On the Derivation of Hooke's Law for Plane State Conditions

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Abstract: We discuss the derivation of Hooke's law for plane stress and plane strain states from its general three-dimensional representation. This means that we consider the anisotropic case to ensure a certain generality of our representation. Thereby, two approaches are examined, namely the tensorial representation involving fourth-order tensors over a three-dimensional vector space, and the Voigt-Mandel-Notation involving second-order tensors over a six-dimensional vector space. The latter reduces to a vector-matrix notation common in engineering applications. It turns out that both approaches have their merits: The tensorial approach is easier to handle symbolically, the matrix approach is easier to handle numerically. Both procedures are applicable for arbitrary material symmetries. Finally, we answer the question why a material under the assumptions of a plane stress state behaves softer and why it behaves stiffer under a plane strain state compared to the three-dimensional state.

Keywords: Plane stress, Plane strain, Elasticity, Hooke's law, Anisotropy

1 Introduction

1.1 Motivation

Commonly, the aim of continuum mechanics is to describe the motion and deformation of an arbitrary body $\mathcal{B}$ in the three-dimensional Euclidian space $\mathcal{E}^3$ under given loading. However, in general it is not possible to solve the arising system of Partial Differential Equations (PDEs) without the help of numerical solution shemes like the Finite Element Method (FEM). But more important is that it is not always necessary to solve the full system of PDEs. Often, it is possible to reduce the problem size because in one or two directions no gradients occur, for example due to the specific boundary conditions or when the extension in this direction is small compared to the other two directions. This reduction is quite remarkable: In simulations, the number of degrees of freedom at similar discretisations d is by a factor of $3d^3/(2d^2) = 1.5d$ lower.

Here we discuss the plane stress state (PT) and the plane strain state (PE). In the first state it is assumed that in the normal direction (e_3) to the plane no stress but a deformation occurs, namely a free lateral contraction. In the PE it is assumed that in the normal direction (e_3) to the plane no strains occur, hence there are out-of-plane reaction stresses to this constraint. Historically, these assumptions can be addressed to the works of Airy (1863), Maxwell (1870), Levy (1899), Flamant (1892) and Golovin (1881) among others (e.g. Michell (1899), Carothers (1914), Love (1944)). We mention these scientists to honour their work because commonly these assumptions are used without any reference. For a deeper insight in the history of the PT and PE we refer the interested reader to the review by Teodorescu (1964). The plane states described under the designations generalized plane stress or generalized plane strain remain out of the scope of present treatise.

A famous example for these two states is a gear wheel with a straight-cut and thick teeth which is presented in Fig. 1 (This is the arbitrary body $\mathcal{B}$, now.). In general, the stress state which arises in a tooth is very complex and to analyse this stress state a three-dimensional analysis is required. However, a faster and more efficient analysis is possible. For a thick teeth under loading the stress state is approximately equal to a PT at the end of the teeth (cf. Fig. 1 left-hand side) and the stress state in the middle of the teeth converges to a PE (cf. Fig. 1 right-hand side). The assumption of a PT at the end of the teeth is possible because at this position no stress in e_3 direction can occur. In the middle, the deformation is symmetric, hence no shear strains E_{13} and E_{23} and normal strain E_{33} occur. Since the teeth can hardly contract along the e_3-direction due to the attachment to the wheel, a plane strain state is a reasonable and conservative approximation in the middle plane.

The saved time could be used e.g. to consider further influences in the simulation or to perform parametric studies of the teeth geometry, etc. Another famous example for the use of plane states is the field of fracture mechanics (e.g. Phase-Field-Method (Miehe et al., 2010; Kuhn et al., 2015), Cohesive-Zone-Model (Barenblatt, 1959; Dugdale, 1960)).

By using the PT or PE assumptions and going from the 3D- to the 2D-case one needs to condense the three-dimensional elasticity law from 3D to 2D as well. Although this is not complicated, there appears to be no systematic derivation of this. While the two-dimensional PT and PE stiffnesses are well known and found in textbooks for isotropic materials (e.g. Altenbach et al. (2018); Chaboche and Lemaitre (1990)), an examination of the general anisotropic case is unknown to the authors.

1.2 Organization

The paper is organized as follows. We firstly consider the general tensorial representation that involves fourth-order tensors for both PT and PE states. We proceed by expressing the PT and PE stiffnesses and compliances in a Voigt-Mandel type
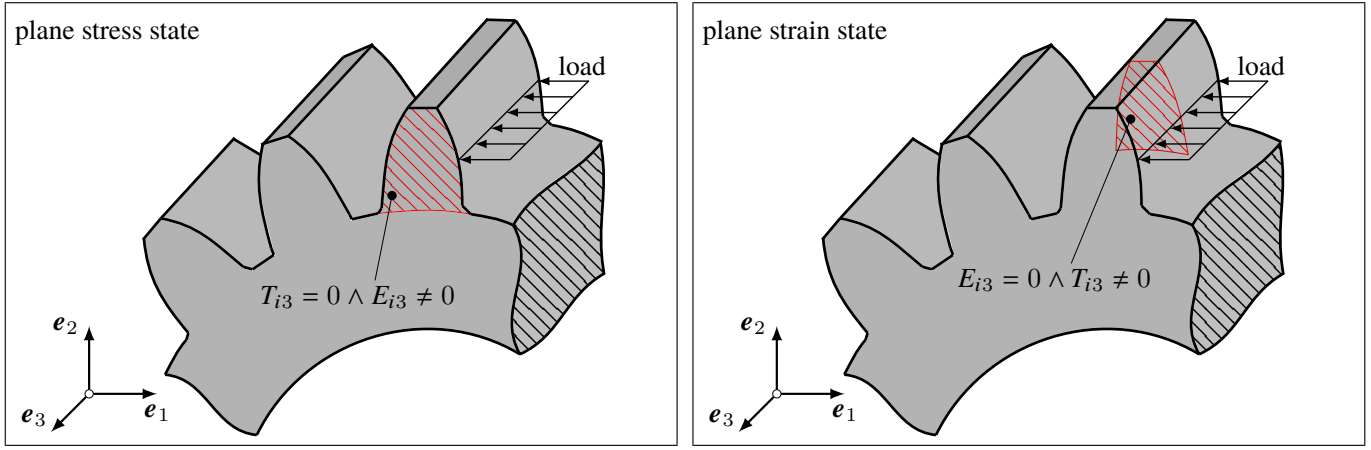
Fig. 1: Gear wheel with a straight-cut under loading based on Wittel et al. (2017) with assumed PT to analyse the stresses at the end of a tooth (left-hand side) and with assumed PE to analyse the stresses in the middle of a tooth (right-hand side) for $i \in \{1, 2, 3\}$

