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On material-convective elasto-plasticity

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Abstract: A material-convective continuum formulation is presented which differs significantly from the finite elasto-plasticity descriptions of general-purpose finite element simulation tools like *Dyna3D*, *Abaqus*, *Marc*, etc. The material-convective continuum formulation offers physical significance in particular with respect to the geometrical interpretation of the (plastic) deformation tensors—in contrast to the so-called Updated Lagrangian Formulation of general-purpose finite element simulation tools which is unphysical due to its inaccurate (directional non-convective) integration of the (plastic) deformation increments: this inaccurately integrated (plastic) deformation does not obey the geometrical interpretation of proper (plastic) deformation tensors and may even lead to a violation of the first fundamental law of thermodynamics, the conservation of energy. The material-convective time integrals are the reverse of the material-convective time derivatives, and the only material-convective time derivative of a symmetric second-order Eulerian tensor is its Green-Naghdi rate which is rotationally and translationally convected with the material.

Keywords: material-convective continuum formulation, Green-Naghdi rate, material-convective time integration, non-material Zaremba-Jaumann rate, geometrical interpretation of deformation tensors

1 Introduction

The modeling of finite elasto-plasticity must be based on proper definitions of the deformation tensors—for the total deformation tensors $\mathsf{b} = \mathsf{R}.\mathsf{C}.\mathsf{R}^{\mathrm{T}}$ as well as for the partial (elastic ${}^{e}\mathsf{b} = \mathsf{R}.{}^{e}\mathsf{C}.\mathsf{R}^{\mathrm{T}}$, plastic ${}^{p}\mathsf{b} = \mathsf{R}.{}^{p}\mathsf{C}.\mathsf{R}^{\mathrm{T}}, \cdots$) ones. The six internal degrees of freedom of properly defined symmetric (total or partial) deformation tensors may be interpreted as three principal values (eigenvalues) and three orthogonal principal axes (eigenvectors), whose eigenvalues are functions of the present and reference edge lengths of the corresponding present $d\hat{v}$ and reference $d\hat{V}$ principal infinitesimal volume elements only [and do not depend on the geometrical path through the whole time history of all deformation configurations]. For rate-type theories of plasticity, the Eulerian plastic deformation tensor ${}^{p}\mathsf{b}$ must be integrated from the Eulerian plastic flow rule ${}^{p}\mathring{\mathsf{b}} = \cdots$ translational- and rotational-convective with the material. For a Lagrangean material description, the *material velocity* vector

$$\mathbf{v} = \dot{\mathbf{x}}(\mathbf{X}, t) = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t} \tag{1}$$

is given by the time derivative of the motion $\mathbf{x}(\mathbf{X}, t)$ of a material point/particle, but what is its material-convective rotation velocity or spin? The questions of the *material-convective rotation*

$$\mathsf{R} = \mathsf{R}^{-\mathrm{T}} = \sqrt{\mathsf{F}^{-\mathrm{T}}.\mathsf{F}^{-1}}.\mathsf{F} = \sqrt{\mathsf{F}.\mathsf{F}^{\mathrm{T}}}.\mathsf{F}^{-\mathrm{T}} = \mathsf{F}.\sqrt{\mathsf{F}^{-1}.\mathsf{F}^{-\mathrm{T}}} = \mathsf{F}^{-\mathrm{T}}.\sqrt{\mathsf{F}^{\mathrm{T}}.\mathsf{F}} \tag{2}$$

and the *material-convective spin*

$$\Omega = \dot{\mathsf{R}}.\mathsf{R}^{\mathrm{T}} = -\mathsf{R}.\dot{\mathsf{R}}^{\mathrm{T}} = -\Omega^{\mathrm{T}} \tag{3}$$

tensors are discussed with respect to the polar decomposition $\mathsf{F} = \mathsf{v}.\mathsf{R} = \mathsf{R}.\mathsf{U}$ of the deformation gradient F into the proper orthogonal (orthonormal $\mathsf{R}^{-1} = \mathsf{R}^{\mathrm{T}}$ and right-handed $|\mathsf{R}| = 1$) material-convective rotation tensor R (2) and the positive definite, symmetric Eulerian *left* or Lagrangean *right stretch tensors*

$$\mathsf{v} = \mathsf{v}^{\mathrm{T}} = \sqrt{\mathsf{F}.\mathsf{F}^{\mathrm{T}}} = \mathsf{R}.\mathsf{U}.\mathsf{R}^{\mathrm{T}} \qquad \text{or} \qquad \mathsf{U} = \mathsf{U}^{\mathrm{T}} = \sqrt{\mathsf{F}^{\mathrm{T}}.\mathsf{F}} = \mathsf{R}^{\mathrm{T}}.\mathsf{v}.\mathsf{R}, \tag{4}$$

where '.' denotes the *dot product operator* (or *single contraction*) and where F^{-1}, F^{T} or $\mathsf{F}^{-\mathrm{T}}$ are, respectively, the *inverse*, the *transpose* or the *inverse transpose* of a second-order tensor F. The spectral representation of the deformation gradient

$$\mathsf{F} = \partial\mathbf{x}/\partial\mathbf{X} = \partial\hat{x}_i/\partial\hat{X}_j\ \hat{\mathbf{E}}_i \otimes \hat{\mathbf{E}}_j = \hat{U}_k\ \underbrace{\hat{\mathbf{e}}_k \otimes \hat{\mathbf{E}}_k}_{\mathsf{R}} = \underbrace{\hat{U}_k \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k}_{\mathsf{v}}.\mathsf{R} = \mathsf{R}.\underbrace{\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k \hat{U}_k}_{\mathsf{U}} = \frac{d\hat{\mathbf{x}}_k \otimes d\hat{\mathbf{X}}_k}{\|d\hat{\mathbf{X}}_k\|^2} \tag{5}$$

unveils the polar decomposition $\mathsf{F} = \mathsf{v}.\mathsf{R} = \mathsf{R}.\mathsf{U}$ as well as the definition of the principal quantities (marked with a *hat*): the Eulerian $d\hat{\mathbf{x}}_k$ and Lagrangean $d\hat{\mathbf{X}}_k$ *eigenvectors* with respect to the present κ and reference κ_0 configurations, the *stretch eigenvalues*

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$\hat{U}_k = \|d\hat{\mathbf{x}}_{(k)}\|/\|d\hat{\mathbf{X}}_{(k)}\|$, the Eulerian *unit eigenvectors* $\hat{\mathbf{e}}_k = d\hat{\mathbf{x}}_{(k)}/\|d\hat{\mathbf{x}}_{(k)}\| = \mathsf{R}.\hat{\mathbf{E}}_k$ with respect to the present configuration κ and the Lagrangean *unit eigenvectors* $\hat{\mathbf{E}}_k = d\hat{\mathbf{X}}_{(k)}/\|d\hat{\mathbf{X}}_{(k)}\| = \mathsf{R}^{\mathrm{T}}.\hat{\mathbf{e}}_k$ with respect to the reference configuration κ_0. The '$\otimes$' operators denote dyadic products, the length of a vector $\mathbf{x}$ is given by the (2-)norm $\|\mathbf{x}\| = \sqrt{x_k\, x_k} = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}$ and, throughout this work, the *summation convention is applied to repeated indices* (if they are not enclosed in brackets).
This work has the following structure: after summarizing the kinematical relations of the finite total deformation in Sections 2 and 3 some finite (partial) deformation measures (like ${}^{@}\mathsf{C} = \{\mathsf{C},\, {}^{e}\mathsf{C},\, {}^{p}\mathsf{C}, \cdots\}$ the total, elastic, plastic, $\cdots$ Cauchy-Green deformation tensors) including their geometrical interpretation are introduced in Sections 4 and 5. These finite deformation measures are defined by their spectral representation, and they may be pushed-forward ${}^{@}\mathsf{b} = \mathsf{R}.{}^{@}\mathsf{C}.\mathsf{R}^{\mathrm{T}}$ or pulled-back ${}^{@}\mathsf{C} = \mathsf{R}^{\mathrm{T}}.{}^{@}\mathsf{b}.\mathsf{R}$ to their Eulerian ${}^{@}\mathsf{b}$ or Lagrangean ${}^{@}\mathsf{C}$ flavors by the polar rotation (2) of the deformation gradient (5). The corresponding partial deformation-rate tensors ${}^{@}\mathsf{d} = \{{}^{e}\mathsf{d},\, {}^{p}\mathsf{d}\} = \frac{1}{2}\mathsf{R}.{}^{@}\mathsf{U}^{-1}.{}^{@}\dot{\mathsf{C}}.{}^{@}\mathsf{U}^{-1}.\mathsf{R}^{\mathrm{T}}$ do not coincide ${}^{@}\mathsf{d} \neq \overline{{}^{@}\mathsf{d}} = \{\overline{{}^{e}\mathsf{d}},\, \overline{{}^{p}\mathsf{d}}\}$ with the additive contributions $\overline{{}^{e}\mathsf{d}} + \overline{{}^{p}\mathsf{d}} = \mathsf{d}$ of the total deformation rate (from the stress power equation) in Section 6. Finally, non-material «co-rotational» rates in conjunction with the hypo-elasticity of Truesdell and the Updated Lagrangian Formulation (often applied for finite plasticity analysis within general-purpose finite element simulation tools) are critically discussed in Section 7.

2 The material-convective Lagrangean description $\mathbf{x}(\mathbf{X}, t)$

From a Lagrangean point of view, the deformation gradient (5) maps the vicinity vector $d\mathbf{X}$ of a position vector $\mathbf{X}$ in the reference configuration κ_0 to the vicinity vector

$$d\mathbf{x} = \mathsf{F}.d\mathbf{X} = d\mathbf{X}.\mathsf{F}^{\mathrm{T}} \tag{6}$$

of a position vector $\mathbf{x}$ in the present configuration κ, and the vicinity vectors $d\mathbf{x}$ and $d\mathbf{X}$ describe the kinematical behavior of infinitesimal material line elements in the present κ and reference κ_0 configurations. The infinitesimal mass element dm [the *unit mass* $dm = \rho_0\, dV = \rho\, dv$ with the mass densities ρ_0, ρ and the infinitesimal volumes dV, dv in the reference κ_0, present κ configurations] depicted in Figure 1 is therefore transformed material-convectively from a cube $(\rho_0\, dV)$ in the reference configuration κ_0 to a skewed parallelepiped $(\rho\, dv)$ in the present configuration κ.
The three arbitrary orthogonal Lagrangean material line elements $d\mathbf{X}_k$ of Figure 1 with the corresponding three arbitrary orthogonal Lagrangean unit vectors $\mathbf{E}_k = d\mathbf{X}_{(k)}/\|d\mathbf{X}_{(k)}\|$ are transformed (6) to the Eulerian material line elements $d\mathbf{x}_k$ which are in general not orthogonal, and the three orthogonal Lagrangean material unit vectors $\mathbf{E}_k$ stay neither orthogonal nor unit vectors when mapped (6) with the material to $\mathsf{F}.\mathbf{E}_k$. Therefore, an arbitrary material-convective Eulerian basis is *defined* by the orthonormal Eulerian base vectors

$$\mathbf{e}_k := \mathsf{R}.\mathbf{E}_k = \mathbf{E}_k.\mathsf{R}^{\mathrm{T}} \tag{7}$$

co-rotated with the material-convective rotation R tensor (2) relative to the arbitrary orthonormal Lagrangean base vectors $\mathbf{E}_k$ so that the component bases refered to in this work can solely be defined as orthogonal unit vector bases

$$\mathbf{E}_i.\mathbf{E}_j = \mathbf{e}_i.\mathbf{e}_j = \hat{\mathbf{E}}_i.\hat{\mathbf{E}}_j = \hat{\mathbf{e}}_i.\hat{\mathbf{e}}_j = \delta_{ij} = \begin{cases} 1 & \{i = j\} \\ 0 & \{i \neq j\} \end{cases} \tag{8}$$

where δ_{ij} denotes Kronecker's delta; the components S_{ij}, $\hat{S}_k$ of corresponding Lagrangean $\mathsf{S} = S_{ij}\mathbf{E}_i \otimes \mathbf{E}_j = \hat{S}_k \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k$ and Eulerian $\mathsf{s} = \mathsf{R}.\mathsf{S}.\mathsf{R}^{\mathrm{T}} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \hat{S}_k \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k$ symmetric tensors are then identical—relative to their material-convective bases $\mathbf{e}_i \otimes \mathbf{e}_j = \{\mathsf{R}.\mathbf{E}_i\} \otimes \{\mathbf{E}_j.\mathsf{R}^{\mathrm{T}}\}$ and $\hat{\mathbf{e}}_{(k)} \otimes \hat{\mathbf{e}}_{(k)} = \{\mathsf{R}.\hat{\mathbf{e}}_{(k)}\} \otimes \{\hat{\mathbf{e}}_{(k)}.\mathsf{R}^{\mathrm{T}}\}$. The arbitrary orthogonal material-convective base unit vectors $\mathbf{E}_k$, $\mathbf{e}_k$ (7); the unit vectors

$$\mathbf{E} = \frac{d\mathbf{X}}{\|d\mathbf{X}\|}\,, \qquad \mathbf{e} := \mathsf{R}.\mathbf{E} = \mathbf{E}.\mathsf{R}^{\mathrm{T}}\,, \qquad \mathbf{i} = \frac{d\mathbf{x}}{\|d\mathbf{x}\|}\,, \qquad \mathbf{I} := \mathsf{R}^{\mathrm{T}}.\mathbf{i} = \mathbf{i}.\mathsf{R} \tag{9}$$

pointing along the edges $d\mathbf{X}$, $d\mathbf{x}$ of the infinitesimal mass elements $dm = \rho_0\, dV$ in the reference configuration κ_0 and $dm = \rho\, dv$ in the present configuration κ;

