2)\,T} A^* e_{n+1} = u^{\,T} A^* e_{n+1} \tag{13}$$

$$KK^{(o)} = K, \tag{14}$$

where $K^{(o)}$ is the matrix K, in the definition of which, the vector ξ from (10) for A = E is chosen. Relation (12) asserts the orthogonality of $w^{(1)}$ and e_{n+1} with respect to the Riemann metric on Int Q_M x R introduced. The geometry notion in (13) is that the projections of $w^{(2)}$ and u^* onto the direction of e_{n+1} coincide (fig. 4).

The vector $w^{(1)}$ is tangent to the submanifold

$B q_{n+1} \times q_{n+1} \subset B$. For $u_{n+1} = 0$, when $v = w^{(1)}$, this vector determines such an admissible motion of the robot arm, so that the end effector remains fixed in the position $\gamma(q_{n+1})$ and the configuration of the robot arm alters continuously. When $u_{n+1} \neq 0$ a second admissible motion with a tangent vector $w^{(2)}$ is defined. In this case, the resulting admissible motion possesses a tangent vector $v^* = w^{(1)} + w^{(2)}$ and induces an alteration in the end effector's position in compliance with the given absolutely defining trajectory.

Denote by

$$\mathbf{U} = \left\{ U = u^*(q^*) : B \longrightarrow R^{n+1} \right\} \tag{15}$$

the set of all smooth vector fields of B. Every element of $\mathbf{U}$ is mapped by the linear mapping K into a tangent vector field of B. We define the elements $U, V \in \mathbf{U}$ equivalent if, and only if, the relation

$$KU = KV \tag{16}$$

holds for all $q^* \in B$. One can prove that this relation is a relation of equivalence. Denote with U/K the quotient set defined by means of the above relation. It follows from (10) and (11) that the elements of U which define ordered pairs of vectors $(w^{(1)}, w^{(2)})$ identical for all $q^* \in B$, belong to the same class of equivalence. Further on, relation (14) implies that every class of equivalence of U/K contains a unique tangent vector field. Hence, the quotient set U/K is identical with the set of al tangent vector fields of B.

Let $\mathbf{U} \in$ U/K be the class of equivalence which contains the vector field U. It can be proved that the unique tangent vector field of $\mathbf{U}$ is

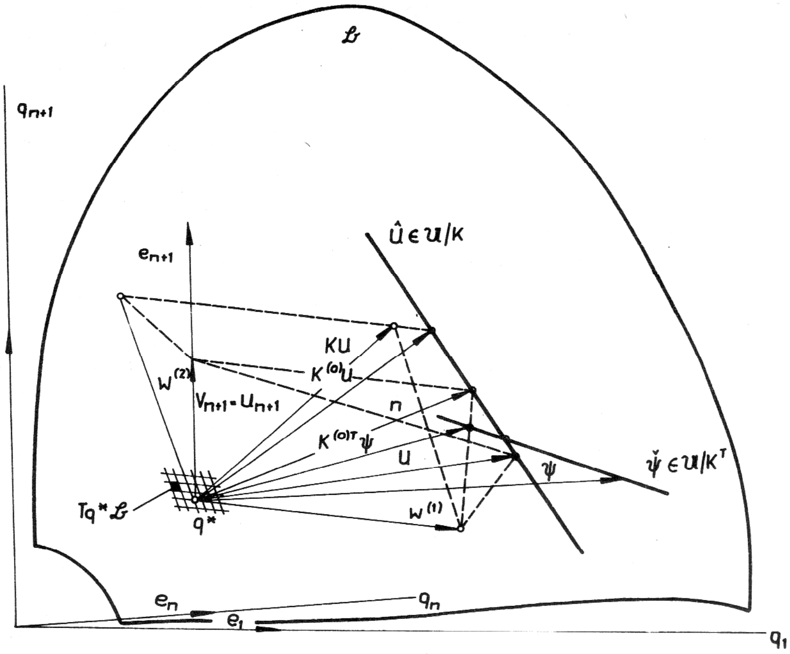
$$K^{(o)} U \tag{17}$$

Besides, the elements of $\mathbf{U}$ can be written as

$$U + \mathrm{diag}(J^T (JA^{-1} J^T)^{-1} J, 1) V \tag{18}$$

where $V = (v^T, 0)^T$ is an arbitrary element of $\mathbf{U}$.