vector-matrix-notation where the indexing is adopted to the plane state conditions. We further on examine simplifications following from different material symmetries. Finally, problems arising of the assumptions of plane states are addressed.

1.3 Notation

We employ a direct tensor notation. Vectors are denoted as $\boldsymbol{a} = a_i \boldsymbol{e}_i$, where i is an implicit summation index and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a set of orthonormal base vectors of a Cartesian coordinate system. The scalar product is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$. For example, for the base vectors we have $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$, with the KRONECKER symbol δ_{ij} being 1 for $i = j$ and zero for $i \neq j$. Higher order tensors are obtained by the dyadic product and denoted by capital letters, where $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b} = A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ is an example for a second-order tensor, $\mathbb{C} = \boldsymbol{A} \otimes \boldsymbol{B} = C_{ijkl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$ is an example for a fourth-order tensor, and $\mathfrak{E} = \boldsymbol{A} \otimes \boldsymbol{B} \otimes \boldsymbol{C} \otimes \boldsymbol{D} = E_{ijklmnop} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \otimes \boldsymbol{e}_m \otimes \boldsymbol{e}_n \otimes \boldsymbol{e}_o \otimes \boldsymbol{e}_p$ is an example for an eighth-order tensor. A tensor-vector product is denoted as $\boldsymbol{A} \cdot \boldsymbol{a} = A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot a_l \boldsymbol{e}_l = A_{ij} a_l \boldsymbol{e}_i (\boldsymbol{e}_j \cdot \boldsymbol{e}_l) = A_{ij} a_l \delta_{jl} \boldsymbol{e}_i = A_{ij} a_j \boldsymbol{e}_i$. The scalar contractions are extended to higher order tensors such that the positivity of the scalar product is maintained, e.g. $\boldsymbol{A} : \boldsymbol{A} = A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j : A_{kl} \boldsymbol{e}_k \otimes \boldsymbol{e}_l = A_{ij} A_{kl} \delta_{ik} \delta_{jl} = A_{ij} A_{ij}$. We additionally introduce the Rayleigh product which maps all basis vectors of a tensor simultaneously without changing components. When applied between dyad and a tetrad, the product is $\boldsymbol{B} \star \mathbb{A} = A_{ijkl} (\boldsymbol{B} \cdot \boldsymbol{e}_i) \otimes (\boldsymbol{B} \cdot \boldsymbol{e}_j) \otimes (\boldsymbol{B} \cdot \boldsymbol{e}_k) \otimes (\boldsymbol{B} \cdot \boldsymbol{e}_l)$ with components $A_{ijkl} = \mathbb{A} :: \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$. A tensor is said to be symmetric when $\boldsymbol{a} \cdot \boldsymbol{A} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{A} \cdot \boldsymbol{a}$ holds. With respect to orthonormal bases, this reduces to the index symmetry $A_{ij} = A_{ji}$. The tensor components may be arranged in a matrix, where the index symmetry becomes a transposition across the main diagonal. Hence, for second-order tensors we denote the symmetry by $\boldsymbol{A} = \boldsymbol{A}^{\mathsf{T}}$, i.e. $\boldsymbol{A} \in \mathcal{S}ym$. For detailed explanations of these operations we refer to e.g. Bertram and Glüge (2015). Also, we make use of numerical vectors and matrices which are both indicated by the superscript index M.

$$\boldsymbol{E}^{\mathrm{M}} = \begin{bmatrix} E_1^{\mathrm{M}} \\ E_2^{\mathrm{M}} \\ E_3^{\mathrm{M}} \end{bmatrix} \qquad\qquad \mathbb{C}^{\mathrm{M}} = \begin{bmatrix} C_{11}^{\mathrm{M}} & C_{12}^{\mathrm{M}} & C_{13}^{\mathrm{M}} \\ C_{21}^{\mathrm{M}} & C_{22}^{\mathrm{M}} & C_{23}^{\mathrm{M}} \\ C_{31}^{\mathrm{M}} & C_{32}^{\mathrm{M}} & C_{33}^{\mathrm{M}} \end{bmatrix}$$

With regard to vector-matrix calculations we refer to Brannon (2018). This representation is used to analyse some properties of the 2D stiffness and compliance matrix, later on.