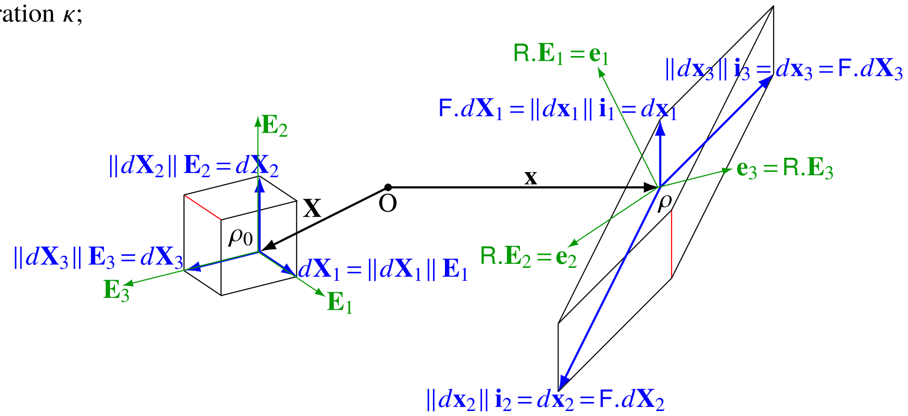

Figure 1. Lagrangean mapping of an infinitesimal mass element $dm = \rho_0\, dV = \rho\, dv$ from a cube $(\rho_0\, dV)$ in the reference configuration κ_0 to a skewed parallelepiped $(\rho\, dv)$ in the present configuration κ within an arbitrary material-convective base vector system $\mathbf{e}_k := \mathsf{R}.\mathbf{E}_k$ (without a *hat*)

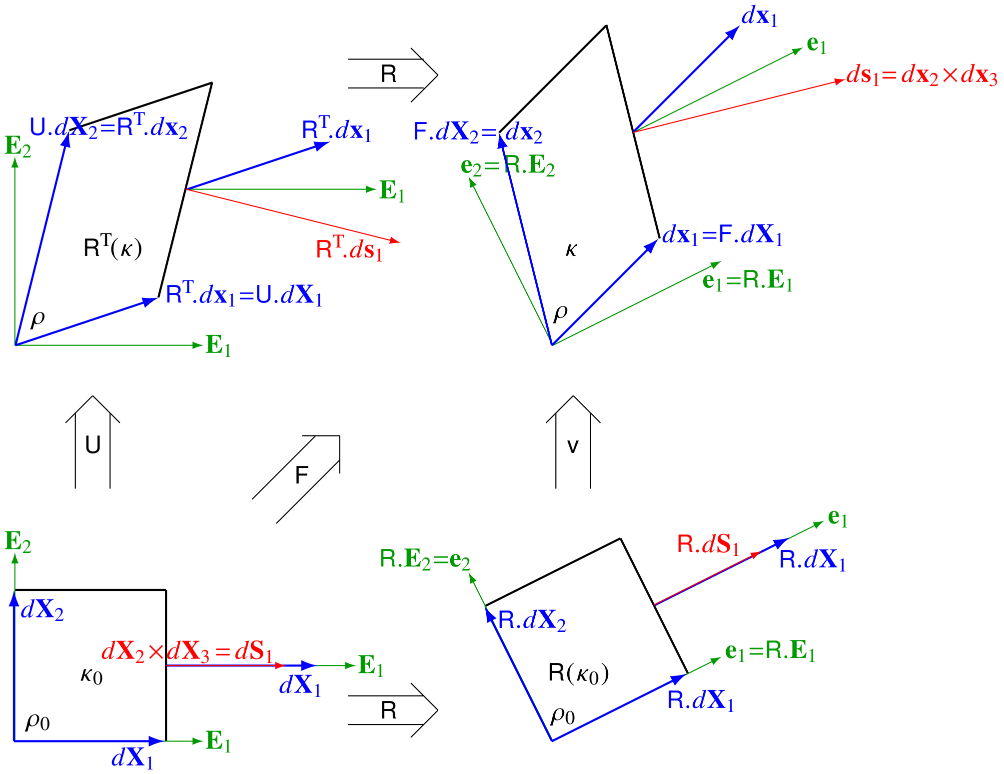

Figure 2. Lagrangean view on the polar decomposition of the deformation gradient $\mathsf{F}=\mathsf{v}.\mathsf{R}=\mathsf{R}.\mathsf{U}$ within an arbitrary material-convective base vector system $\mathbf{e}_k{:=}\,\mathsf{R}.\mathbf{E}_k$ (without a *hat*) exemplified for a plane finite deformation of an infinitesimal mass element $dm=\rho_0\,dV=\rho\,dv$ mapped (6) from a cube ($\rho_0\,dV$) in the reference configuration κ_0 to a skewed parallelepiped ($\rho\,dv$) in the present configuration κ

and the unit vectors

$$\mathbf{N}=\frac{d\mathbf{S}}{\|d\mathbf{S}\|}\,,\qquad \mathbf{n}:=\mathsf{R}.\mathbf{N}=\mathbf{N}.\mathsf{R}^{\mathrm{T}}\,,\qquad \mathbf{m}=\frac{d\mathbf{s}}{\|d\mathbf{s}\|}\,,\qquad \mathbf{M}:=\mathsf{R}^{\mathrm{T}}.\mathbf{m}=\mathbf{m}.\mathsf{R} \tag{10}$$

of the surface normals $d\mathbf{S}$, $d\mathbf{s}=\frac{\rho_0}{\rho}\mathsf{F}^{-\mathrm{T}}.d\mathbf{S}$ [Nanson's formula] differ for the mapping (6) in their Eulerian flavor from each other, $\mathbf{e}\neq\mathbf{i}\neq\mathbf{m}$, as shown in Figure 1 and exemplified for a plane finite deformation in Figure 2 [which also illustrates the polar decomposition of the deformation gradient $\mathsf{F}=\mathsf{v}.\mathsf{R}=\mathsf{R}.\mathsf{U}$].
Within a principal vector basis (unit eigenvector system marked with a *hat*)

$$\hat{\mathbf{e}}_k=\mathsf{R}.\hat{\mathbf{E}}_k=\hat{\mathbf{E}}_k.\mathsf{R}^{\mathrm{T}}, \tag{11}$$

the eigenvectors are represented by the edges of infinitesimal mass elements dm, see Figure 3 for the Lagrangean view. These edges are mapped (6) material-convectively from a cube $dm=\rho_0\,d\hat{V}$ in the reference configuration κ_0 to a rectangular parallelepiped $dm=\rho\,d\hat{v}$ in the present configuration κ. The corresponding unit eigenvectors/principal base vectors $\hat{\mathbf{E}}_k$, $\hat{\mathbf{e}}_k=\mathsf{R}.\hat{\mathbf{E}}_k$; the edge vectors $d\hat{\mathbf{X}}_k$, $d\hat{\mathbf{x}}_k=\hat{U}_{(k)}\mathsf{R}.d\hat{\mathbf{X}}_{(k)}$; and the surface normal vectors $d\hat{\mathbf{S}}_k$, $d\hat{\mathbf{s}}_k=\frac{\hat{U}_1\hat{U}_2\hat{U}_3}{\hat{U}_{(k)}}\mathsf{R}.d\hat{\mathbf{S}}_{(k)}$ [from Eq.(5) and Nanson's formula] of the infinitesimal mass elements $dm=\rho_0\,d\hat{V}$ in the reference configuration κ_0 and $dm=\rho\,d\hat{v}$ in the present configuration κ are therefore collinear

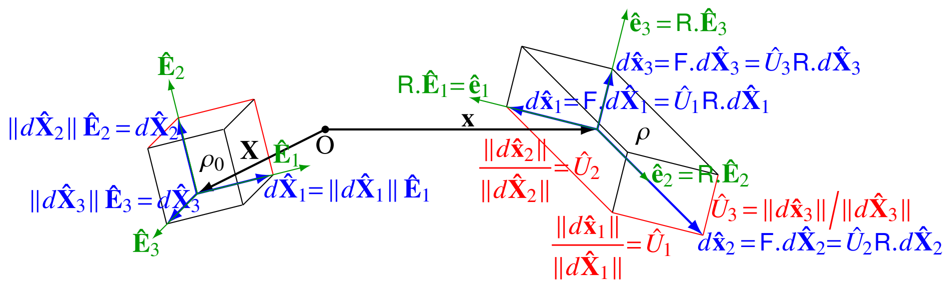

Figure 3. Lagrangean mapping of the eigenvectors relative to the principal base vector systems $\hat{\mathbf{e}}_k=\mathsf{R}.\hat{\mathbf{E}}_k$ (marked with a *hat* and represented by the edges of infinitesimal mass elements $dm=\rho_0\,d\hat{V}=\rho\,d\hat{v}$) from a Lagrangean cube ($\rho_0\,d\hat{V}$) in the reference configuration κ_0 to an Eulerian rectangular parallelepiped ($\rho\,d\hat{v}$) in the present configuration κ

such that

$$\hat{\mathbf{I}}_k = \hat{\mathbf{E}}_k = \frac{d\hat{\mathbf{X}}_{(k)}}{\|d\hat{\mathbf{X}}_{(k)}\|} = \hat{\mathbf{M}}_k = \hat{\mathbf{N}}_k = \frac{d\hat{\mathbf{S}}_{(k)}}{\|d\hat{\mathbf{S}}_{(k)}\|}\,, \qquad \hat{\mathbf{e}}_k = \hat{\mathbf{i}}_k = \frac{d\hat{\mathbf{x}}_{(k)}}{\|d\hat{\mathbf{x}}_{(k)}\|} = \hat{\mathbf{n}}_k = \hat{\mathbf{m}}_k = \frac{d\hat{\mathbf{s}}_{(k)}}{\|d\hat{\mathbf{s}}_{(k)}\|} \tag{12}$$

holds.

2.1 Superposed rigid body motions

Under a superposed rigid body motion (hereafter denoted by SRBM and marked with a *subscript plus*)

$$_+\mathbf{x}(\mathbf{X}, {}_+t) = \mathbf{a}(t) + \mathsf{Q}(t).\mathbf{x}(\mathbf{X}, t)\,, \qquad {}_+t = t - a \tag{13}$$

characterized by the translation vector $\mathbf{a}(t)$ and second-order rotation tensor $\mathsf{Q}(t)$ [both functions of time t only]

- *Lagrangean tensors (written in uppercase) are invariant*, like the reference position vector $_+\mathbf{X} = \mathbf{X}$ [which specifies a material point/particle by its position in the reference configuration κ_0 at $t = 0$] or the symmetric second-order right stretch tensor $_+\mathsf{U} = \mathsf{U}$
- *Eulerian tensors (written in lowercase) are altered* in particular through the rotation $\mathsf{Q} = \mathsf{Q}^{-\mathrm{T}}$ of the SRBM, like the present position vector (13) or the symmetric second-order left stretch tensor $_+\mathsf{v} = \mathsf{Q}.\mathsf{v}.\mathsf{Q}^{\mathrm{T}}$

2.2 Time derivatives of Lagrangean tensor fields

By its physical definition, the Lagrangean description is convected with the material. Therefore, the time derivative

$$\dot{\mathsf{S}} = \dot{S}_{ij}\mathbf{E}_i \otimes \mathbf{E}_j = \mathring{\mathsf{S}} \tag{14}$$

of a symmetric second-order Lagrangean tensor $\mathsf{S} = S_{ij}\mathbf{E}_i \otimes \mathbf{E}_j = \mathsf{S}^{\mathrm{T}} = S_{ij}\mathbf{E}_j \otimes \mathbf{E}_i$ is identical to its material-convective rate $\mathring{\mathsf{S}}$ since the time derivatives $\dot{\mathbf{E}}_k = \mathbf{0}$ of arbitrary Lagrangean base unit vectors $\mathbf{E}_k$ vanish. In a Lagrangean description $\dot{\mathsf{S}}$ and $\mathring{\mathsf{S}}$ need *not* to be distinguished. The time derivative (14) of a symmetric second-order Lagrangean tensor $\mathsf{S} = \hat{S}_k \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k$ reads in spectral representation

$$\dot{\mathsf{S}} = \dot{\hat{S}}_k \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \hat{S}_k \dot{\hat{\mathbf{E}}}_k \otimes \hat{\mathbf{E}}_k + \hat{S}_k \hat{\mathbf{E}}_k \otimes \dot{\hat{\mathbf{E}}}_k = \dot{\hat{S}}_k \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \Lambda.\mathsf{S} - \mathsf{S}.\Lambda\,, \tag{15}$$

where the time derivatives

$$\dot{\hat{\mathbf{E}}}_k = \Lambda.\hat{\mathbf{E}}_k = -\hat{\mathbf{E}}_k.\Lambda = \underline{\lambda} \times \hat{\mathbf{E}}_k = -\hat{\mathbf{E}}_k \times \underline{\lambda} = -\epsilon_{ijk}\,\hat{\lambda}_i\,\hat{\mathbf{E}}_j \tag{16}$$

of the Lagrangean principal base vectors $\hat{\mathbf{E}}_k$ (unit eigenvectors) can either be expressed as dot products with the antisymmetric second-order Lagrangean principal spin tensor

$$\Lambda = -\Lambda^{\mathrm{T}} = -\epsilon_{ijk}\,\lambda_k\,\mathbf{E}_i \otimes \mathbf{E}_j \tag{17}$$

or as cross products with the dual Lagrangean principal spin vector $\underline{\lambda}$. The components of dual tensors $\underline{\lambda} = \lambda_k \mathbf{E}_k$ and $\Lambda = \Lambda_{ij}\mathbf{E}_i \otimes \mathbf{E}_j = -\Lambda^{\mathrm{T}} = -\Lambda_{ji}\mathbf{E}_i \otimes \mathbf{E}_j$ obey

$$\lambda_k = -\tfrac{1}{2}\,\epsilon_{ijk}\,\Lambda_{ij}\,, \qquad \Lambda_{ij} = -\Lambda_{ji} = -\epsilon_{ijk}\,\lambda_k \tag{18}$$

where

$$\epsilon_{ijk} = \tfrac{(i-j)(j-k)(k-i)}{2} = \begin{cases} 1 & \{ijk = 123,\ 231,\ 312\} \\ -1 & \{ijk = 321,\ 132,\ 213\} \\ 0 & \{otherwise\} \end{cases} \tag{19}$$

denotes the Levi-Civita (1925) epsilon [also known as alternating unit symbol].