Fig. 4



The sum of classes and the scalar multiple of a class by a real number are defined in the quotient set U/K by the corresponding linear operations with the tangent fields of the classes. For example, the sum of the classes U_1 and U_2 is defined as the sum of $K^{(o)} U_1$ and $K^{(o)} U_2$ in the tangent space $T_{q*}B$ for all $q^* \in B$. Therefore, the linear operations above introduced, emerge as the well familiar linear operations with vectors from the tangent space T_{q*} B for all $q^* \in B$. Finally, one can establish that these linear operations introduce a structure of linear vector space in U/K.

By means of the transpose matrix of K, a quotient set U/K^T is defined. The elements $\underline{\Psi}$, $\Theta \in U$ are regarded equivalent, i. e. they belong to the same class of equivalence of U/K^T, if and only if $K^T \underline{\Psi} = K^T \Theta$ holds for all $q^* \in B$. Thus, the vector fields $\underline{\Psi} = \psi^* (q^*) \in U$; $\psi^* = (\psi^T, \psi_{n+1})^T$ that define identical ordered pairs of vectors

$$[(P\psi)^T, 0]^T \quad , \quad [0^T, \alpha^T \psi^*]^T \tag{19}$$

for all $q^* \in B$ form a class of equivalence of U/K^T. Denote with $\check{\underline{\Psi}}$ the class of U/K^T defined by $\underline{\Psi}$.

Since $K^T K^{(o)} \underline{\Psi} = K^T \underline{\Psi}$ therefore

$$K^{(o)\,T} \underline{\Psi} \in \check{\underline{\Psi}} \tag{20}$$

and for all $q^* \in B$ the representative of this vector field lies in the subspace generated by the vectors (19). (fig. 4) Thus, the quotient set U/K^T is identical with the set of all vector fields (20), where $\underline{\Psi}$ is an arbitrary element of U.

The sum of classes and the scalar multiple of a class by a real number is defined in the quotient set U/K^T as the similar linear operations applied to the representatives (20) of these classes. The linear operations, above introduced, define a structure of linear vector space in the quotient set U/K^T.

Since

$$K\mathrm{diag}(P, 1) = K \tag{21}$$

it is apparent that the vector field $\Pi = \mathrm{diag}(P, 1)\, U$ belongs to $\hat{U} \in U/K$ (fig. 4). Moreover, for all $q^* \in B$ its representative lies in the subspace generated by the vectors $[(Pu)^T, 0]^T$ and $[0^T, u_{n+1}]^T$, where $U = (u^T, u_{n+1})^T$.

Definition 2. The class of equivalence $\check{\underline{\Psi}} \in U/K^T$ is called conjugate to the class $\hat{U} \in U/K$, if

$$\mathrm{diag}(P, 1)\, u = K^T \check{\underline{\Psi}} \tag{22}$$

It can be proved that relation (22) defines isomorphism between the quotient spaces U/K and U/K^T. If a class $\underline{\Psi}$ is given, then its conjugate $\hat{U} \in U/K$ can be found. Obviously,

$$\mathrm{diag}(P, 1)\, K^T \underline{\Psi} = K^T \underline{\Psi} \tag{23}$$

and from (22) and (16) it follows that $\Gamma = (\psi^T, \alpha^T \underline{\Psi})^T$ is a representative of the class $\hat{U}$, conjugate to $\check{\underline{\Psi}}$. Thus, the conjugate to $\check{\underline{\Psi}}$ is fully defined. Conversely, if $\hat{U}$ is given, then

$$K^T (u^T, u_{n+1} - \xi^T u) = \mathrm{diag}(P, 1)\, U \tag{24}$$

Hence, the conjugate class $\check{\underline{\Psi}}$ to $\hat{U}$ possesses a representative $\check{\underline{\Psi}} = (u^T, u_{n+1} - \xi^T u)^T$. Thus, the vector field Π defines in (22) a bijection between U/K^T and U/K.