2 Continuum Mechanical Propaedeutics

The formulation of the common quasi-static boundary value problem under small deformations for a linear-elastic body is given as follows.

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = \boldsymbol{0} \qquad\qquad \text{bal. of linear momentum} \qquad (1)$$

$$\boldsymbol{T} = \mathbb{C} : \boldsymbol{E} \qquad\qquad \text{constitutive law} \qquad (2)$$

$$\boldsymbol{E} = \frac{1}{2} \left[\boldsymbol{\nabla} \otimes \boldsymbol{u} + (\boldsymbol{\nabla} \otimes \boldsymbol{u})^{\top} \right] \qquad\qquad \text{strain-displacement-relation} \qquad (3)$$

Herein, $\boldsymbol{T} = T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ is the stress tensor, $\mathbb{C} = C_{ijkl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$ is the constitutive tensor, $\boldsymbol{E} = E_{kl} \boldsymbol{e}_k \otimes \boldsymbol{e}_l$ is the linearized strain tensor and $\boldsymbol{\nabla} = \partial/\partial x_i$ represents the nabla operator with respect to the reference configuration. The first index of the stress and strain tensor refers to the force, respectively, displacement direction and the second index refers to the surface normal.

2.1 Three-Dimensional Hooke's Law

We focus on Hooke's law (2), which contains the stiffness tetrad.

$$\mathbb{C} = C_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{4}$$

Further on, $\boldsymbol{T} \in \mathcal{S}ym$ and $\boldsymbol{E} \in \mathcal{S}ym$ hold true, along with the following symmetry properties for the constitutive tensor.

$\boldsymbol{A}\!:\!\mathbb{C}\!:\!\boldsymbol{B} = \boldsymbol{B} : \mathbb{C}\!:\!\boldsymbol{A}$	$C_{ijkl} = C_{klij}$	major symmetry
$\boldsymbol{A}\!:\!\mathbb{C}\quad = \boldsymbol{A}^\top\!:\!\mathbb{C}$	$C_{ijkl} = C_{jikl}$	left subsymmetry
$\mathbb{C}\!:\!\boldsymbol{A} = \quad \mathbb{C}\!:\!\boldsymbol{A}^\top$	$C_{ijkl} = C_{ijlk}$	right subsymmetry

Therein, $\boldsymbol{A}$ and $\boldsymbol{B}$ are chosen arbitrary. The subsymmetries basically denote the restriction to the subspace of symmetric second-order tensors. The major symmetry is the integrability condition to guarantee that the energy density yields following form.

$$w = \frac{1}{2}\,\boldsymbol{E}\!:\!\mathbb{C}\!:\!\boldsymbol{E} \tag{5}$$

For the potential relation subsequent expression holds.

$$\boldsymbol{T} = \frac{\partial w}{\partial \boldsymbol{E}} \tag{6}$$

Carrying out the inversion of $\mathbb{C}$ on the subspace of symmetric tensors due to the positive definiteness of the stiffness tetrad w.r.t. this space, we obtain the inverse form of Hooke's law.

$$\boldsymbol{E} = \mathbb{S}\!:\!\boldsymbol{T} \tag{7}$$

Herein, the fourth-order compliance tetrad is used.

$$\mathbb{S} = \mathbb{C}^{-1} = S_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l, \tag{8}$$

For $\mathbb{S}$, the same symmetries hold as for $\mathbb{C}$. The complementary energy density is defined as follows.

$$w^* = \frac{1}{2}\,\boldsymbol{T}\!:\!\mathbb{S}\!:\!\boldsymbol{T}, \tag{9}$$

The potential relation then yields following expression.

$$\boldsymbol{E} = \frac{\partial w^*}{\partial \boldsymbol{T}}. \tag{10}$$

In linear elasticity we have $w = w^*$. In nonlinear elasticity this does not hold, and it is often not possible to obtain the explicit expression of the corresponding dual energy, formally related to each other by the Legendre transform.

In absence of material symmetries, $\mathbb{C}$ has 21 independent components. If a material symmetry is present, it is usually formalized as a subgroup of the special orthogonal group (Weyl, 1939; Zheng and Boehler, 1994). Then, the symmetry condition $\mathbb{C} = \boldsymbol{Q}_i * \mathbb{C}$ is used to derive constraints on $\mathbb{C}$ for a set of generators $\boldsymbol{Q}_i$ of the symmetry group, where the product $*$ is the rotation of the stiffness with the material. Other approaches are possible, for example by using mirror planes. It is known that eight elastic symmetries can be distinguished (Voigt, 1910; Nye, 1985; Cowin and Mehrabadi, 1987; Forte and Vianello, 1996)[1]. These can be found, for example, in Weber et al. (2018) in terms of the tensorial notation and in Ting (1996, 2003) in the Voigt notation.