3 The spatial Eulerian description $\mathbf{X}(\mathbf{x}, t)$ with material convection

From an Eulerian point of view, the inverse deformation gradient

$$\mathsf{F}^{-1} = \partial\mathbf{X}/\partial\mathbf{x} = \partial\hat{X}_i/\partial\hat{x}_j\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \frac{1}{\hat{U}_k}\underbrace{\hat{\mathbf{E}}_k \otimes \hat{\mathbf{e}}_k}_{\mathsf{R}^{\mathrm{T}}} = \mathsf{R}^{\mathrm{T}}.\underbrace{\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k\ 1/\hat{U}_k}_{\mathsf{v}^{-1}} = \underbrace{1/\hat{U}_k\,\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k}_{\mathsf{U}^{-1}}.\mathsf{R}^{\mathrm{T}} = \frac{d\hat{\mathbf{X}}_k \otimes d\hat{\mathbf{x}}_k}{\|d\hat{\mathbf{x}}_k\|^2} \tag{20}$$

maps the vicinity vector $d\mathbf{x}$ of a position vector $\mathbf{x}$ in the present configuration κ back to the vicinity vector

$$d\mathbf{X} = \mathsf{F}^{-1}.d\mathbf{x} = d\mathbf{x}.\mathsf{F}^{-\mathrm{T}} \tag{21}$$

of a position vector $\mathbf{X}$ in the reference configuration κ_0, and the vicinity vectors $d\mathbf{X}$ and $d\mathbf{x}$ describe [like in Eq.(6)] the kinematical behavior of infinitesimal material line elements in the reference κ_0 and present κ configurations. The infinitesimal mass element $dm = \rho\,dv = \rho_0\,dV$ [the *unit mass*] depicted in Figure 4 is therefore transformed material-convectively from a cube $(\rho\,dv)$ in the present configuration κ back to a skewed parallelepiped $(\rho_0\,dV)$ in the reference configuration κ_0.

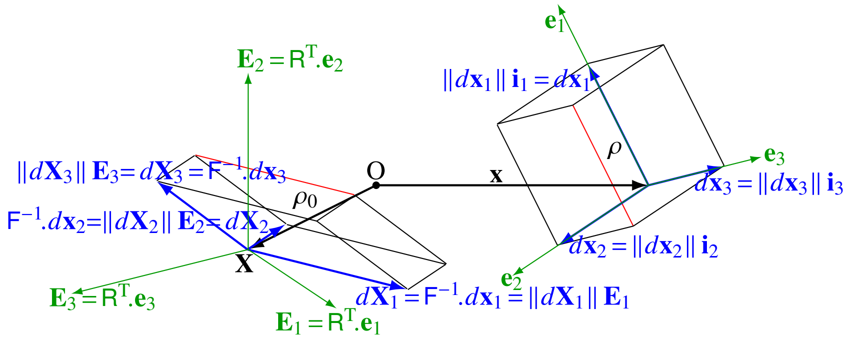

Figure 4. Eulerian mapping of an infinitesimal mass element $dm = \rho\, dv = \rho_0\, dV$ from a cube $(\rho\, dv)$ in the present configuration κ back to a skewed parallelepiped $(\rho_0\, dV)$ in the reference configuration κ_0 within an arbitrary material-convective base vector system $\mathbf{E}_k := \mathsf{R}^{\mathrm{T}}.\mathbf{e}_k$ (without a *hat*)

The arbitrary orthogonal material-convective base unit vectors $\mathbf{E}_k := \mathsf{R}^{\mathrm{T}}.\mathbf{e}_k$; the unit vectors $\mathbf{i}$ (9c), $\mathbf{E}$ (9a) pointing along the edges $d\mathbf{x}$, $d\mathbf{X}$ of the infinitesimal mass elements $dm = \rho\, dv$ in the present configuration κ and $dm = \rho_0\, dV$ in the reference configuration κ_0; and the unit vectors $\mathbf{m}$ (10c), $\mathbf{N}$ (10a) of the surface normals $d\mathbf{s}$, $d\mathbf{S} = \frac{\rho}{\rho_0}\mathsf{F}^{\mathrm{T}}.d\mathbf{s}$ [Nanson's inverse formula] differ for the reverse mapping (21) in their Lagrangean flavor from each other, $\mathbf{E} \neq \mathbf{I} \neq \mathbf{N}$, as shown in Figure 4 and exemplified for a plane finite deformation in Figure 5 [which also illustrates the polar decomposition of the inverse deformation gradient $\mathsf{F}^{-1} = \mathsf{R}^{\mathrm{T}}.\mathsf{v}^{-1} = \mathsf{U}^{-1}.\mathsf{R}^{\mathrm{T}}$].

Within a principal vector basis (unit eigenvector system marked with a *hat*) the eigenvectors are represented by the edges of infinitesimal mass elements dm, see Figure 6 for the Eulerian view. These edges are mapped back (21) material-convectively from a cube $dm = \rho\, d\hat{v}$ in the present configuration κ to a rectangular parallelepiped $dm = \rho_0\, d\hat{V}$ in the reference configuration κ_0. The corresponding unit eigenvectors $\hat{\mathbf{e}}_k$, $\hat{\mathbf{E}}_k = \mathsf{R}^{\mathrm{T}}.\hat{\mathbf{e}}_k$; the edge vectors $d\hat{\mathbf{x}}_k$, $d\hat{\mathbf{X}}_k = {}^{1}/_{\hat{U}_{(k)}}\mathsf{R}^{\mathrm{T}}.d\hat{\mathbf{x}}_{(k)}$; and the surface normal vectors $d\hat{\mathbf{s}}_k$, $d\hat{\mathbf{S}}_k = \frac{\hat{U}_{(k)}}{\hat{U}_1\hat{U}_2\hat{U}_3}\mathsf{R}^{\mathrm{T}}.d\hat{\mathbf{s}}_{(k)}$ [from Eq.(20) and Nanson's inverse formula] of the infinitesimal mass elements $dm = \rho\, d\hat{v}$ in the present configuration κ and $dm = \rho_0\, d\hat{V}$ in the reference configuration κ_0 are therefore collinear, such that (12) holds.

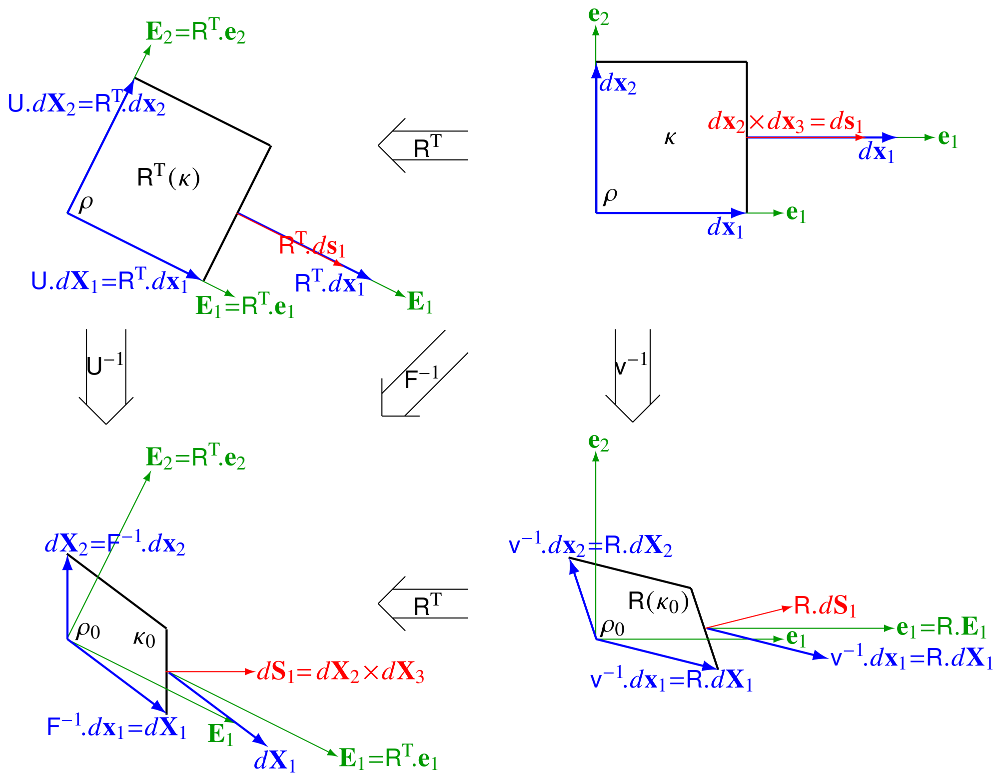

Figure 5. Eulerian view on the polar decomposition of the inverse deformation gradient $\mathsf{F}^{-1} = \mathsf{R}^{\mathrm{T}}.\mathsf{v}^{-1} = \mathsf{U}^{-1}.\mathsf{R}^{\mathrm{T}}$ within an arbitrary material-convective base vector system $\mathbf{E}_k := \mathsf{R}^{\mathrm{T}}.\mathbf{e}_k$ (without a *hat*) exemplified for a plane finite deformation of an infinitesimal mass element $dm = \rho\, dv = \rho_0\, dV$ mapped (21) from a cube $(\rho\, dv)$ in the present configuration κ back to a skewed parallelepiped $(\rho_0\, dV)$ in the reference configuration κ_0

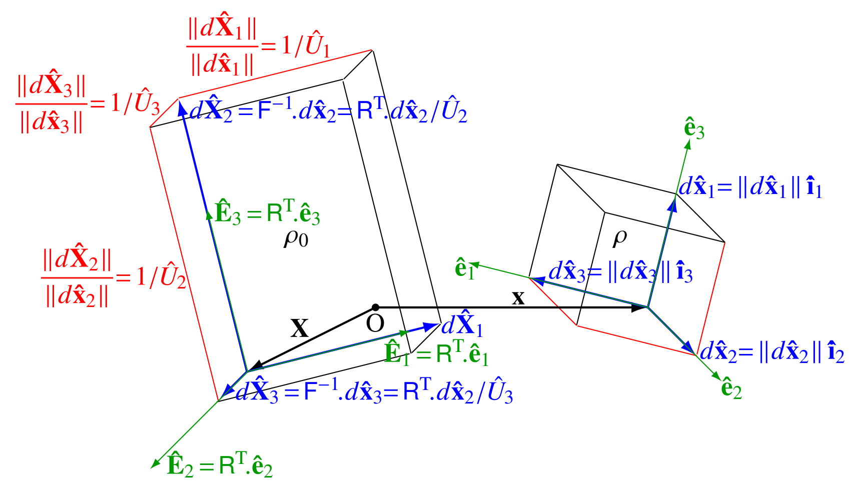

Figure 6. Eulerian mapping of the eigenvectors relative to the principal base vector systems $\hat{\mathbf{E}}_k = \mathsf{R}^{\mathrm{T}}.\hat{\mathbf{e}}_k$ (marked with a *hat* and represented by the edges of infinitesimal mass elements $dm = \rho\, d\hat{v} = \rho_0\, d\hat{V}$) from an Eulerian cube $(\rho\, d\hat{v})$ in the present configuration κ back to a Lagrangean rectangular parallelepiped $(\rho_0\, d\hat{V})$ in the reference configuration κ_0

3.1 Time derivatives of Eulerian scalar $\varphi(\mathbf{x},t)$, vector $\mathbf{a}(\mathbf{x},t)$ and symmetric second-order tensor $\mathsf{s}(\mathbf{x},t)$ fields

The non-material time derivative

$$\frac{d\varphi(\mathbf{x},t)}{dt} = \left.\frac{\partial\varphi}{\partial t}\right|_{\mathbf{x}} + \frac{\partial\varphi}{\partial\mathbf{x}}.\tilde{\mathbf{c}} \tag{22}$$

of a scalar $\varphi(\mathbf{x},t)$ field follows from the product rule with a non-material translation velocity vector $\tilde{\mathbf{c}} = \partial\mathbf{x}/\partial t$. If $\tilde{\mathbf{c}}$ is replaced by the material translation velocity $\mathbf{v}$ vector (1) then the time derivative (22) turns into the translational-convective time derivative

$$\dot{\varphi}(\mathbf{x},t) = \left.\frac{\partial\varphi}{\partial t}\right|_{\mathbf{x}} + \frac{\partial\varphi}{\partial\mathbf{x}}.\mathbf{v} = \mathring{\varphi}(\mathbf{x},t) \tag{23}$$

which is, for a scalar $\varphi(\mathbf{x},t)$ with no directional orientation, identical to the material-convective rate $\mathring{\varphi}(\mathbf{x},t)$ field. Similarly, the non-material time derivative of an Eulerian vector $\mathbf{a}(\mathbf{x},t) = \tilde{a}_k\tilde{\mathbf{e}}_k$ field follows as

$$\frac{d\mathbf{a}(\mathbf{x},t)}{dt} = \left.\frac{\partial\mathbf{a}}{\partial t}\right|_{\mathbf{x}} + \frac{\partial\mathbf{a}}{\partial\mathbf{x}}.\tilde{\mathbf{c}} = \underbrace{\left(\left.\frac{\partial\tilde{a}_k}{\partial t}\right|_{\mathbf{x}} + \frac{\partial\tilde{a}_k}{\partial\mathbf{x}}.\tilde{\mathbf{c}}\right)}_{d\tilde{a}_k(\mathbf{x},t)/dt}\tilde{\mathbf{e}}_k + \tilde{a}_k\underbrace{\overbrace{\left(\left.\frac{\partial\tilde{\mathbf{e}}_k}{\partial t}\right|_{\mathbf{x}} + \frac{\partial\tilde{\mathbf{e}}_k}{\partial\mathbf{x}}.\tilde{\mathbf{c}}\right)}^{\dot{\tilde{\mathbf{e}}}_k = \tilde{\Theta}.\tilde{\mathbf{e}}_k = -\tilde{\mathbf{e}}_k.\tilde{\Theta}}}_{\tilde{\Theta}.\mathbf{a} = -\mathbf{a}.\tilde{\Theta}} \tag{24}$$

where the non-material time derivatives of the non-convective base unit vectors $\tilde{\mathbf{e}}_k$ are given by