Each class $\hat{U} \in U/K$, such that KU is nonzero for all $q^* \in B$ can be interpreted as a tangent vector field of B. Suppose that for every $t \in I$ the element $U_t \in U$ is given as a continuous piecewise smooth function of t. Then, a family of trajectories $q_j^* : B \rightarrow B$ of the field KU_t is defined, i. e. the elements of q_t^* satisfy the following system of ordinary differential equations

$$\dot{q}^* = KU_t \; ; \; t \in I \tag{25}$$

Indeed, let q^* (t) be an arbitrary trajectory of $KU_t \in U$ such that $q^* (t_o) \in B$. Then [15], relation (1) implies that q^* (t) $\in B$ for all $t \in I$. For example, the resolved motion rate control method [3], [4] for robot arms with redundancy is obtained from (25), where $U_t = (0^T, 1)^T$ for all $q^* \in B$ and $t \in I$. In this case, (25) takes the form

$$\dot{q} = A^{-1} J^T (JA^{-1} J^T)^{-1} \frac{\partial \gamma}{\partial q_{n+1}}$$

$$\dot{q}_{n+1} = 1$$

Obviously, this system is equivalent to

$$\dot{q} = A^{-1} J^T (J A^{-1} J^T)^{-1} \frac{\partial \gamma}{\partial t}$$

$t \in I$, the latter being the desired final result, if the matrix A is interpreted as a „positive definite weighting matrix".

Conversely, any admissible motion $(q(t), \chi(t)) = q^*(t)$ may be interpreted as a trajectory of (25), where the vector field U_t is appropriately chosen. Suppose N is an open subset of $F(0)^{-1}$ and $N \supset B$. Hence [13], a smooth function $g: F(0)^{-1} \to [0,1]$ with the following properties exists. It satisfies $g(q^*) \equiv 1$ for all $q^* \in B$ and it is $g(q^*) \equiv 0$ for all q^* out of N. Let $\xi^{(o)}$ be the vector ξ from (10), defined for A = E. Then it follows from (14) and (11) that $q^*(t)$ is a trajectory of (25) where U_t is defined as the restriction to B of

$$\left(\begin{array}{c|c} P & \xi - \xi^{(o)} \\ \hline 0 & 1 \end{array}\right) \; g\dot{q}^*(t) \qquad (26)$$

for all $t \in I$.

In particular, let the vector of the joint rates $q(t)$ be determined by a method that presupposes maximum rank of the mapping $F: Q_M \to R^m$ [1], [3] – [9]. By implication, this method employs $\dot{\chi}(t) \equiv 1$ for all $t \in I$. Thus, in view of the above considerations, we conclude that $\dot{q}(t)$ is defined by the first n-components of KU_t, where U_t is the restriction of (26) for $\dot{q}^*((t) = (\dot{q}(t)^T, 1)^T$.

Finally, we come to the conclusion, that the solution of the problem here considered is a trajectory of a tangent vector field $V_t = KU_t$, $t \in I$ of the smooth manifold B. The method employed needs no explicit definition of B and thus it has assumed a form suitable for numerical interpretation. The control of the admissible motion is reduced to an appropriate assignment of the element $U_t \in U$, $t \in I$ in the system of ordinary differential equations (25).

Definition 3. We call a class of motion control each continuous piecewise smooth function $\hat{U}_t : I \to U/K$, such that KU_t is nonzero for all $q^* \in B$, $t \in I$.

In view of this definition, the set $\underline{X}_\gamma$ can be considered as the union of the families q_t^* of solution curves of the system (25), where $\hat{U}_t$ describes the set of all classes of motion control.

Let $\hat{V}_t^{(o)}$, $KU_t^{(o)} \in \hat{V}_t^{(o)}$ be a class of motion control, which possesses a trajectory $q^{(o)*}(t) = (q^{(o)}(t), \chi^{(o)}(t))$, $t \in I$ that is an optimal admissible motion. Further on $\hat{U}_t^{(o)}$ is called a class of optimal motion control. If the functional (3) is given in the form $L(\dot{q}, \dot{\chi}) = \int_{t_o}^{t_1} f(\dot{q}, \dot{\chi})\, dt$, then the class $\hat{U}_t^{(o)}$ of optimal motion control may be determined by means of the maximum principle of Pontrjagin [16]. Replace U_t in the system (25) and its conjugate by the representative $U_t^{(o)}$ of a class of optimal motion control, determined by the maximum principle. Then, the optimal admissible motion can be found by integrating the system of ordinary differential equations, thus obtained under boundary conditions deriving from the specific formulation of the problem stated. From a computational point of view the easiest case is when in this formulation the initial and terminal configurations of the robot arm are assigned, i. e. the admissible motions under consideration are with fixed end points. However, in the most general case of the problem formulation only the initial and terminal positions of the end effector are known. Another posibility is that the initial configuration of the robot arm and the terminal position of the end effector be given in advance [8]. In order to determine the boundary conditions of the system obtained, the transversality condition [16] is to be employed in both cases. This condition defines the optimal initial and respectively the terminal configuration of the robot arm.