2.2 Two-Dimensional Hooke's Law

As it is mentioned in Sect. 1 many problems allow the reduction of the system of PDEs (Eqs. (1)-(3)) by assuming a PT or PE. However, these assumptions lead to changes of the components of the constitutive tensor $\mathbb{C}$ because under the PT a deformation in the third direction is possible but it results no stress, vice versa, under the PE no deformation in the third direction results but it remains a stress. However, the form of Eqs. (1)-(3) does not change under a PT or a PE, but we have one less independent equation. Hence, we can write the plane system of PDEs.

$$\boldsymbol{\nabla}^{2D}\!\cdot\!\boldsymbol{T}^{2D} = \boldsymbol{0} \qquad\qquad \text{bal. of linear momentum} \tag{11}$$

$$\boldsymbol{T}^{2D} = \mathbb{C}^{2D}\!:\!\boldsymbol{E}^{2D} \qquad\qquad \text{constitutive law} \tag{12}$$

$$\boldsymbol{E}^{2D} = \frac{1}{2}\left[\boldsymbol{\nabla}^{2D}\otimes\boldsymbol{u}^{2D} + (\boldsymbol{\nabla}^{2D}\otimes\boldsymbol{u}^{2D})^\top\right] \qquad\qquad \text{strain-displacement-relation} \tag{13}$$

The inverse form of Hooke's law is then given as follows.

$$\boldsymbol{E}^{2D} = \mathbb{S}^{2D}\!:\!\boldsymbol{T}^{2D} \tag{14}$$

The quantities with the superscript 2D are reduced by negligence of the third direction (we project in the $\boldsymbol{e}_1-\boldsymbol{e}_2$ plane with normal vector $\boldsymbol{n} = \boldsymbol{e}_3$), hence, the indices i, j, k, l are substituted by $\alpha, \beta, \gamma, \delta$ and run through the values $1, 2$ instead of $1, 2, 3$. Here, we present a general representation of the stiffness tetrad $\mathbb{C}^{2D}$ which is fully derived from the common anisotropic stiffness tetrad $\mathbb{C}$, thus, it is possible to use Eqs. (11)-(13).

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[1]Note that Gurtin (1972) gives 10 different symmetries, namely isotropy and 9 different anisotropic matrices on pages 87 to 89.

3 Generalized Representation of Plane State Conditions

3.1 Plane Stress State

The first step is to rewrite Eq. (1) as follows.

$$T_{\alpha\beta} = 2C_{\alpha\beta 13}E_{13} + 2C_{\alpha\beta 23}E_{23} + C_{\alpha\beta 33}E_{33} + C_{\alpha\beta\gamma\delta}E_{\gamma\delta} \tag{15}$$

$$T_{13} = 2C_{1313}\,E_{13} + 2C_{1323}\,E_{23} + C_{1333}\,E_{33} + C_{13\alpha\beta}\,E_{\alpha\beta} \tag{16}$$

$$T_{23} = 2C_{2313}\,E_{13} + 2C_{2323}\,E_{23} + C_{2333}\,E_{33} + C_{23\alpha\beta}\,E_{\alpha\beta} \tag{17}$$

$$T_{33} = 2C_{3313}\,E_{13} + 2C_{3323}\,E_{23} + C_{3333}\,E_{33} + C_{33\alpha\beta}\,E_{\alpha\beta} \tag{18}$$

Hence, all quantities for the third direction (index 3) are separated. The PT claims that $T_{13} = T_{23} = T_{33} = 0$, hence, we solve Eqs. (16)-(18) for E_{13}, E_{23} and E_{33}. Keep in mind that the stiffness tetrad has the major symmetry, thus, the subsequent components are identically equal which leads to simplifications.

$$C_{3313} = C_{1333} \qquad\qquad C_{3323} = C_{2333} \qquad\qquad C_{1323} = C_{2313}$$

Further on, we introduce following abbreviations a_{ijkl} for a more compact notation.

$$a_{1313} = C_{3323}^2 \quad\;\; -C_{2323}C_{3333}, \quad a_{2313} = C_{1323}C_{3333} - C_{3313}C_{3323}, \quad a_{3313} = C_{2323}C_{3313} - C_{1323}C_{3323},$$

$$a_{1323} = C_{1323}C_{3333} - C_{3313}C_{3323}, \quad a_{2323} = C_{3313}^2 \quad\;\; -C_{1313}C_{3333}, \quad a_{3323} = C_{1313}C_{3323} - C_{1323}C_{3313},$$

$$a_{1333} = C_{2323}C_{3313} - C_{1323}C_{3323}, \quad a_{2333} = C_{1313}C_{3323} - C_{1323}C_{3313}, \quad a_{3333} = C_{1323}^2 \quad\;\; -C_{1313}C_{2323}.$$

The third and fourth index of these abbreviations refer to the indices of the strain component while the first and second index are used for summation. Next to that a normalisation of the correction term Θ is introduced.

$$\Theta = C_{3333}a_{3333} - 2C_{1323}C_{3313}C_{3323} + C_{2323}C_{3313}^2 + C_{1313}C_{3323}^2$$

This term is used due to differences between 3D and plane states which are resulting from constraints of the assumptions of PT and PE. The following results upon solving Eqs. (16)-(18).

$$E_{13} = -\frac{1}{2\Theta}\left(a_{1313}C_{13\alpha\beta} + a_{2313}C_{23\alpha\beta} + a_{3313}C_{33\alpha\beta}\right)E_{\alpha\beta} \tag{19}$$

$$E_{23} = -\frac{1}{2\Theta}\left(a_{1323}C_{13\alpha\beta} + a_{2323}C_{23\alpha\beta} + a_{3323}C_{33\alpha\beta}\right)E_{\alpha\beta} \tag{20}$$

$$E_{33} = -\frac{1}{\Theta}\left(a_{1333}C_{13\alpha\beta} + a_{2333}C_{23\alpha\beta} + a_{3333}C_{33\alpha\beta}\right)E_{\alpha\beta} \tag{21}$$

Inserting Eqs. (19)-(21) into Eq. (15) leads to three fourth-order tensors which are the corrections of the stiffness tetrad.