$$\dot{\tilde{\mathbf{e}}}_k = \left.\frac{\partial\tilde{\mathbf{e}}_k}{\partial t}\right|_{\mathbf{x}} + \frac{\partial\tilde{\mathbf{e}}_k}{\partial\mathbf{x}}.\tilde{\mathbf{c}} = \tilde{\Theta}.\tilde{\mathbf{e}}_k = -\tilde{\mathbf{e}}_k.\tilde{\Theta} \tag{25}$$

with a non-material antisymmetric second-order spin tensor $\tilde{\Theta} = -\tilde{\Theta}^{\mathrm{T}}$. If the non-material translation velocity $\tilde{\mathbf{c}}$ vector and the non-material spin $\tilde{\Theta}$ tensor are, respectively, replaced by the material translation velocity $\mathbf{v}$ vector (1) and the antisymmetric second-order Eulerian tensor (3) of material-convective spin Ω [see Dienes (1979, 1986)] then the time derivative $d\tilde{a}_k/dt\,\tilde{\mathbf{e}}_k$ relative to the $\tilde{\Theta}$-co-rotated basis $\tilde{\mathbf{e}}_k$ in Eq.(24) turns into the *material-convective vector rate* $\mathring{\mathbf{a}}(\mathbf{x},t)$, the first underbraced term of the translational-convective time derivative

$$\dot{\mathbf{a}}(\mathbf{x},t) = \left.\frac{\partial\mathbf{a}}{\partial t}\right|_{\mathbf{x}} + \frac{\partial\mathbf{a}}{\partial\mathbf{x}}.\mathbf{v} = \underbrace{\overbrace{\left(\left.\frac{\partial a_k}{\partial t}\right|_{\mathbf{x}} + \frac{\partial a_k}{\partial\mathbf{x}}.\mathbf{v}\right)}^{\dot{a}_k(\mathbf{x},t)}\mathbf{e}_k}_{\mathring{\mathbf{a}}(\mathbf{x},t)} + \underbrace{a_k\overbrace{\left(\left.\frac{\partial\mathbf{e}_k}{\partial t}\right|_{\mathbf{x}} + \frac{\partial\mathbf{e}_k}{\partial\mathbf{x}}.\mathbf{v}\right)}^{\dot{\mathbf{e}}_k = \Omega.\mathbf{e}_k = -\mathbf{e}_k.\Omega}}_{\Omega.\mathbf{a} = -\mathbf{a}.\Omega} \tag{26}$$

of a materially co-rotated (2) Eulerian vector $\mathbf{a}(\mathbf{x},t) = a_k\mathbf{e}_k = \mathsf{R}.\mathbf{A}$ with respect to its corresponding Lagrangean vector $\mathbf{A} = a_k\mathbf{E}_k$. The overbraced

$$\dot{\mathbf{e}}_k = \left.\frac{\partial\mathbf{e}_k}{\partial t}\right|_{\mathbf{x}} + \frac{\partial\mathbf{e}_k}{\partial\mathbf{x}}.\mathbf{v} = \Omega.\mathbf{e}_k = -\mathbf{e}_k.\Omega \tag{27}$$

at the r.h.s. of Eq.(26) denote the time derivatives of the material-convective base unit vectors (7) defined with the antisymmetric second-order Eulerian tensor (3) of material-convective spin Ω, which follow from $\dot{\mathbf{e}}_k = \dot{\mathsf{R}}.\mathbf{E}_k = \dot{\mathsf{R}}.\mathsf{R}^{\mathrm{T}}.\mathbf{e}_k$ and the inverse of Eq.(7).

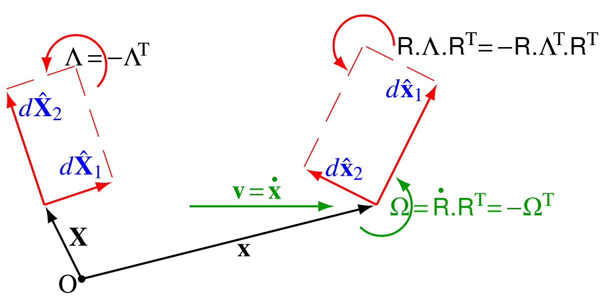

Figure 7. Kinematics of Lagrangean $d\hat{\mathbf{X}}_k$ and Eulerian $d\hat{\mathbf{x}}_k$ eigenvectors

In the same vein, the non-material time derivative

$$\frac{d\mathsf{s}(\mathbf{x},t)}{dt} = \left.\frac{\partial \mathsf{s}}{\partial t}\right|_{\mathbf{x}} + \frac{\partial \mathsf{s}}{\partial \mathbf{x}}.\tilde{\mathbf{c}} = \underbrace{\left(\left.\frac{\partial \tilde{s}_{ij}}{\partial t}\right|_{\mathbf{x}} + \frac{\partial \tilde{s}_{ij}}{\partial \mathbf{x}}.\tilde{\mathbf{c}}\right)}_{d\tilde{s}_{ij}(\mathbf{x},t)/dt}\tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_j + \underbrace{\tilde{s}_{ij}\tilde{\Theta}.\tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_j}_{\tilde{\Theta}.\mathsf{s}} - \underbrace{\tilde{s}_{ij}\tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_j.\tilde{\Theta}}_{\mathsf{s}.\tilde{\Theta}} \tag{28}$$

of a symmetric second-order Eulerian tensor $\mathsf{s}(\mathbf{x},t) = \tilde{s}_{ij}\tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_j$ field is defined with a non-material translation velocity $\tilde{\mathbf{c}}$ and a non-material time derivative (25) of the non-convective base unit vectors $\tilde{\mathbf{e}}_k$. If the non-material translation velocity $\tilde{\mathbf{c}}$ vector and the non-material spin $\tilde{\Theta}$ tensor are, respectively, replaced by the material translation velocity $\mathbf{v}$ vector (1) and the tensor (3) of material-convective spin Ω then the time derivative $d\tilde{s}_{ij}/dt\ \tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_j$ relative to the $\tilde{\Theta}$-co-rotated basis $\tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_j$ in Eq.(28) turns into the *material-convective tensor rate* $\mathring{\mathsf{s}}(\mathbf{x},t)$, the first underbraced term of the translational-convective time derivative

$$\dot{\mathsf{s}}(\mathbf{x},t) = \left.\frac{\partial \mathsf{s}}{\partial t}\right|_{\mathbf{x}} + \frac{\partial \mathsf{s}}{\partial \mathbf{x}}.\mathbf{v} = \underbrace{\overbrace{\left(\left.\frac{\partial s_{ij}}{\partial t}\right|_{\mathbf{x}} + \frac{\partial s_{ij}}{\partial \mathbf{x}}.\mathbf{v}\right)}^{\dot{s}_{ij}(\mathbf{x},t)}\mathbf{e}_i \otimes \mathbf{e}_j}_{\mathring{\mathsf{s}}(\mathbf{x},t)} + \underbrace{s_{ij}\Omega.\mathbf{e}_i \otimes \mathbf{e}_j}_{\Omega.\mathsf{s}} - \underbrace{s_{ij}\mathbf{e}_i \otimes \mathbf{e}_j.\Omega}_{\mathsf{s}.\Omega} \tag{29}$$

of a symmetric second-order Eulerian tensor $\mathsf{s}(\mathbf{x},t) = s_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathsf{R.S.R}^T$ with respect to its corresponding Lagrangean tensor $\mathsf{S} = s_{ij}\mathbf{E}_i \otimes \mathbf{E}_j$. The time derivative (29) of a symmetric second-order Eulerian tensor $\mathsf{s} = \hat{S}_k\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k = \mathsf{R.S.R}^T$ reads in spectral representation

$$\dot{\mathsf{s}} = \dot{\hat{S}}_k\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k + \hat{S}_k\dot{\hat{\mathbf{e}}}_k \otimes \hat{\mathbf{e}}_k + \hat{S}_k\hat{\mathbf{e}}_k \otimes \dot{\hat{\mathbf{e}}}_k = \dot{\hat{S}}_k\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k + \Gamma.\mathsf{s} - \mathsf{s}.\Gamma \tag{30}$$

where the time derivatives

$$\dot{\hat{\mathbf{e}}}_k = (\mathsf{R}.\hat{\mathbf{E}}_k)^{\bullet} = \dot{\mathsf{R}}.\hat{\mathbf{E}}_k + \mathsf{R}.\dot{\hat{\mathbf{E}}}_k = (\underbrace{\dot{\mathsf{R}}.\mathsf{R}^T}_{\Omega} + \mathsf{R}.\Lambda.\mathsf{R}^T).\hat{\mathbf{e}}_k = \Gamma.\hat{\mathbf{e}}_k = -\hat{\mathbf{e}}_k.\Gamma \tag{31}$$

of the Eulerian principal base vectors $\hat{\mathbf{e}}_k$ (unit eigenvectors) follow with the antisymmetric second-order Eulerian principal spin tensor

$$\Gamma = -\Gamma^T = \Omega + \mathsf{R}.\Lambda.\mathsf{R}^T = -\epsilon_{ijk}\,\underbrace{(\omega_k + \lambda_k)}_{\gamma_k}\,\mathbf{e}_i \otimes \mathbf{e}_j \tag{32}$$

compiled from $\Omega = -\epsilon_{ijk}\,\omega_k\,\mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathsf{R}.\Lambda.\mathsf{R}^T = -\epsilon_{ijk}\,\lambda_k\,\mathbf{e}_i \otimes \mathbf{e}_j$ [cf. Eqs.(7), (17) and Figure 7].

3.2 The material-convective Green-Naghdi rate of a symmetric second-order Eulerian tensor $\mathsf{s}(\mathbf{x},t)$ field

Since the material-convective rate $\mathring{\mathsf{S}}$ of a symmetric second-order Lagrangean tensor S is identical (14) to its time derivative $\dot{\mathsf{S}}$, the material-convective rate

$$\mathring{\mathsf{s}} = \mathsf{R}.\underbrace{(\mathsf{R}^T.\mathsf{s}.\mathsf{R})^{\bullet}}_{\dot{\mathsf{S}}}.\mathsf{R}^T = \dot{\mathsf{s}} - \Omega.\mathsf{s} + \mathsf{s}.\Omega \tag{33}$$

of its corresponding material-convectively forward-rotated symmetric second-order Eulerian tensor $\mathsf{s} = \mathsf{R.S.R}^T$ is given by the forward-rotation $\mathsf{R}.\dot{\mathsf{S}}.\mathsf{R}^T$ of the time derivative $\dot{\mathsf{S}} = (\mathsf{R}^T.\mathsf{s}.\mathsf{R})^{\bullet} = \mathring{\mathsf{S}}$ of the back-rotated Lagrangean tensor $\mathsf{S} = \mathsf{R}^T.\mathsf{s}.\mathsf{R}$—consistent with Eq.(29) where the $\mathbf{e}_i \otimes \mathbf{e}_j$ components $\dot{s}_{ij}$ of the material-convective tensor rate $\mathring{\mathsf{s}} = \dot{s}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ are defined relative to the material-convective (7) basis $\mathbf{e}_i \otimes \mathbf{e}_j = \{\mathsf{R}.\mathbf{E}_i\} \otimes \{\mathbf{E}_j.\mathsf{R}^T\}$. The only material-convective time derivative (denoted with a *superscript ring*) of a symmetric second-order Eulerian tensor is given by the *Green-Naghdi rate* (33) [see Eqs.(8.20)–(8.23) of Green&Naghdi (1965) p.273] which convects the material rotationally with the antisymmetric tensor Ω of the material-convective spin (3) and translationally with the material velocity $\mathbf{v}$ vector (1). With (30) and (32), the material-convective rate (33) of the symmetric second-order Eulerian tensor $\mathsf{s}=\hat{S}_k\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_k=\mathsf{R.S.R}^\mathrm{T}$ reads in spectral representation

$$\mathring{\mathsf{s}} = \dot{\hat{S}}_k\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_k+(\mathsf{R}.\Lambda.\mathsf{R}^\mathrm{T}).\mathsf{s}-\mathsf{s}.(\mathsf{R}.\Lambda.\mathsf{R}^\mathrm{T}) = \mathsf{R}.\underbrace{(\dot{\hat{S}}_k\hat{\mathbf{E}}_k\otimes\hat{\mathbf{E}}_k+\Lambda.\mathsf{S}-\mathsf{S}.\Lambda)}_{\dot{\mathsf{S}}}.\mathsf{R}^\mathrm{T}, \tag{34}$$

cf. Eq.(15).