IV. EXAMPLE. In this section, the technique of the method for kinematic path control above described, is demonstrated in the case when the quality criterion (3) is given in the form

$$L(\dot{q}, \dot{\chi}) = \int_{t_o}^{t_1} \dot{q}^T S \dot{q}\, dt \longrightarrow \min_{\underline{X}\gamma} \qquad (27)$$

where $S = S(q)$ is a n x n positive definite matrix, continuous with respect to q. For instance, when the end effector is considered as a moving rigid body the quantity $(1/2)\, \dot{q}^T S \dot{q}$ may express its kinetic energy in the inertial space [4]. This quantity may represent also the kinetic energy of the robotic system, when it is viewed upon as a linkage of rigid bodies.

Let the matrix S define the Riemann metric on Int Q_M and the matrix $S^* = \text{diag}(S, 1)$ define the Riemann metric on Int $Q_M \times R$. The substitution of the first n equations of (25) into (27) yields

$$L(U_t) = \int_{t_o}^{t_1} U_t^T D\, U_t \longrightarrow \min_{U_t} \qquad (28)$$

Owing to the Riemann metric introduced, the matrix D is obtained in the form $D = \text{diag}(PSP, k)$, $k = \xi^T \xi > 0$. From the formulation of the problem stated, it follows that the end points $q^{(o)*}(t_o)$, $q^{(o)*}(t_1)$ of the optimal admissible motion belong to certain smooth submanifolds in R^{n+1} (fig. 2), i. e.

$$q^{(o)*}(t_o) \in B_{\lambda_o} \times \lambda_o \; ; \; q^{(o)*}(t_1) \in B_{\lambda_1} \times \lambda_1 \qquad (29)$$

while $q^{(o)*}(t)$ satisfies the system of equations (25) for $t \in [t_o, t_1]$. Further on, the optimal admissible motion is found by applying the maximum principle of Pontrjagin to the system (25) and to the conditions (28, 29). The Hamiltonian H is defined as $H = -U_t^T D\, U_t + \underline{\Psi}^T K\, U_t$, where the vector $\underline{\Psi} = (\psi^T, \psi_{n+1})^T$, $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T$ of auxiliary functions satisfies the following equations

$$\dot{\psi}_i = U_t^T \frac{\partial D}{\partial q_i} U_t - \underline{\Psi}^T \frac{\partial K}{\partial q_i} U_t \qquad (30)$$

$i = 1, 2, \ldots, n+1$

The class of optimal motion·control $\hat{U}_t^{(o)}$ is a continuous, piecewise smooth function of t into the quotient space U/K. Hence, it is sufficient to determine from the maximum condition

$$\frac{\partial H}{\partial U_t} = -2D\,U_t + K^T \underline{\Psi} = 0,\ t\in[t_o, t_1] \qquad (31)$$

one representative $U_t^{(o)}$ of the class of optimal motion control $U_t^{(o)}$.