$$A_{\alpha\beta\gamma\delta}^{13} = C_{\alpha\beta 13}\left(a_{1313}C_{13\gamma\delta} + a_{2313}C_{23\gamma\delta} + a_{3313}C_{33\gamma\delta}\right)$$

$$A_{\alpha\beta\gamma\delta}^{23} = C_{\alpha\beta 23}\left(a_{1323}C_{13\gamma\delta} + a_{2323}C_{23\gamma\delta} + a_{3323}C_{33\gamma\delta}\right)$$

$$A_{\alpha\beta\gamma\delta}^{33} = C_{\alpha\beta 33}\left(a_{1333}C_{13\gamma\delta} + a_{2333}C_{23\gamma\delta} + a_{3333}C_{33\gamma\delta}\right)$$

The superscript index $i3$ here refers to the corresponding strain measures. These three tensors can be used to build a general correction tensor.

$$\mathbb{A} = \mathbb{A}^{13} + \mathbb{A}^{23} + \mathbb{A}^{33} \tag{22}$$

In index notation this tensor is represented as follows.

$$A_{\alpha\beta\gamma\delta} = a_{i3k3}C_{\alpha\beta k3}C_{i3\gamma\delta}$$

A direct notation is possible, also.

$$\mathbb{A} = a_{i3k3}\left(\mathbb{C}\colon e_k \otimes e_3\right) \otimes \left(e_i \otimes e_3\colon \mathbb{C}\right) \tag{23}$$

Finally, it follows the representation of the stiffness tetrad for the PT.

$$\mathbb{C}^{\mathrm{PT}} = \mathbb{C} - \frac{1}{\Theta}\mathbb{A} \qquad\qquad\Longleftrightarrow\qquad\qquad C_{\alpha\beta\gamma\delta}^{\mathrm{PT}} = C_{\alpha\beta\gamma\delta} - \frac{1}{\Theta}a_{i3k3}C_{\alpha\beta k3}C_{i3\gamma\delta} \tag{24}$$

With this corrected stiffness tetrad HOOKE's law under a PT takes subsequent form.

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$$T^{\text{PT}} = \mathbb{C}^{\text{PT}} : E^{\text{PT}} = \left[\mathbb{C} - \frac{1}{\Theta}\mathbb{A}\right] : E^{\text{PT}}$$

$$\updownarrow \tag{25}$$

$$T^{\text{PT}}_{\alpha\beta} = C^{\text{PT}}_{\alpha\beta\gamma\delta} E^{\text{PT}}_{\gamma\delta} = \left[C_{\alpha\beta\gamma\delta} - \frac{1}{\Theta} a_{i3k3} C_{\alpha\beta k3} C_{i3\gamma\delta}\right] E^{\text{PT}}_{\gamma\delta}$$

3.2 Plane Strain State

For the PE we begin with Eq. (7). Up next, the third direction (index 3) is separated from this set of equations, like it was done before. Subsequent relations result.

$$E_{\alpha\beta} = 2S_{\alpha\beta13}T_{13} + 2S_{\alpha\beta23}T_{23} + S_{\alpha\beta33}T_{33} + S_{\alpha\beta\gamma\delta}T_{\gamma\delta} \tag{26}$$

$$E_{13} = 2S_{1313}\,T_{13} + 2S_{1323}\,T_{23} + S_{1333}\,T_{33} + S_{13\alpha\beta}T_{\alpha\beta} \tag{27}$$

$$E_{23} = 2S_{2313}\,T_{13} + 2S_{2323}\,T_{23} + S_{2333}\,T_{33} + S_{23\alpha\beta}T_{\alpha\beta} \tag{28}$$

$$E_{33} = 2S_{3313}\,T_{13} + 2S_{3323}\,T_{23} + S_{3333}\,T_{33} + S_{33\alpha\beta}T_{\alpha\beta} \tag{29}$$

The PE claims that $E_{13} = E_{23} = E_{33} = 0$, hence, we solve Eqs. (27)-(29) for T_{13}, T_{23} and T_{33}. Keep in mind that the compliance tetrad has the major symmetry (like the stiffness tetrad), thus, following components are identically equal which leads to simplifications.

$$S_{3313} = S_{1333} \qquad\qquad S_{3323} = S_{2333} \qquad\qquad S_{1323} = S_{2313}$$

Further on, some abbreviations for a more compact notation are introduced.

$$b_{1313} = S^2_{3323} \qquad -S_{2323}S_{3333} \qquad b_{2313} = S_{1323}S_{3333} - S_{3313}S_{3323} \qquad b_{3313} = S_{2323}S_{3313} - S_{1323}S_{3323}$$

$$b_{1323} = S_{1323}S_{3333} - S_{3313}S_{3323} \qquad b_{2323} = S^2_{3313} \qquad -S_{1313}S_{3333} \qquad b_{3323} = S_{1313}S_{3323} - S_{1323}S_{3313}$$

$$b_{1333} = S_{2323}S_{3313} - S_{1323}S_{3323} \qquad b_{2333} = S_{1313}S_{3323} - S_{1323}S_{3313} \qquad b_{3333} = S^2_{1323} \qquad -S_{1313}S_{2323}$$

We furthermore introduce a normalisation of the correction term.

$$\Phi = S_{3333}b_{3333} - 2S_{1323}S_{3313}S_{3323} + S_{2323}S^2_{3313} + S_{1313}S^2_{3323}$$