4 Finite deformation kinematics

From the polar decomposition of the deformation gradient $\mathsf{F}=\mathsf{R.U}$ into the material-convective rotation R tensor (2) and the right stretch $\mathsf{U}=\hat{U}_k\hat{\mathbf{E}}_k\otimes\hat{\mathbf{E}}_k$ tensor (4b) and with the material-convective spin Ω tensor (3) and the spin tensors Λ, Γ of the Lagrangean (17), Eulerian (32) principal axes, the velocity gradient

$$\begin{aligned}\dot{\mathsf{F}}.\mathsf{F}^{-1} &= (\dot{\mathsf{R}}.\mathsf{U}+\mathsf{R}.\dot{\mathsf{U}}).(\mathsf{U}^{-1}.\mathsf{R}^\mathrm{T}) = (\dot{\hat{U}}_k\,\hat{\mathbf{e}}_k\otimes\hat{\mathbf{E}}_k+\hat{U}_k\,\dot{\hat{\mathbf{e}}}_k\otimes\hat{\mathbf{E}}_k+\hat{U}_k\,\hat{\mathbf{e}}_k\otimes\dot{\hat{\mathbf{E}}}_k).(\hat{\mathbf{E}}_\ell\otimes\hat{\mathbf{e}}_\ell\,{}^1\!/\hat{U}_\ell)\\ &= \underbrace{\dot{\mathsf{R}}.\mathsf{R}^\mathrm{T}}_{\Omega}+\mathsf{R}.(\dot{\mathsf{U}}.\mathsf{U}^{-1}).\mathsf{R}^\mathrm{T}=\frac{\dot{\hat{U}}_k}{\hat{U}_k}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_k+\underbrace{\Omega+\mathsf{R}.\Lambda.\mathsf{R}^\mathrm{T}}_{\Gamma}-\mathsf{F}.\Lambda.\mathsf{F}^{-1}=\frac{\dot{\hat{U}}_k}{\hat{U}_k}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_k+\Omega+\mathsf{R}.(\Lambda-\mathsf{U}.\Lambda.\mathsf{U}^{-1}).\mathsf{R}^\mathrm{T}\end{aligned} \tag{35}$$

may additively be split into (its symmetric part) the Eulerian deformation-rate tensor

$$\begin{aligned}\mathsf{d} &= \tfrac{1}{2}(\dot{\mathsf{F}}.\mathsf{F}^{-1}+\mathsf{F}^{-\mathrm{T}}.\dot{\mathsf{F}}^\mathrm{T}) = \tfrac{1}{2}\mathsf{R}.(\dot{\mathsf{U}}.\mathsf{U}^{-1}+\mathsf{U}^{-1}.\dot{\mathsf{U}}).\mathsf{R}^\mathrm{T} = \frac{\dot{\hat{U}}_k}{\hat{U}_k}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_k-\tfrac{1}{2}\mathsf{R}.(\mathsf{U}.\Lambda.\mathsf{U}^{-1}-\mathsf{U}^{-1}.\Lambda.\mathsf{U}).\mathsf{R}^\mathrm{T}\\ &= \tfrac{1}{2}(\mathring{\mathsf{v}}.\mathsf{v}^{-1}+\mathsf{v}^{-1}.\mathring{\mathsf{v}}) = \tfrac{1}{2}\mathsf{v}^{-1}.\mathring{\mathsf{b}}.\mathsf{v}^{-1} = \tfrac{1}{2}\mathsf{F}^{-\mathrm{T}}.\dot{\mathsf{C}}.\mathsf{F}^{-1}\end{aligned} \tag{36}$$

and into (its antisymmetric part) the Eulerian vorticity tensor

$$\mathsf{w} = \tfrac{1}{2}(\dot{\mathsf{F}}.\mathsf{F}^{-1}-\mathsf{F}^{-\mathrm{T}}.\dot{\mathsf{F}}^\mathrm{T}) = \Omega+\tfrac{1}{2}\mathsf{R}.(\dot{\mathsf{U}}.\mathsf{U}^{-1}-\mathsf{U}^{-1}.\dot{\mathsf{U}}).\mathsf{R}^\mathrm{T} = \Omega-\tfrac{1}{2}\mathsf{R}.(\mathsf{U}.\Lambda.\mathsf{U}^{-1}+\mathsf{U}^{-1}.\Lambda.\mathsf{U}-2\Lambda).\mathsf{R}^\mathrm{T}. \tag{37}$$

The $\mathring{\mathsf{b}}$ and $\dot{\mathsf{C}}$ at the r.h.s. of Eq.(36) are material-convective rates [cf. (33) and (14)] of the positive definite, symmetric left and right Cauchy (1827b)-Green (1839) deformation tensors

$$\mathsf{b} = \mathsf{b}^\mathrm{T} = \mathsf{F}.\mathsf{F}^\mathrm{T} = \mathsf{v}^2 = \mathsf{R}.\mathsf{C}.\mathsf{R}^\mathrm{T} \qquad \text{and} \qquad \mathsf{C} = \mathsf{C}^\mathrm{T} = \mathsf{F}^\mathrm{T}.\mathsf{F} = \mathsf{U}^2 = \mathsf{R}^\mathrm{T}.\mathsf{b}.\mathsf{R}, \tag{38}$$

respectively. The Lagrangean deformation-rate tensor

$$\mathsf{D} = \mathsf{R}^\mathrm{T}.\mathsf{d}.\mathsf{R} = \tfrac{1}{2}(\dot{\mathsf{U}}.\mathsf{U}^{-1}+\mathsf{U}^{-1}.\dot{\mathsf{U}}) = \frac{\dot{\hat{U}}_k}{\hat{U}_k}\hat{\mathbf{E}}_k\otimes\hat{\mathbf{E}}_k-\tfrac{1}{2}(\mathsf{U}.\Lambda.\mathsf{U}^{-1}-\mathsf{U}^{-1}.\Lambda.\mathsf{U}) = \tfrac{1}{2}\mathsf{U}^{-1}.\dot{\mathsf{C}}.\mathsf{U}^{-1} \tag{39}$$

follows from the material-convective backward rotation of Eq.(36). The principal components $\hat{w}_k$, $\hat{\omega}_k$, $\hat{\lambda}_k$ of the antisymmetric tensors $\mathsf{w}=-\epsilon_{ijk}\,\hat{w}_k\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j$, $\Omega=-\epsilon_{ijk}\,\hat{\omega}_k\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j$, $\Lambda=-\epsilon_{ijk}\,\hat{\lambda}_k\,\hat{\mathbf{E}}_i\otimes\hat{\mathbf{E}}_j$ [cf. (17)] are, from the r.h.s. of Eq.(37), related by

$$\hat{w}_1 = \hat{\omega}_1-\tfrac{1}{2}(\tfrac{\hat{U}_2}{\hat{U}_3}+\tfrac{\hat{U}_3}{\hat{U}_2}-2)\,\hat{\lambda}_1, \qquad \hat{w}_2 = \hat{\omega}_2-\tfrac{1}{2}(\tfrac{\hat{U}_3}{\hat{U}_1}+\tfrac{\hat{U}_1}{\hat{U}_3}-2)\,\hat{\lambda}_2, \qquad \hat{w}_3 = \hat{\omega}_3-\tfrac{1}{2}(\tfrac{\hat{U}_1}{\hat{U}_2}+\tfrac{\hat{U}_2}{\hat{U}_1}-2)\,\hat{\lambda}_3. \tag{40}$$

4.1 Symmetric total or partial Cauchy-Green deformation tensors and their rates

The left ${}^@\mathsf{b}=\{\mathsf{b}, {}^e\mathsf{b}, {}^p\mathsf{b}, \cdots\}$ and right ${}^@\mathsf{C}=\{\mathsf{C}, {}^e\mathsf{C}, {}^p\mathsf{C}, \cdots\}$ total or partial (elastic, plastic, $\cdots$) Cauchy-Green deformation tensors are [analogously to the definitions (38)] defined through the *same* material-convective rotation R tensor (2) as

$${}^@\mathsf{b} = ({}^@\hat{U}_k)^2\;{}^@\hat{\mathbf{e}}_k\otimes{}^@\hat{\mathbf{e}}_k = \mathsf{R}.{}^@\mathsf{C}.\mathsf{R}^\mathrm{T}, \qquad {}^@\mathsf{C} = ({}^@\hat{U}_k)^2\;{}^@\hat{\mathbf{E}}_k\otimes{}^@\hat{\mathbf{E}}_k\,, \qquad {}^@\hat{\mathbf{e}}_k = \mathsf{R}.{}^@\hat{\mathbf{E}}_k = {}^@\hat{\mathbf{E}}_k.\mathsf{R}^\mathrm{T}, \tag{41}$$

with the total or partial (elastic, plastic, $\cdots$) eigenvalues ${}^@\hat{U}_k=\{\hat{U}_k, {}^e\hat{U}_k, {}^p\hat{U}_k, \cdots\}$, i.e. the principal stretch ratios, and the unit eigenvectors ${}^@\hat{\mathbf{E}}_k=\{\hat{\mathbf{E}}_k, {}^e\hat{\mathbf{E}}_k, {}^p\hat{\mathbf{E}}_k, \cdots\}$, i.e. the principal directions. Under SRBM the tensors of Eq.(41) obey: ${}^@_+\mathsf{b}=\mathsf{Q}.{}^@\mathsf{b}.\mathsf{Q}^\mathrm{T}$, ${}^@_+\hat{\mathbf{e}}_k=\mathsf{Q}.{}^@\hat{\mathbf{e}}_k={}^@\hat{\mathbf{e}}_k.\mathsf{Q}^\mathrm{T}$ and ${}^@_+\mathsf{C}={}^@\mathsf{C}$, ${}^@_+\hat{\mathbf{E}}_k={}^@\hat{\mathbf{E}}_k$. With the left ${}^@\mathsf{v}=\{\mathsf{v}, {}^e\mathsf{v}, {}^p\mathsf{v}, \cdots\}$ and right ${}^@\mathsf{U}=\{\mathsf{U}, {}^e\mathsf{U}, {}^p\mathsf{U}, \cdots\}$ total or partial (elastic, plastic, $\cdots$) stretch tensors given by

$${}^@\mathsf{v} = {}^@\hat{U}_k\;{}^@\hat{\mathbf{e}}_k\otimes{}^@\hat{\mathbf{e}}_k = \mathsf{R}.{}^@\mathsf{U}.\mathsf{R}^\mathrm{T} \qquad \text{and} \qquad {}^@\mathsf{U} = {}^@\hat{U}_k\;{}^@\hat{\mathbf{E}}_k\otimes{}^@\hat{\mathbf{E}}_k = \mathsf{R}^\mathrm{T}.{}^@\mathsf{v}.\mathsf{R}, \tag{42}$$

the total or partial (elastic, plastic, $\cdots$) Eulerian deformation-rate tensors may be defined by

$${}^@\mathsf{d} = \tfrac{1}{2}({}^@\mathring{\mathsf{v}}.{}^@\mathsf{v}^{-1}+{}^@\mathsf{v}^{-1}.{}^@\mathring{\mathsf{v}}) = \tfrac{1}{2}\,{}^@\mathsf{v}^{-1}.{}^@\mathring{\mathsf{b}}.{}^@\mathsf{v}^{-1} = \tfrac{1}{2}\mathsf{R}.({}^@\dot{\mathsf{U}}.{}^@\mathsf{U}^{-1}+{}^@\mathsf{U}^{-1}.{}^@\dot{\mathsf{U}}).\mathsf{R}^\mathrm{T} = \tfrac{1}{2}\mathsf{R}.{}^@\mathsf{U}^{-1}.{}^@\dot{\mathsf{C}}.{}^@\mathsf{U}^{-1}.\mathsf{R}^\mathrm{T} \tag{43}$$

[cf. Eq.(36)], the corresponding material-convectively back-rotated Lagrangean deformation-rate tensors by

$${}^@\mathsf{D} = \mathsf{R}^\mathrm{T}.{}^@\mathsf{d}.\mathsf{R} = \tfrac{1}{2}({}^@\dot{\mathsf{U}}.{}^@\mathsf{U}^{-1}+{}^@\mathsf{U}^{-1}.{}^@\dot{\mathsf{U}}) = \tfrac{1}{2}\,{}^@\mathsf{U}^{-1}.{}^@\dot{\mathsf{C}}.{}^@\mathsf{U}^{-1} \tag{44}$$

[cf. Eq.(39)], and the total or partial (elastic, plastic, $\cdots$) Eulerian vorticity tensors by

$${}^@\mathsf{w} = \Omega+\tfrac{1}{2}({}^@\mathring{\mathsf{v}}.{}^@\mathsf{v}^{-1}-{}^@\mathsf{v}^{-1}.{}^@\mathring{\mathsf{v}}) = \Omega+\tfrac{1}{2}\mathsf{R}.({}^@\dot{\mathsf{U}}.{}^@\mathsf{U}^{-1}-{}^@\mathsf{U}^{-1}.{}^@\dot{\mathsf{U}}).\mathsf{R}^\mathrm{T} \tag{45}$$

[cf. Eq.(37)]. Under SRBM the Eulerian tensor of material-convective spin (3) obeys ${}_+\Omega=\dot{\mathsf{Q}}.\mathsf{Q}^\mathrm{T}+\mathsf{Q}.\Omega.\mathsf{Q}^\mathrm{T}$, the Eulerian vorticity tensors (45) obey ${}^@_+\mathsf{w}=\dot{\mathsf{Q}}.\mathsf{Q}^\mathrm{T}+\mathsf{Q}.{}^@\mathsf{w}.\mathsf{Q}^\mathrm{T}$, the left or right stretch tensors obey ${}^@_+\mathsf{v}=\mathsf{Q}.{}^@\mathsf{v}.\mathsf{Q}^\mathrm{T}$ or ${}^@_+\mathsf{U}={}^@\mathsf{U}$, and the rate of deformation tensors (43) or (44) obey ${}^@_+\mathsf{d}=\mathsf{Q}.{}^@\mathsf{d}.\mathsf{Q}^\mathrm{T}$ or ${}^@_+\mathsf{D}={}^@\mathsf{D}$, respectively.