In the general case it would be a difficult computational task to obtain $U_t^{(o)}$ from (31) because the matrix D is irregular. Suppose that Ψ is defined as $\underline{\Psi} : [t_o, t_1] \rightarrow U$ and denote with $\check{\underline{\Psi}}$ the corresponding class in U/K^T. Then the maximum condition (31) can be rendered to the conjugation condition (22) of U/K and U/K^T. Apparently, the equation (31) can be written in the form

$$\mathrm{diag}\,(P, 1)\, S^*\, \mathrm{diag}\,(P, 1)\, U_t = (1/2) \begin{pmatrix} P\,|\,0 \\ \hline \alpha^T k^{-1} \end{pmatrix} \underline{\Psi} \qquad (32)$$

Besides,

$$1/2\, K^T (\psi^T, \alpha^T \underline{\Psi}\, k^{-1} - \xi^T \psi)^T = (1/2) \begin{pmatrix} P\,|\,0 \\ \hline \alpha^T k^{-1} \end{pmatrix} \underline{\Psi} \qquad (33)$$

From (22), (32) and (33) it follows that for an arbitrary representative $U_t^{(o)}$ of the class of optimal motion control the vector field $S^*\,\mathrm{diag}\,(P, 1)\, U_t^{(o)}$ defines a class of U/K which is conjugate to the class of U/K^T with the representative $(1/2)\, \check{(}\psi^T, \alpha^T \underline{\Psi}\, k^{-1} - \xi^T \psi)^T$. Then, as it was shown in (23) the class $\hat{\Gamma} \in U/K$ of the vector field S^* diag (P, 1) U_t is determined by the known vector field $\Gamma = (1/2)\,(\psi^T, \alpha^T \underline{\Psi}\, k^{-1})^T$. Moreover, there exists a vector field (18) $V = (v^T, 0)^T$ so that

$$S^*\,\mathrm{diag}\,(P, 1)\, U_t^{(o)} = \Gamma + \mathrm{diag}\,(J^T (JS^{-1} J^T)^{-1} J, 1)\, V \qquad (34)$$

It is easily established that all elements of an optimal class $\hat{U}_t^{(o)}$ of motion control satisfy this equation and besides, to different classes of optimal motion control in (34) there correspond different vector fields V of the form $(v^T, 0)^T$. The equation (34) multiplied leftwise by the matrix S^{*-1} yields

$$\mathrm{diag}\,(P,1)\, U_t^{(o)} = S^{*-1} (\Gamma + \mathrm{diag}\,(J^T (JS^{-1} J^T)^{-1} J, 1)\, V) \qquad (35)$$

This equation holds, if and only if

$$S^{-1} \psi + S^{-1} J^T (JS^{-1} J^T)^{-1} Jv \in N(J) \qquad (36)$$

holds over $q^{(o)*}(t)$. We employ this condition to determine the vector field V in (34).

Since $S^{-1} J^T (JS^{-1} J^T)^{-1} Jv \in N_S(J)^{\perp}$ for an arbitrary v, then $(1/2)\, S^{-1} \psi$ can be represented as a direct sum of its projections onto N(J) and $N_S(J)^{\perp}$ i. e. $(1/2)\, S^{-1} \psi = (1/2)\, P_S S^{-1} \psi + (1/2)\, S^{-1} J^T (JS^{-1} J^T)^{-1} JS^{-1} \psi$ where $P_S = E - S^{-1} J^T (JS^{-1} J^T)^{-1} J$. Hence, in order to satisfy (36) the components v of the vector field V are to be defined as $v = -1/2\, S^{-1} \psi$. Consequently, from (21), (17) and (35) the otpimal class of motion control $\hat{U}_t^{(o)}$ is defined by its representative

$$U_t^{(o)} = \mathrm{diag}\,(P_S, 1)\, S^{*-1}\, \Gamma \qquad (37)$$

It can be proved that the class of optimal motion control is invariant with respect to the choice of the representative Γ of the class $\hat{\Gamma}$. Hence, the class of optimal motion control is unique.

Thus, the optimal admissible motion can be found by means of integrating systems (25) and (30), where U_t is substituted by the expression (37) of $U_t^{(o)}$, given as a function of q^* and $\underline{\Psi}$. Substitute $\Theta = (1/2)\,\underline{\Psi}$; $\Theta = (\vartheta^T, \vartheta_{n+1})^T$ into the system obtained. Then the optimal admissible motion satisfies the system of ordinary differential equations

$$\dot{q}^* = K\bar{\Theta} \qquad t\in[t_o, t_1]$$

$$\dot{\vartheta}_i = \bar{\Theta}^T \frac{\partial D}{\partial q_i} \bar{\Theta} - \Theta^T \frac{\partial K}{\partial q_i} \bar{\Theta} \qquad (38)$$