It results the solution of Eqs. (27)-(29)

$$T_{13} = -\frac{1}{2\Phi}\left(b_{1313}S_{13\alpha\beta} + b_{2313}S_{23\alpha\beta} + b_{3313}S_{33\alpha\beta}\right)T_{\alpha\beta} \tag{30}$$

$$T_{23} = -\frac{1}{2\Phi}\left(b_{1323}S_{13\alpha\beta} + b_{2323}S_{23\alpha\beta} + b_{3323}S_{33\alpha\beta}\right)T_{\alpha\beta} \tag{31}$$

$$T_{33} = -\frac{1}{\Phi}\left(b_{1333}S_{13\alpha\beta} + b_{2333}S_{23\alpha\beta} + b_{3333}S_{33\alpha\beta}\right)T_{\alpha\beta} \tag{32}$$

Inserting Eqs. (30)-(32) into Eq. (26) leads to three fourth-order tensors which are the corrections of the compliance tetrad.

$$B^{13}_{\alpha\beta\gamma\delta} = S_{\alpha\beta13}\left(b_{1313}S_{13\gamma\delta} + b_{2313}S_{23\gamma\delta} + b_{3313}S_{33\gamma\delta}\right)$$

$$B^{23}_{\alpha\beta\gamma\delta} = S_{\alpha\beta23}\left(b_{1323}S_{13\gamma\delta} + b_{2323}S_{23\gamma\delta} + b_{3323}S_{33\gamma\delta}\right)$$

$$B^{33}_{\alpha\beta\gamma\delta} = S_{\alpha\beta33}\left(b_{1333}S_{13\gamma\delta} + b_{2333}S_{23\gamma\delta} + b_{3333}S_{33\gamma\delta}\right)$$

The superscript index $i3$ here refer to the corresponding stress measures. These three tensors are used to build a correction tensor for plane strain state.

$$\mathbb{B} = \mathbb{B}^{13} + \mathbb{B}^{23} + \mathbb{B}^{33} \tag{33}$$

In index notation this tensor is represented as follows.

$$B_{\alpha\beta\gamma\delta} = b_{i3k3}S_{\alpha\beta k3}S_{i3\gamma\delta}$$

A direct notation is possible, also.

$$\mathbb{B} = b_{i3k3}\left(\mathbb{S} : e_k \otimes e_3\right) \otimes \left(e_i \otimes e_3 : \mathbb{S}\right) \tag{34}$$

Finally, it follows the representation of the compliance tetrad for the plane strain state as

$$\mathbb{S}^{\text{PE}} = \mathbb{S} - \frac{1}{\Phi}\mathbb{B} \qquad\qquad \Longleftrightarrow \qquad\qquad S^{\text{PE}}_{\alpha\beta\gamma\delta} = S_{\alpha\beta\gamma\delta} - \frac{1}{\Phi}b_{i3k3}S_{\alpha\beta k3}S_{i3\gamma\delta} \tag{35}$$

With this corrected compliance tetrad, inverse HOOKE's law under a plane strain state takes subsequent form.

$$E^{\mathrm{PE}} = \mathbb{S}^{\mathrm{PE}} \colon T^{\mathrm{PE}} = \left[\mathbb{S} - \frac{1}{\Phi} \mathbb{B} \right] \colon T^{\mathrm{PE}}$$

$$\updownarrow \tag{36}$$

$$E^{\mathrm{PE}}_{\alpha\beta} = S^{\mathrm{PE}}_{\alpha\beta\gamma\delta} T^{\mathrm{PE}}_{\gamma\delta} = \left[S_{\alpha\beta\gamma\delta} - \frac{1}{\Phi} b_{i3k3} S_{\alpha\beta k3} S_{i3\gamma\delta} \right] T^{\mathrm{PE}}_{\gamma\delta}$$

4   Matrix Representation

4.1   Normalized 6D Basis

We have seen that the plane stiffnesses and compliances can be obtained from solving three scalar equations, with successive backward substitution. Another approach is to consider sub-matrices in the matrix-vector notation of HOOKE's law. But care must be taken: For the usual rules of algebra to hold, the basis needs to be normalized. This leads to the so called MANDEL-notation (index $\square^{\mathrm{M}}$), see for example Brannon (2018) Sects. 26.2 and 26.3, Helnwein (2001) or Cowin and Mehrabadi (1992). The normalization was popularized by Mandel (1965), but has been used much earlier by LORD KELVIN, cf. Thomson (1856). Since all involved second-order tensors are symmetric, we introduce a symmetrized basis $B_\Gamma \, \forall \, \Gamma \in \{1, \dots, 6\}$.

$$B_1 = e_1 \otimes e_1 \qquad\qquad\qquad B_4 = e_3 \otimes e_3$$

$$B_2 = e_2 \otimes e_2 \qquad\qquad\qquad B_5 = \frac{1}{\sqrt{2}} \, (e_1 \otimes e_3 + e_3 \otimes e_1) \tag{37}$$

$$B_3 = \frac{1}{\sqrt{2}} \, (e_1 \otimes e_2 + e_2 \otimes e_1) \qquad\qquad\qquad B_6 = \frac{1}{\sqrt{2}} \, (e_2 \otimes e_3 + e_3 \otimes e_2)$$

Herein we have introduced capital greek indices having the values $1 \ldots 6$. The indices of the tensor components ij are substituted with them (Γ) as follows.

$$\{11 \leftrightarrow 1, 22 \leftrightarrow 2, 12 \leftrightarrow 3, 33 \leftrightarrow 4, 13 \leftrightarrow 5, 23 \leftrightarrow 6\}$$