4.2 Symmetric total or partial generalized strain tensors, their rates and their work-conjugate stresses

By applying total or partial Eulerian ${}^{@}\mathfrak{b} = {}^{@}B_{ijk\ell}\,\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_\ell = {}^{@}\mathfrak{a}^{-1}$, ${}^{@}\mathfrak{a} = {}^{@}A_{ijk\ell}\,\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_\ell = {}^{@}\mathfrak{b}^{-1}$ or Lagrangean ${}^{@}\mathcal{B} = {}^{@}B_{ijk\ell}\,\mathbf{E}_i\otimes\mathbf{E}_j\otimes\mathbf{E}_k\otimes\mathbf{E}_\ell = {}^{@}\mathcal{A}^{-1}$, ${}^{@}\mathcal{A} = {}^{@}A_{ijk\ell}\,\mathbf{E}_i\otimes\mathbf{E}_j\otimes\mathbf{E}_k\otimes\mathbf{E}_\ell = {}^{@}\mathcal{B}^{-1}$ fourth-order transformation tensors, which are inverse to each other ${}^{@}B_{ijmn}\,{}^{@}A_{mnk\ell} = \frac{1}{2}(\delta_{ik}\,\delta_{j\ell}+\delta_{i\ell}\,\delta_{jk})$, which obey the symmetries ${}^{@}B_{ijk\ell} = {}^{@}B_{ij\ell k} = {}^{@}B_{jik\ell} = {}^{@}B_{k\ell ij}$ and ${}^{@}A_{ijk\ell} = {}^{@}A_{ij\ell k} = {}^{@}A_{jik\ell} = {}^{@}A_{k\ell ij}$ and which are defined through the distinct eigenvalues ${}^{@}\hat{U}_\lambda = \{\hat{U}_\lambda, {}^{e}\hat{U}_\lambda, {}^{p}\hat{U}_\lambda, \cdots\}$ and the corresponding symmetric second-order Eulerian ${}^{@}\mathsf{v}_\lambda = {}^{@}\mathfrak{b}_\lambda = \{\mathfrak{b}_\lambda, {}^{e}\mathfrak{b}_\lambda, {}^{p}\mathfrak{b}_\lambda, \cdots\}$ or Lagrangean ${}^{@}\mathsf{U}_\lambda = {}^{@}\mathsf{C}_\lambda = \{\mathsf{C}_\lambda, {}^{e}\mathsf{C}_\lambda, {}^{p}\mathsf{C}_\lambda, \cdots\}$ eigenprojection tensors, kinematical relations similar to Eqs.(43)–(44) of Eulerian

$$ {}^{@}\mathsf{d} = {}^{@}\mathfrak{a}..{}^{@}\mathring{\mathsf{e}}\,, \qquad {}^{@}\mathring{\mathsf{e}} = {}^{@}\mathfrak{b}..{}^{@}\mathsf{d} \tag{46} $$

or Lagrangean

$$ {}^{@}\mathsf{D} = {}^{@}\mathcal{A}..{}^{@}\dot{\mathsf{E}}\,, \qquad {}^{@}\dot{\mathsf{E}} = {}^{@}\mathcal{B}..{}^{@}\mathsf{D} \tag{47} $$

type may be specified for the symmetric total or partial Eulerian ${}^{@}\mathring{\mathsf{e}} = \mathsf{R}.{}^{@}\dot{\mathsf{E}}.\mathsf{R}^{\mathrm{T}}$ or Lagrangean ${}^{@}\dot{\mathsf{E}} = \mathsf{R}^{\mathrm{T}}.{}^{@}\mathring{\mathsf{e}}.\mathsf{R}$ material-convective (Green-Naghdi) rates of generalized strain tensors [for a comprehensive discussion of the definition, calculation and algorithmic treatment of the generalized strain-rate kinematics see Heiduschke (2019)], where '..' denotes the *double dot product operator* $\mathsf{a}..\mathsf{b} = \mathrm{tr}(\mathsf{a}.\mathsf{b}^{\mathrm{T}}) = a_{ij}\,b_{ij} = \mathrm{tr}(\mathsf{a}^{\mathrm{T}}.\mathsf{b}) = \mathsf{b}..\mathsf{a}$, that is the *double contraction* defined here by the traces of dot products of the second-order tensors a and b.

The relations of the symmetric Eulerian tensor t of Cauchy (1823, 1827a) stress or of the symmetric Lagrangean tensor $\mathsf{T} = \frac{\rho_0}{\rho}\mathsf{R}^{\mathrm{T}}.\mathsf{t}.\mathsf{R}$ of back-rotated Kirchhoff (1852) stress to the symmetric total or partial Eulerian ${}^{@}\mathsf{s} = \frac{\rho}{\rho_0}\mathsf{R}.{}^{@}\mathsf{S}.\mathsf{R}^{\mathrm{T}}$ or Lagrangean ${}^{@}\mathsf{S} = \frac{\rho_0}{\rho}\mathsf{R}^{\mathrm{T}}.{}^{@}\mathsf{s}.\mathsf{R}$ work-conjugate tensors of generalized stress are given through the same fourth-order transformation tensors ${}^{@}\mathfrak{a}$, ${}^{@}\mathfrak{b}$, ${}^{@}\mathcal{A}$, ${}^{@}\mathcal{B}$ in Eulerian form by

$$ {}^{@}\mathsf{s} = {}^{@}\mathfrak{a}..\mathsf{t}\,, \qquad \mathsf{t} = {}^{@}\mathfrak{b}..{}^{@}\mathsf{s} \tag{48} $$

or in Lagrangean form by

$$ {}^{@}\mathsf{S} = {}^{@}\mathcal{A}..\mathsf{T}\,, \qquad \mathsf{T} = {}^{@}\mathcal{B}..{}^{@}\mathsf{S} \tag{49} $$

respectively.

4.3 The non-material Zaremba-Jaumann rate

The Zaremba-Jaumann rate tensor

$$ {}^{ZJ}\mathring{\mathsf{s}} = \dot{\mathsf{s}} - \mathsf{w}.\mathsf{s} + \mathsf{s}.\mathsf{w} \tag{50} $$

[see Zaremba (1903) eq.(32) on p.607, eq.(37) on p.610; and Jaumann (1911) eq.(11c) on p.395] has about the same structure as the material-convective Green-Naghdi rate $\mathring{\mathsf{s}} = \dot{\mathsf{s}} - \Omega.\mathsf{s} + \mathsf{s}.\Omega$ tensor (33) just with the Eulerian vorticity w tensor (37) instead of the material-convective spin Ω tensor (3). Therefore the Zaremba-Jaumann rate ${}^{ZJ}\mathring{\mathsf{s}}$ is, in general, not rotationally convected with the material and thus a *non-material rate*. The difference between the material-convective Green-Naghdi rate $\mathring{\mathsf{s}}$ and the non-material Zaremba-Jaumann rate ${}^{ZJ}\mathring{\mathsf{s}}$ is [from (37)] given by

$$ \mathring{\mathsf{s}} - {}^{ZJ}\mathring{\mathsf{s}} = -(\Omega-\mathsf{w}).\mathsf{s} + \mathsf{s}.(\Omega-\mathsf{w}) = \tfrac{1}{2}\mathsf{R}.(\underbrace{\dot{\mathsf{U}}.\mathsf{U}^{-1}-\mathsf{U}^{-1}.\dot{\mathsf{U}}}_{2\Lambda-\mathsf{U}.\Lambda.\mathsf{U}^{-1}-\mathsf{U}^{-1}.\Lambda.\mathsf{U}}).\mathsf{R}^{\mathrm{T}}.\mathsf{s} - \tfrac{1}{2}\mathsf{s}.\mathsf{R}.(\underbrace{\dot{\mathsf{U}}.\mathsf{U}^{-1}-\mathsf{U}^{-1}.\dot{\mathsf{U}}}_{2\Lambda-\mathsf{U}.\Lambda.\mathsf{U}^{-1}-\mathsf{U}^{-1}.\Lambda.\mathsf{U}}).\mathsf{R}^{\mathrm{T}} \tag{51} $$

[cf. Green&McInnis (1967) eq.(2.17) on p.222].

4.4 Material-convective rates and corresponding time integrals with respect to the plastic flow rules

For the rate-type theories of plasticity, the (translational-convective) time derivative

$$ {}^{p}\dot{\mathsf{C}} = {}^{p}\dot{C}_{ij}\,\mathbf{E}_i\otimes\mathbf{E}_j = {}^{p}\mathring{\mathsf{C}} = \underbrace{{}^{p}\dot{\hat{C}}_k\,{}^{p}\hat{\mathbf{E}}_k\otimes{}^{p}\hat{\mathbf{E}}_k}_{{}^{p}\Lambda\ \textit{co-rotated with the plastic Lagrangean principal axes}} + \overbrace{{}^{p}\hat{C}_k\,{}^{p}\dot{\hat{\mathbf{E}}}_k\otimes{}^{p}\hat{\mathbf{E}}_k}^{{}^{p}\Lambda.{}^{p}\mathsf{C}} + \overbrace{{}^{p}\hat{C}_k\,{}^{p}\hat{\mathbf{E}}_k\otimes{}^{p}\dot{\hat{\mathbf{E}}}_k}^{-{}^{p}\mathsf{C}.{}^{p}\Lambda} \tag{52} $$

of the symmetric Lagrangean right Cauchy-Green plastic deformation tensor ${}^{p}\mathsf{C}$ (41b) [which is identical to the material-convective plastic rate ${}^{p}\mathring{\mathsf{C}}$] constitutes the *plastic flow rule* ${}^{p}\dot{\mathsf{C}} = \cdots$ which specifies the material flow behavior of plasticity; the plastic flow rule must be properly time integrated in order to obtain a proper plastic deformation tensor/measure

$$ \begin{aligned} {}^{p}\mathsf{C}(t) &= \int_{\tau=0}^{t} {}^{p}\dot{C}_{ij}(\tau)\,d\tau\,\mathbf{E}_i\otimes\mathbf{E}_j + {}^{p}\mathsf{C}(0) = {}^{p}C_{ij}(t)\,\mathbf{E}_i\otimes\mathbf{E}_j \\ &= \int_{\tau=0}^{t} \underbrace{{}^{p}\dot{\hat{C}}_k(\tau)\,{}^{p}\hat{\mathbf{E}}_k(\tau)\otimes{}^{p}\hat{\mathbf{E}}_k(\tau)}_{{}^{p}\Lambda\ \textit{co-rotated with the plastic Lagrangean principal axes}}\,d\tau + {}^{p}\mathsf{C}(0) = {}^{p}\hat{C}_k(t)\,{}^{p}\hat{\mathbf{E}}_k(t)\otimes{}^{p}\hat{\mathbf{E}}_k(t)\,. \end{aligned} \tag{53} $$

The material-convective time integral ${}^p\mathsf{C} = \int {}^p\dot{\mathsf{C}}\, dt + {}^p\mathsf{C}_0 = {}^pC_{ij}\, \mathbf{E}_i \otimes \mathbf{E}_j = (\int {}^p\dot{C}_{ij}\, dt + {}^pC_{ij}(0))\mathbf{E}_i \otimes \mathbf{E}_j$ of a symmetric Lagrangean plastic flow tensor ${}^p\dot{\mathsf{C}}$ is simply given by its arbitrary $\mathbf{E}_i \otimes \mathbf{E}_j$ components ${}^pC_{ij} = \int {}^p\dot{C}_{ij}\, dt + {}^pC_{ij}(0)$ which follow from the time integrals of each single plastic flow component. The spectral representation at the r.h.s. of Eq.(52) reveals that the plastic Lagrangean unit eigenvectors ${}^p\hat{\mathbf{E}}_k$ are spinning with ${}^p\Lambda$ [cf. Eqs.(16)–(19)]; thus the terms ${}^p\dot{\hat{C}}_k\, {}^p\hat{\mathbf{E}}_k \otimes {}^p\hat{\mathbf{E}}_k$ [in matrix component notation placed on the diagonal] represent the ${}^p\Lambda$ co-rotated tensor rate [underbraced in Eq.(52)] which corresponds to the ${}^p\Lambda$ co-rotated time integral [underbraced in Eq.(53)]. *Only if* the plastic deformation is *integrated material-convectively* from the corresponding plastic flow rule ${}^p\dot{\mathsf{C}} = \mathsf{R}^{\mathrm{T}}.{}^p\mathring{\mathsf{b}}.\mathsf{R} = \cdots$ then the resulting *plastic deformation* ${}^p\mathsf{C} = \mathsf{R}^{\mathrm{T}}.{}^p\mathsf{b}.\mathsf{R}$ is a *proper tensor* with *geometrical interpretation*—as further discussed in (the next) section 5.
The translational-convective time derivative

$$
{}^p\dot{\mathsf{b}} = \overbrace{{}^p\dot{C}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j}^{{}^p\mathring{\mathsf{b}}\ [\textit{materially } \Omega \textit{ co-rotated Green-Naghdi rate}]} + \underbrace{{}^pC_{ij}\dot{\mathbf{e}}_i \otimes \mathbf{e}_j}_{\Omega.{}^p\mathsf{b}} + \underbrace{{}^pC_{ij}\mathbf{e}_i \otimes \dot{\mathbf{e}}_j}_{-{}^p\mathsf{b}.\Omega} = \underbrace{{}^p\dot{\hat{C}}_k\, {}^p\hat{\mathbf{e}}_k \otimes {}^p\hat{\mathbf{e}}_k}_{{}^p\Gamma \textit{ co-rotated with the plastic Eulerian principal axes}} + \overbrace{{}^p\hat{C}_k\, {}^p\dot{\hat{\mathbf{e}}}_k \otimes {}^p\hat{\mathbf{e}}_k}^{{}^p\Gamma.{}^p\mathsf{b}} + \overbrace{{}^p\hat{C}_k\, {}^p\hat{\mathbf{e}}_k \otimes {}^p\dot{\hat{\mathbf{e}}}_k}^{-{}^p\mathsf{b}.{}^p\Gamma} \tag{54}
$$

of the symmetric Eulerian left Cauchy-Green plastic deformation tensor ${}^p\mathsf{b}$ (41a) shows at the l.h.s. of Eq.(54) that the material-convectively co-rotated Eulerian unit vectors $\mathbf{e}_k$ (7) spin (27) with Ω (3); thus the ${}^p\dot{C}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ terms [cf. Eq.(29)] represent the materially Ω co-rotated Green-Naghdi rate [overbraced at the l.h.s. of Eq.(54)] which corresponds to the materially Ω co-rotated time integral

$$
\begin{aligned}
{}^p\mathsf{b}(t) &= \int_{\tau=0}^{t} \overbrace{{}^p\dot{C}_{ij}(\tau)\, \mathbf{e}_i(\tau) \otimes \mathbf{e}_j(\tau)}^{\Omega \textit{ co-rotated with the material}}\, d\tau + {}^p\mathsf{b}(0) = {}^pC_{ij}(t)\, \mathbf{e}_i \otimes \mathbf{e}_j \\
&= \int_{\tau=0}^{t} \underbrace{{}^p\dot{\hat{C}}_k(\tau)\, {}^p\hat{\mathbf{e}}_k(\tau) \otimes {}^p\hat{\mathbf{e}}_k(\tau)}_{{}^p\Gamma \textit{ co-rotated with the plastic Eulerian principal axes}}\, d\tau + {}^p\mathsf{C}(0) = {}^p\hat{C}_k(t)\, {}^p\hat{\mathbf{e}}_k(t) \otimes {}^p\hat{\mathbf{e}}_k(t)
\end{aligned} \tag{55}
$$

[overbraced at the l.h.s. of Eq.(55)]. The spectral representation at the r.h.s. of Eq.(54) reveals that the plastic Eulerian unit eigenvectors ${}^p\hat{\mathbf{e}}_k$ are spinning with ${}^p\Gamma$ [cf. Eqs.(31)–(32)]; thus the terms ${}^p\dot{\hat{C}}_k\, {}^p\hat{\mathbf{e}}_k \otimes {}^p\hat{\mathbf{e}}_k$ [in matrix component notation placed on the diagonal] represent the ${}^p\Gamma$ co-rotated tensor rate [underbraced at the r.h.s. of Eq.(54)] which corresponds to the ${}^p\Gamma$ co-rotated time integral [underbraced at the r.h.s. of Eq.(55)].