$i = 1, 2, \ldots, n+1$

where $\bar{\Theta} = \mathrm{diag}\,(P_S, 1)\, S^{*-1} (\vartheta^T, \alpha^T \Theta\, k^{-1})^T$

The conditions for transversality [16] corresponding to (29) yield $P\,\vartheta\,(t_o) = 0$; $P\,\vartheta\,(t_1) = 0$. These equations define 2n boundary conditions for the system (38). The remaining two boundary conditions are $q_{n+1}(t_o) = \lambda_o$ and $q_{n+1}(t_1) = \lambda_1$. They follow directly from definition (2) of the function $\chi(t)$ and illustrate the point that the end effector's initial position is γ_o, while γ_1 is its terminal position.

V. CONCLUSIONS. The mathematical apparatus of the method described, gives ground for a common approach to the kinematic path control of robot arms with redundancy, based on the assumption that the Jacobian matrix of the generalized function of the end effector's position possesses maximum rank. It has also been established that similar methods for kinematic path control follow from the method, here proposed. The final results are in a mode suitable for application in computer programmes. Therefore, this technique for kinematic path control can be employed in the field of robotics for preprogramming the movement of the end effector and for investigating the movements of real antropomorphic structures in biomechanics, as well.

REVERENCES

[1] Konstantinov, M. S., Markov, M. D., Nenchev, D. N.: Kinematic Control of Redundant Manipulators. Proc. of the 11-th ISIR. Japan, Tokyo (1981) pp. 561 – 568.

[2] Мальшев, В.: Необходимые условия существования программых движений манипулятора в задаче отслеживания. Техническая кибернетика. Изв. АН СССР (1982) №. 4, стр. 230 – 239.

[3] Whitney, D.: Resolved Motion Rate Control of Manipulators and Human Prostheses. IEEE Trans. on MMS (June 1969), No. 2 pp. 47 – 53.

[4] Whitney, D.: The Mathematics of Coordinated Control of Prosthetic Arms and Manipulators. Trans. of the ASME, ser. G (Dec. 1972) pp. 303 – 309.

[5] Hanafusa, H., Yoshikawa, T., Nakamura, Y.: Analysis and Control of Articulated Robot Arms with Proc. of the 8-th Triennial World Congress of IFAC. Japan, Kyoto (1981) pp. 1927 – 1932.

[6] Konstantinov, M. M., Toshev, Y. E.: On the Problem of Control Regional Structures. Proc. of the 4-th National Congress on Theoretical and Applied Mechanics. Bulgaria, Varna (1981) pp. 411 – 416.

[7] Попов, Е., Верещагин, В., Зенкевич, С.: Манипуляционные роботы. Динамика и алгоритмы. Наука, Москва (1978) стр. 230 – 239

[8] Konstantinov, M. S., Patarinski, S. P., Nenchev, D. N., Zamanov, V. B.: A Contribution to the Inverse Kinematic Problem for Industrial Robots. Proc. of the 12-th ISIR. France, Paris (1982) pp. 456 – 469.

[9] Denk, W., Maißer, P., Rothe, H.: Bahnsteuerung von Manipulatoren. Z. Technische Mechanik 2 (1981) H. 2, S. 1 – 16.

[10] Хирш, М.: Дифференциальная топология. Мир, Москва (1979).

[11] Лилов, Л.: Структура, кинематика и динамика систем твердых тел. Успехи механики (1983) №. 1, 2. (Готовится к печати).

[12] Luenberger, D.: Optimization by Vector Space Methods. John Wiley & Sons Inc.. New York (1969).

[13] Дубровин, В., Новиков, С., Фоменко, А.: Современная геометрия. Наука, Москва (1979).

[14] Парлетт, Б.: Симетричная проблема собственных значений. Численные методы. Мир, Москва (1983).

[15] Торп, Дж.: Начальные главы дифференциальной геометрии. Мир, Москва (1982).

[16] Понтрягин, Л. С., Болтянский, В. Г., Гамкрелидзе, Г. В., Мищенко, Е. Ф.: Математическая теория оптимальных процессов. Наука, Москва (1976).

Anschrift der Verfasser:

E. Krustev
L. Lilov
Institute of Mechanics
and Biomechanics

Sofia, Bulgaria