This basis is orthogonal due to the normalizing factor $1/\sqrt{2}$.

$$B_\Gamma \colon B_\Lambda = \delta_{\Gamma\Lambda} \qquad\qquad\qquad \forall \, \Gamma, \Lambda \in \{1, \dots, 6\} \tag{38}$$

Our ordering deviates from the VOIGT ordering. It is such that the in- and out-of-plane components of the 1-2-plane have the indices $1 \ldots 3$ and $4 \ldots 6$, respectively. Comparing coefficients with the $e_i \otimes e_j$-basis we can now write a compact form.

$$E = E_{ij} \, e_i \otimes e_j \qquad\qquad\qquad \forall \, i, j \in \{1, \dots, 3\} \tag{39}$$

$$ = E^{\mathrm{M}}_\Lambda B_\Lambda \qquad\qquad\qquad \forall \, \Lambda \in \{1, \dots, 6\} \tag{40}$$

In case of the strain tensor, the components appear as follows.

$$E^{\mathrm{M}}_1 = E_{11} \qquad\qquad\qquad E^{\mathrm{M}}_4 = E_{33} \tag{41}$$

$$E^{\mathrm{M}}_2 = E_{22} \qquad\qquad\qquad E^{\mathrm{M}}_5 = \sqrt{2} \, E_{13} \tag{42}$$

$$E^{\mathrm{M}}_3 = \sqrt{2} \, E_{12} \qquad\qquad\qquad E^{\mathrm{M}}_6 = \sqrt{2} \, E_{23} \tag{43}$$

The same factors apply when going from T_{ij} to T^{M}_Γ. Regarding the stiffness and compliance tensors, we can also find a compact expression.

$$\mathbb{C} = C_{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l \qquad\qquad\qquad \forall \, i, j, k, l \in \{1, \dots, 3\} \tag{44}$$

$$\phantom{\mathbb{C}} = C^{\mathrm{M}}_{\Gamma\Lambda} B_\Gamma \otimes B_\Lambda \qquad\qquad\qquad \forall \, \Gamma, \Lambda \in \{1, \dots, 6\} \tag{45}$$

The components $C^{\mathrm{M}}_{\Gamma\Lambda}$ appear as follows, while $C^{\mathrm{M}}_{\Gamma\Lambda} = C^{\mathrm{M}}_{\Lambda\Gamma}$ holds.

$$
\begin{array}{llllll}
C^{\mathrm{M}}_{11}{=}C_{1111} & C^{\mathrm{M}}_{12}{=}C_{1122} & C^{\mathrm{M}}_{13}{=}\sqrt{2}C_{1112} & C^{\mathrm{M}}_{14}{=}\phantom{\sqrt{2}}C_{1133} & C^{\mathrm{M}}_{15}{=}\sqrt{2}C_{1113} & C^{\mathrm{M}}_{16}{=}\sqrt{2}C_{1123} \\
& C^{\mathrm{M}}_{22}{=}C_{2222} & C^{\mathrm{M}}_{23}{=}\sqrt{2}C_{2212} & C^{\mathrm{M}}_{24}{=}\phantom{\sqrt{2}}C_{2233} & C^{\mathrm{M}}_{25}{=}\sqrt{2}C_{2213} & C^{\mathrm{M}}_{26}{=}\sqrt{2}C_{2223} \\
& & C^{\mathrm{M}}_{33}{=}\phantom{\sqrt{}}2C_{1212} & C^{\mathrm{M}}_{34}{=}\sqrt{2}C_{1233} & C^{\mathrm{M}}_{35}{=}\phantom{\sqrt{}}2C_{1213} & C^{\mathrm{M}}_{36}{=}\phantom{\sqrt{}}2C_{1223} \\
& & & C^{\mathrm{M}}_{44}{=}\phantom{\sqrt{2}}C_{3333} & C^{\mathrm{M}}_{45}{=}\sqrt{2}C_{3313} & C^{\mathrm{M}}_{46}{=}\sqrt{2}C_{3323} \\
& & & & C^{\mathrm{M}}_{55}{=}\phantom{\sqrt{}}2C_{1313} & C^{\mathrm{M}}_{56}{=}\phantom{\sqrt{}}2C_{1323} \\
& & & & & C^{\mathrm{M}}_{66}{=}\phantom{\sqrt{}}2C_{2323}
\end{array}
\tag{46}
$$

The same holds for $\mathbb{S}$.

4.2 Matrix-Representation of HOOKE's Law

HOOKE's law can now be written as follows.

$$
\begin{bmatrix} T_1^M \\ T_2^M \\ T_3^M \\ T_4^M \\ T_5^M \\ T_6^M \end{bmatrix} = \left[\begin{array}{ccc|ccc} C_{11}^M & C_{12}^M & C_{13}^M & C_{14}^M & C_{15}^M & C_{16}^M \\ & C_{22}^M & C_{23}^M & C_{24}^M & C_{25}^M & C_{26}^M \\ & & C_{33}^M & C_{34}^M & C_{35}^M & C_{36}^M \\ \hline & & & C_{44}^M & C_{45}^M & C_{46}^M \\ & & & & C_{55}^M & C_{56}^M \\ \text{sym} & & & & & C_{66}^M \end{array} \right] \begin{bmatrix} E_1^M \\ E_2^M \\ E_3^M \\ E_4^M \\ E_5^M \\ E_6 \end{bmatrix}
$$

$$
= \left[\begin{array}{c|c} \mathbb{C}_{UL}^M & \mathbb{C}_{UR}^M \\ \hline \mathbb{C}_{LL}^M & \mathbb{C}_{LR}^M \end{array} \right] \begin{bmatrix} E_{ip}^M \\ E_{op}^M \end{bmatrix} \qquad \qquad \mathbb{C}_{UR}^M = \left[\mathbb{C}_{LL}^M \right]^T \tag{47}
$$