5 The geometrical interpretation of total and partial Cauchy-Green deformation tensors

The total or partial (elastic, plastic, $\cdots$) reference ${}^{@}\underline{L} = \{\underline{L}{=}d\mathbf{X}, {}^e\underline{L}, {}^p\underline{L}, \cdots\}$ and present ${}^{@}\underline{\ell} = \{\underline{\ell}{=}d\mathbf{x}, {}^e\underline{\ell}, {}^p\underline{\ell}, \cdots\}$ vicinity vectors of a particle's reference $\mathbf{X}$ and present $\mathbf{x}$ position are related to each other [analogously to Eq.(6)] by

$$
{}^{@}\underline{\ell} = (\mathsf{R}.{}^{@}\mathsf{U}).{}^{@}\underline{L} = ({}^{@}\mathsf{v}.\mathsf{R}).{}^{@}\underline{L} = {}^{@}\underline{L}.(\mathsf{R}^{\mathrm{T}}.{}^{@}\mathsf{v}) = {}^{@}\underline{L}.({}^{@}\mathsf{U}.\mathsf{R}^{\mathrm{T}}) \tag{56}
$$

and [analogously to Eq.(21)] inversely by

$$
{}^{@}\underline{L} = ({}^{@}\mathsf{U}^{-1}.\mathsf{R}^{\mathrm{T}}).{}^{@}\underline{\ell} = (\mathsf{R}^{\mathrm{T}}.{}^{@}\mathsf{v}^{-1}).{}^{@}\underline{\ell} = {}^{@}\underline{\ell}.({}^{@}\mathsf{v}^{-1}.\mathsf{R}) = {}^{@}\underline{\ell}.(\mathsf{R}.{}^{@}\mathsf{U}^{-1})\,. \tag{57}
$$

For a Lagrangean description with (56) the total and partial (elastic, plastic, $\cdots$) left (41a) and right (41b) Cauchy-Green deformation tensors may be projected onto the Lagrangean $\mathbf{E} = \dfrac{{}^{@}\underline{L}}{\|{}^{@}\underline{L}\|} = \dfrac{\mathsf{R}^{\mathrm{T}}.\{{}^{@}\mathsf{v}^{-1}.{}^{@}\underline{\ell}\}}{\|{}^{@}\mathsf{v}^{-1}.{}^{@}\underline{\ell}\|} = \mathsf{R}^{\mathrm{T}}.\mathbf{e} = \mathbf{e}.\mathsf{R}$ and Eulerian $\mathbf{e} = \mathsf{R}.\mathbf{E} = \mathbf{E}.\mathsf{R}^{\mathrm{T}}$ unit vector directions [cf. Eqs.(9a) and (9b)] of the total or partial reference vicinities ${}^{@}\underline{L} = \{\underline{L}, {}^e\underline{L}, {}^p\underline{L}\}$ in order to result in the (scalar) quadratic stretch ratios (without a *hat*)

$$
\left({}^{@}U\right)^2 = \left(\frac{\|{}^{@}\underline{\ell}\|}{\|{}^{@}\underline{L}\|}\right)^2 = \frac{{}^{@}\underline{\ell}.{}^{@}\underline{\ell}}{{}^{@}\underline{L}.{}^{@}\underline{L}} = \underbrace{\frac{1}{\|{}^{@}\underline{L}\|}{}^{@}\underline{L}}_{\mathbf{E}}.{}^{@}\mathsf{C}.\underbrace{{}^{@}\underline{L}\frac{1}{\|{}^{@}\underline{L}\|}}_{\mathbf{E}} = \mathbf{E}.{}^{@}\mathsf{C}.\mathbf{E} = \mathbf{e}.{}^{@}\mathsf{b}.\mathbf{e}\,. \tag{58}
$$

The Lagrangean deformation tensor projection (58) [multiplied by $\|{}^{@}\underline{L}\|^2$] of a right Cauchy-Green (partial) deformation tensor ${}^{@}\mathsf{C}$ exhibits according to [Ogden (1984) p.95 with Ogden's $\mathsf{A} := \mathsf{F}$] the following geometrical interpretation: since ${}^{@}\mathsf{C}$ is symmetric and positive definite, the quadratic form

$$
{}^{@}\underline{L}.{}^{@}\mathsf{C}.{}^{@}\underline{L} = \|{}^{@}\underline{\ell}\|^2 = \textit{constant} \tag{59}
$$

represents a *reciprocal Lagrangean deformation ellipsoid* with principal axes $\{{}^{@}\hat{\mathbf{E}}_1, {}^{@}\hat{\mathbf{E}}_2, {}^{@}\hat{\mathbf{E}}_3\}$ and semi-axes proportional to $\{1/{}^{@}\hat{U}_1, 1/{}^{@}\hat{U}_2, 1/{}^{@}\hat{U}_3\}$ formed from the reference vicinity vectors ${}^{@}\underline{L}$ centered at the particle's reference position vector $\mathbf{X}$ which material is mapped onto an *Eulerian sphere*

$$
{}^{@}\underline{\ell}.{}^{@}\underline{\ell} = \|{}^{@}\underline{\ell}\|^2 = \textit{constant} \tag{60}
$$

of radius $||{}^{@}\underline{\ell}||$ centered at the particle's present position vector $\mathbf{x}$. If the quadratic form (59) is forward-rotated with R and expressed by the left Cauchy-Green (partial) deformation tensors ${}^{@}\mathsf{b} = \mathsf{R}.{}^{@}\mathsf{C}.\mathsf{R}^{\mathsf{T}}$ as

$$\{{}^{@}\underline{L}.\mathsf{R}^{\mathsf{T}}\}.{}^{@}\mathsf{b}.\{\mathsf{R}.{}^{@}\underline{L}\} = ({}^{@}\mathsf{b}.\{\mathsf{R}.{}^{@}\underline{L}\}).\{\mathsf{R}.{}^{@}\underline{L}\} = ||{}^{@}\underline{\ell}||^2 = constant \tag{61}$$

then the ${}^{@}\mathsf{b}$ may geometrically be interpreted as a *reciprocal Eulerian deformation ellipsoid* with principal axes $\{{}^{@}\hat{\mathbf{e}}_1,\ {}^{@}\hat{\mathbf{e}}_2,\ {}^{@}\hat{\mathbf{e}}_3\}$ and semi-axes proportional to $\{1/{}^{@}\hat{U}_1,\ 1/{}^{@}\hat{U}_2,\ 1/{}^{@}\hat{U}_3\}$ centered at the particle's present position vector $\mathbf{x}$.
For an Eulerian description with (57) the total or partial (elastic, plastic, $\cdots$) left and right inverse Cauchy-Green deformation tensors may be projected onto the Eulerian $\mathbf{i} = \dfrac{{}^{@}\underline{\ell}}{||{}^{@}\underline{\ell}||} = \dfrac{\mathsf{R}.\{{}^{@}\mathsf{U}.{}^{@}\underline{L}\}}{||{}^{@}\mathsf{U}.{}^{@}\underline{L}||} = \mathsf{R}.\mathbf{I} = \mathbf{I}.\mathsf{R}^{\mathsf{T}}$ and Lagrangean $\mathbf{I} = \mathsf{R}^{\mathsf{T}}.\mathbf{i} = \mathbf{i}.\mathsf{R}$ unit vector directions [cf. Eqs.(9c) and (9d)] of the total or partial present vicinities ${}^{@}\underline{\ell} = \{\underline{\ell},\ {}^{e}\underline{\ell},\ {}^{p}\underline{\ell}\}$ in order to result in the (scalar) inverse quadratic stretch ratios (without a *hat*)

$$\left(\frac{1}{{}^{@}U}\right)^2 = \left(\frac{||{}^{@}\underline{L}||}{||{}^{@}\underline{\ell}||}\right)^2 = \frac{{}^{@}\underline{L}.{}^{@}\underline{L}}{{}^{@}\underline{\ell}.{}^{@}\underline{\ell}} = \underbrace{\frac{1}{||{}^{@}\underline{\ell}||}{}^{@}\underline{\ell}}_{\mathbf{i}}.{}^{@}\mathsf{b}^{-1}.\underbrace{{}^{@}\underline{\ell}\frac{1}{||{}^{@}\underline{\ell}||}}_{\mathbf{i}} = \mathbf{i}.{}^{@}\mathsf{b}^{-1}.\mathbf{i} = \mathbf{I}.{}^{@}\mathsf{C}^{-1}.\mathbf{I}\,. \tag{62}$$

The Eulerian deformation tensor projection (62) [multiplied by $||{}^{@}\underline{\ell}||^2$] of an inverse left Cauchy-Green (partial) deformation tensor ${}^{@}\mathsf{b}^{-1}$ exhibits according to [Ogden (1984) pp.94–95 with Ogden's $\mathsf{B} := \mathsf{F}^{-T}$] the following geometrical interpretation: since ${}^{@}\mathsf{b}^{-1}$ is symmetric and positive definite, the quadratic form

$${}^{@}\underline{\ell}.{}^{@}\mathsf{b}^{-1}.{}^{@}\underline{\ell} = ||{}^{@}\underline{L}||^2 = constant \tag{63}$$

represents an *Eulerian deformation ellipsoid* with principal axes $\{{}^{@}\hat{\mathbf{e}}_1,\ {}^{@}\hat{\mathbf{e}}_2,\ {}^{@}\hat{\mathbf{e}}_3\}$ and semi-axes proportional to $\{{}^{@}\hat{U}_1,\ {}^{@}\hat{U}_2,\ {}^{@}\hat{U}_3\}$ formed from the present vicinity vectors ${}^{@}\underline{\ell}$ centered at the particle's present position vector $\mathbf{x}$ which material is mapped back onto the *Lagrangean sphere*

$${}^{@}\underline{L}.{}^{@}\underline{L} = ||{}^{@}\underline{L}||^2 = constant \tag{64}$$

of radius $||{}^{@}\underline{L}||$ centered at the particle's reference position vector $\mathbf{X}$ [cf. Finger (1892) pp.1105–1122]. If the quadratic form (63) is back-rotated with R^{T} and expressed by the inverse right Cauchy-Green (partial) deformation tensors ${}^{@}\mathsf{C}^{-1} = \mathsf{R}^{\mathsf{T}}.{}^{@}\mathsf{b}^{-1}.\mathsf{R}$ as

$$\{{}^{@}\underline{\ell}.\mathsf{R}\}.{}^{@}\mathsf{C}^{-1}.\{\mathsf{R}^{\mathsf{T}}.{}^{@}\underline{\ell}\} = ({}^{@}\mathsf{C}^{-1}.\{\mathsf{R}^{\mathsf{T}}.{}^{@}\underline{\ell}\}).\{\mathsf{R}^{\mathsf{T}}.{}^{@}\underline{\ell}\} = ||{}^{@}\underline{L}||^2 = constant \tag{65}$$

then the ${}^{@}\mathsf{C}^{-1}$ may geometrically be interpreted as a *Lagrangean deformation ellipsoid* with principal axes $\{{}^{@}\hat{\mathbf{E}}_1,\ {}^{@}\hat{\mathbf{E}}_2,\ {}^{@}\hat{\mathbf{E}}_3\}$ and semi-axes proportional to $\{{}^{@}\hat{U}_1,\ {}^{@}\hat{U}_2,\ {}^{@}\hat{U}_3\}$ centered at the particle's reference position vector $\mathbf{X}$.
The total and partial (elastic, plastic, $\cdots$) stretch ratios (without a *hat*)

$${}^{@}U = \frac{||{}^{@}\underline{\ell}||}{||{}^{@}\underline{L}||} = \sqrt{\mathbf{E}.{}^{@}\mathsf{C}.\mathbf{E}} = \sqrt{\mathbf{e}.{}^{@}\mathsf{b}.\mathbf{e}} = \frac{1}{\sqrt{\mathbf{i}.{}^{@}\mathsf{b}^{-1}.\mathbf{i}}} = \frac{1}{\sqrt{\mathbf{I}.{}^{@}\mathsf{C}^{-1}.\mathbf{I}}} \tag{66}$$

of the $\mathbf{E}$, $\mathbf{e}$ or $\mathbf{i}$, $\mathbf{I}$ unit directions (9) follow from the Lagrangean (58) or Eulerian (62) deformation tensor projections, respectively.

6 The additivity of the (partial) stress power and Truesdell's hypo-elasticity

The additive split of the Eulerian (total) deformation-rate tensor

$$\mathsf{d} = \overline{{}^{e}\mathsf{d}} + \overline{{}^{p}\mathsf{d}} + \cdots \tag{67}$$

into the partial (elastic, plastic, $\cdots$) deformation rates $\{\overline{{}^{e}\mathsf{d}},\ \overline{{}^{p}\mathsf{d}},\ \cdots\}$—collectively referred to as

$$\overline{{}^{@}\mathsf{d}} = \{\mathsf{d}, \overline{{}^{e}\mathsf{d}}, \overline{{}^{p}\mathsf{d}}, \cdots\} = \tfrac{1}{2}\mathsf{F}^{-\mathsf{T}}.\dot{\overline{{}^{@}\mathsf{C}}}.\mathsf{F}^{-1} \tag{68}$$

—is a physical consequence of the *additivity of the (partial) stress power per unit mass*

$$p = \frac{1}{\rho}\,\mathsf{t}..\mathsf{d} = \overbrace{\frac{1}{\rho}\,\mathsf{t}..\overline{{}^{e}\mathsf{d}}}^{{}^{e}p} + \overbrace{\frac{1}{\rho}\,\mathsf{t}..\overline{{}^{p}\mathsf{d}}}^{{}^{p}p} + \cdots = \frac{1}{2\rho_0}\,\mathfrak{S}..\dot{\mathsf{C}} = \frac{1}{2\rho_0}\underbrace{(\mathsf{U}^{-1}.\mathsf{T}.\mathsf{U}^{-1})}_{\mathfrak{S}}..\dot{\overline{{}^{e}\mathsf{C}}} + \frac{1}{2\rho_0}\underbrace{(\mathsf{U}^{-1}.\mathsf{T}.\mathsf{U}^{-1})}_{\mathfrak{S}}..\dot{\overline{{}^{p}\mathsf{C}}} + \cdots \tag{69}$$

with the reversible elastic ${}^{e}p$ and the irreversible (plastic, $\cdots$) contributions $\{{}^{p}p, \cdots\}$—collectively referred to as

$${}^{@}p = \{p, {}^{e}p, {}^{p}p, \cdots\} = \frac{1}{\rho}\,\mathsf{t}..\overline{{}^{@}\mathsf{d}} = \frac{1}{2\rho_0}(\mathsf{U}^{-1}.\mathsf{T}.\mathsf{U}^{-1})..\dot{\overline{{}^{@}\mathsf{C}}} = \frac{1}{2\rho_0}\,\mathfrak{S}..\dot{\overline{{}^{@}\mathsf{C}}} \tag{70}$$

—where

'..' denotes the *double dot product operator* (or the *double contraction*),

t denotes the symmetric Eulerian tensor of Cauchy (1823, 1827a) stress,

$T = \frac{\rho_0}{\rho} R^T.t.R$ denotes the symmetric Lagrangean tensor of back-rotated Kirchhoff (1852) stress,

$\mathfrak{S} = U^{-1}.T.U^{-1} = \frac{\rho_0}{\rho} F^{-1}.t.F^{-T}$ denotes the symmetric Lagrangean tensor of second Piola (1825)-Kirchhoff (1852) stress

and the partial (elastic, plastic, $\cdots$) contributions of ${}^{@}d$ (43) and ${}^{@}d \neq \overline{{}^{@}d}$ (68) as well as ${}^{@}\dot{C} = \{\dot{C}, {}^{e}\dot{C}, {}^{p}\dot{C}, \cdots\} = {}^{@}\dot{U}.{}^{@}U + {}^{@}U.{}^{@}\dot{U} = 2\,{}^{@}U.R^T.{}^{@}d.R.{}^{@}U$ and ${}^{@}\dot{C} \neq \dot{\overline{{}^{@}C}} = \{\dot{C}, \dot{\overline{{}^{e}C}}, \dot{\overline{{}^{p}C}}, \cdots\} = 2\,F^T.\overline{{}^{@}d}.F$ differ in general from each other.

The notion of *hypo-elasticity* [Truesdell (1955)] for the modeling of hypo-elastic material is taken as a synonym for incremental stress-strain realations (of stress-rate and strain-rate type) [like eq.(99.4) of Truesdell&Noll (1965) p.403], which reads in our notation

$$ {}^{ZJ}\hat{\mathring{t}} = \dot{t} - w.t + t.w = \mathfrak{h}(t, d) \tag{71} $$

and which is defined with the non-material Zaremba-Jaumann rate ${}^{ZJ}\hat{\mathring{t}}$ tensor of Cauchy stress [cf. (50)]. The modeling of hypo-elastic material may then be generalized to hypo-{elastic, plastic, $\cdots$} material by applying the additive split (67) to the tensors $\overline{{}^{@}d}$ of the (partial) deformation rate (68). But how should the corresponding non-material Zaremba-Jaumann rate be defined—with the Eulerian tensor (37) of total vorticity w as

$$ {}^{ZJ}\hat{\mathring{t}} = \dot{t} - w.t + t.w = \mathfrak{h}(t, \overline{{}^{e}d}) \tag{72} $$

or with the Eulerian tensor of elastic vorticity ${}^{e}w$ [cf. Eq.(45)] as

$$ {}^{eZJ}\hat{\mathring{t}} = \dot{t} - {}^{e}w.t + t.{}^{e}w = \mathfrak{h}(t, \overline{{}^{e}d})\ ? \tag{73} $$

Since both Zaremba-Jaumann rates ${}^{ZJ}\hat{\mathring{t}}$ and ${}^{eZJ}\hat{\mathring{t}}$ are non-material rate tensors, they are both inappropriate for the modeling of anisotropic (elastic) material behavior—as pointed out by Green&McInnis (1967). A proper material description should be based on material-convective rates and Truesdell's generalized hypo-elastic material equations should better be written with the Green-Naghdi rate $\mathring{t}$ [cf. (33)] and the material-convective spin Ω tensor (3) as

$$ \mathring{t} = \dot{t} - \Omega.t + t.\Omega = \mathfrak{h}(t, \overline{{}^{e}d})\ . \tag{74} $$

For a material-convective rate (33) of a symmetric second-order Eulerian tensor [cf. (54)] the corresponding material-convective time integral of that tensor is well defined [cf. (55)]. This is *not* the case with Zaremba-Jaumann rates and other non-material rate tensors which corresponding time integrals «lose» the orientation of the material and, therefore, are inappropriate for the formulation of material anisotropy.

7 Critical discussion of non-material «co-rotational» rates and the Updated Lagrangian Formulation

The non-material Zaremba-Jaumann rate ${}^{ZJ}\hat{\mathring{t}}$ tensor of Cauchy stress [cf. (50)] only differs from the material-convective Green-Naghdi rate $\mathring{t}$ tensor [cf. (33)] if the Eulerian vorticity w tensor (37) differs from the material-convective spin Ω tensor (3) and this is, from the spectral representation at the r.h.s. of Eq.(37), only the case for moving Lagrangean principal axes, i.e. if their spin $\Lambda \neq 0$ does not vanish. The following three finite-deformation examples with moving Lagrangean principal axes exhibit flaws of the non-material Zaremba-Jaumann rate—as generally pointed out by Green&McInnis (1967) with respect to Truesdell's hypo-elasticity [see Truesdell (1955); Truesdell&Noll (1965)].

7.1 Simple shear

For the example of monotonically increasing simple finite shear [see e.g. Fig.1 of Lee et al. (1983) p.554] unphysical oscillatory shear stress is predicted for the time integrals of the Zaremba-Jaumann stress rate only [see Figs.2–3 of Lee et al. (1983) p.555; Figs.1–4 of Dafalias (1983) pp.563–564; Fig.1 of Johnson&Bammann (1984) p.736; Fig.2 of Flanagan&Taylor (1987) p.311; Figs.1–5 of Bruhns et al. (2001) pp.678–679; and many others]. These examples emphasize that not every «co-rotational» rate is appropriate for a proper material formulation.

7.2 Closed elastic deformation cycles

For the examples of closed elastic deformation cycles the corresponding tensors of the somehow time integrated Zaremba-Jaumann stress rate at the beginning and the end of a cycle deviate from each other, allegorizing a *perpetuum mobile* and, therefore, violating the conservation of energy [see e.g. Figs.1 and 2 of Kojic&Bathe (1987) pp.176 and 178; Figs.1–4, 7, 10, 12 of Meyers et al. (2003) pp.95–101; Figs.1–7 of Bruhns (2009) pp.196–203]. These examples show for closed elastic deformation cycles that Truesdell's hypo-elasticity with the Zaremba-Jaumann stress rate ${}^{ZJ}\hat{\mathring{t}}$ [cf. (50)] or with other non-material «co-rotational» rate tensors can violate the first fundamental law of thermodynamics.

7.3 Updated Lagrangian Formulation

The comparison of the deformation, depicted in Figure 8, with its time derivation and subsequent integration [based on various time integration procedures] may, especially for the so-called *Updated Lagrangian Formulation*, lead to a violation of the first fundamental law of thermodynamics [see Figs.1–5 of Heiduschke (1995a) pp.2167, 2171 and Heiduschke (1996) pp.749, 752–753].

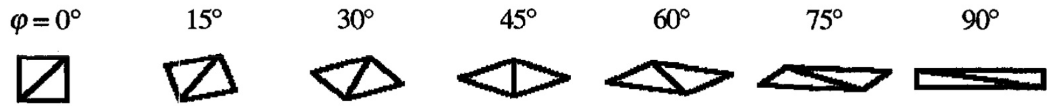

Figure 8. Two-dimensional homogeneos pure distortional (isochoric, equivoluminal) finite deformation with a material-convective rotation angle φ and constantly moving Lagrangean principal axes

When the Updated Lagrangian Formulation is applied within general-purpose finite element simulation tools (like *Dyna3D* and its derivates, *Abaqus*, *Marc*, $\cdots$) then the resulting tensor-rate integrals are also *not* integrated *convective with the material*, and these simulations do not obey the conservation of energy so that the stress and plastic strain tensor components provided just reflect unphysical house numbers.

8 Conclusion

A sound formulation of continuum mechanics requires a geometrical interpretation of the involved deformation tensors describing the total and partial (elastic, plastic, $\cdots$) deformation with respect to the tensorial orientations and magnitudes; deformation tensors which possess such a geometrical interpretation are *proper deformation tensors*. In particular the plastic Cauchy-Green deformation measures ${}^p\mathsf{b} = \mathsf{R}.{}^p\mathsf{C}.\mathsf{R}^T$ and ${}^p\mathsf{C}$ are proper deformation tensors only if they are integrated translational- and rotational-convective with the material from the corresponding plastic flow rules ${}^p\overset{\circ}{\mathsf{b}} = \mathsf{R}.{}^p\dot{\mathsf{C}}.\mathsf{R}^T = \cdots$ and ${}^p\dot{\mathsf{C}} = \cdots$ of a Green-Naghdi type. There are many non-material formulations (including Treusdell's hypo-elasticity and the Updated Lagrangian Formulation of the general-purpose finite element simulation tools {*Dyna3D* and its derivates, *Abaqus*, *Marc*, and the like} which do not follow the material translational- and rotational-convectively; these non-material formulations may even violate the energy conservation balance. A proper continuum formulation must therefore be described translational- and rotational-convective with the material, as suggested in the work at hand, where the total and partial deformation tensors are rotated with the *same* material-convective rotation R tensor (2) back and forth to their Lagrangean and Eulerian flavors. For a material-convective formulation the time derivatives of the total and partial Eulerian deformation tensors should be defined with the Green-Naghdi rate (33) which is co-rotated with the material by the material-convective rotation R tensor (2) and its associated spin Ω tensor (3). For incremental Eulerian material laws [like the hypo-elasticity of Truesdell (1955) or Truesdell&Noll (1965)], where the rates of deformation tensors are specified within the constitutive equations, the time integrals of these rates only result in proper deformation tensors when they are integrated translational- and rotational-convective with the material. Otherwise, the resulting inaccurately integrated deformation tensor components are just unphysical house numbers, which may even lead to a violation of the first fundamental law of thermodynamics, the conservation of energy, as pointed out in the critical discussion of Section 7.

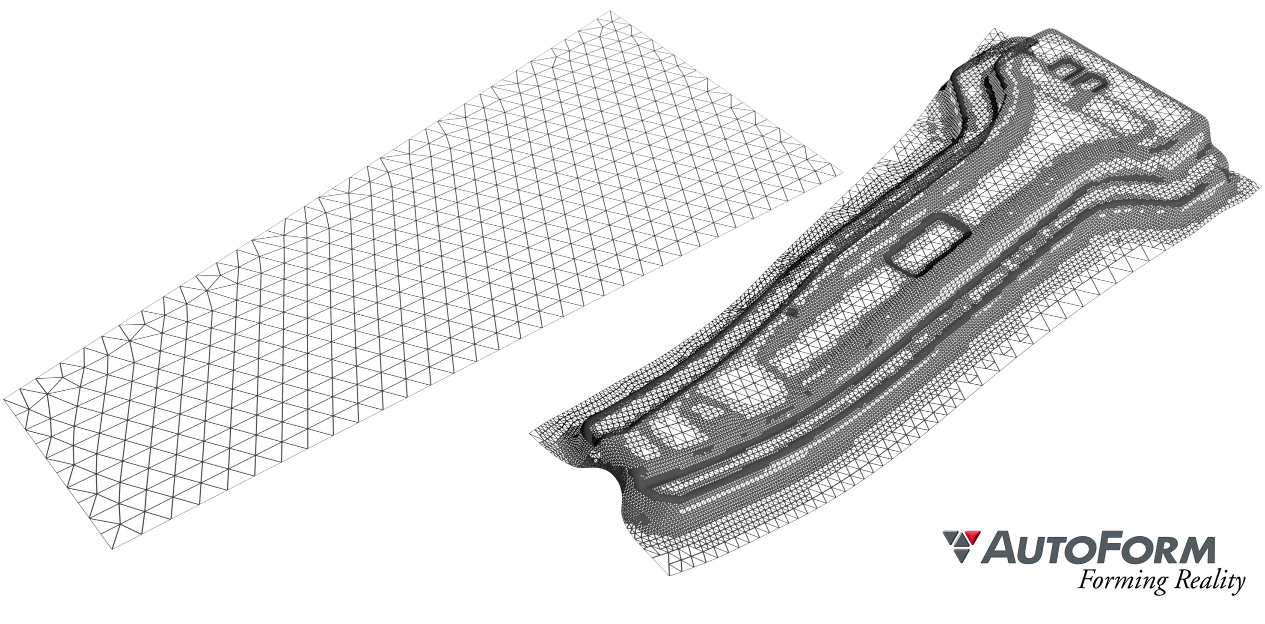

Figure 9. Initial and final configurations/finite element triangular meshes (recursively refined within the critical forming domains) for the deep drawing process of a B-pillar reinforcement modeled through a material-convective logarithmic strain space formulation

Anisotropic material behavior requires a material-convective continuum formulation as presented in this work. The logarithmic strain space formulation—which is implemented into the special-purpose finite element simulation tools *Urmel* [see Heiduschke (1998)], *Pafix* (subsequently renamed to *Hynamic*) [see Anderheggen et al. (1993); Heiduschke (1995b)], and *AutoForm* [see Anderheggen (1991); Heiduschke et al. (1991); Kubli (1996); Heiduschke (1997)]—is such a material-convective description which has proven as most accurate, stable and efficient.

A sheet metal forming process of a B-pillar reinforcement is simulated through a material-convective logarithmic strain space formulation from the initial configuration κ_0 with a triangular mesh for the plane metal sheet to the final configuration κ where the finite element mesh is automatically refined recursively within the critical forming domains, see Figure 9.

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