For the compact form of Eq. (47) we have introduced the subscripts UL for upper left, UR for upper right, LL for lower left, LR for lower right, ip for in-plane, and op for out-of-plane. We can also write this law in terms of components.

$$
T_\Gamma^M = C_{\Gamma\Lambda}^M E_\Lambda^M \tag{48}
$$

$$
E_\Gamma^M = S_{\Gamma\Lambda}^M T_\Lambda^M \qquad\qquad \text{with} \quad S_{\Gamma\Lambda}^M = \left[C_{\Gamma\Lambda}^M \right]^{-1} \tag{49}
$$

Note that $S_{\Gamma\Lambda}^M$ is the usual matrix inverse. In case of a PE or a PT we have either $E_4^M = 0, E_5^M = 0, E_6^M = 0$ (PE) or $T_4^M = 0, T_5^M = 0, T_6^M = 0$ (PT). Therefore, the in-plane components of T (PE) or E (PT) are obtained by restricting $\mathbb{C}$ in HOOKE's law to the components of the upper left 3×3 block matrix $\mathbb{C}_{UL}^M$ or by restricting $\mathbb{S}$ in the inverse HOOKE's law to the components of the upper left 3×3 block matrix $\mathbb{S}_{UL}^M$, respectively. We denote this restriction by $\mathfrak{P}$, where "P" stands for orthogonal projection onto a subspace.

$$
\mathfrak{P}(\mathbb{C}) = \begin{bmatrix} C_{11}^M & C_{12}^M & C_{13}^M \\ & C_{22}^M & C_{23}^M \\ \text{sym} & & C_{33}^M \end{bmatrix} B_\Gamma \otimes B_\Lambda = C_{\Gamma\Lambda}^M B_\Gamma \otimes B_\Lambda \qquad\qquad \forall \Gamma, \Lambda \quad \in \{1, \dots, 3\} \tag{50}
$$

If we pad the kernel or Null space, $\mathfrak{P}$ is an 8$^\text{th}$ order tensor.

$$
\mathfrak{P} = \frac{1}{2} (\delta_{\Theta\Gamma}\delta_{\Lambda\Delta} + \delta_{\Theta\Delta}\delta_{\Lambda\Gamma}) B_\Theta \otimes B_\Lambda \otimes B_\Gamma \otimes B_\Delta \qquad\qquad \forall \Theta, \Lambda, \Gamma, \Delta \in \{1, \dots, 3\} \tag{51}
$$

The projection is obtained as a scalar contraction $\mathfrak{P} :: \mathbb{C}$. We can then restrict attention to the in-plane components, which can easily be inverted.

$$
\text{PE}: \begin{bmatrix} T_1^M \\ T_2^M \\ T_3^M \end{bmatrix} = \begin{bmatrix} C_{11}^M & C_{12}^M & C_{13}^M \\ & C_{22}^M & C_{23}^M \\ \text{sym} & & C_{33}^M \end{bmatrix} \begin{bmatrix} E_1^M \\ E_2^M \\ E_3^M \end{bmatrix} \leftrightarrow \begin{bmatrix} E_1^M \\ E_2^M \\ E_3^M \end{bmatrix} = \begin{bmatrix} C_{11}^M & C_{12}^M & C_{13}^M \\ & C_{22}^M & C_{23}^M \\ \text{sym} & & C_{33}^M \end{bmatrix}^{-1} \begin{bmatrix} T_1^M \\ T_2^M \\ T_3^M \end{bmatrix} \tag{52}
$$

$$
\text{PT}: \begin{bmatrix} E_1^M \\ E_2^M \\ E_3^M \end{bmatrix} = \begin{bmatrix} S_{11}^M & S_{12}^M & S_{13}^M \\ & S_{22}^M & S_{23}^M \\ \text{sym} & & S_{33}^M \end{bmatrix} \begin{bmatrix} T_1^M \\ T_2^M \\ T_3^M \end{bmatrix} \leftrightarrow \begin{bmatrix} T_1^M \\ T_2^M \\ T_3^M \end{bmatrix} = \begin{bmatrix} S_{11}^M & S_{12}^M & S_{13}^M \\ & S_{22}^M & S_{23}^M \\ \text{sym} & & S_{33}^M \end{bmatrix}^{-1} \begin{bmatrix} E_1^M \\ E_2^M \\ E_3^M \end{bmatrix} \tag{53}
$$

Thus, in terms of the 3D stiffness C_{ij}^M, the 3D compliance $S_{ij}^M = \left[C_{ij}^M \right]^{-1}$ and the projection $\mathfrak{P}$ we have the following realtions.

$$
\text{PE}: \quad \begin{aligned} C_{ij}^{2D^M} &= \left(C_{ij}^M \right) = C_{ULij}^M \\ S_{ij}^{2D^M} &= \left[\mathfrak{P}\left(\left[S_{ij}^M \right]^{-1} \right) \right]^{-1} = \left[C_{ULij}^M \right]^{-1} \end{aligned} \qquad\qquad \forall \; i, j \in \{1, 2, 3\} \tag{54}
$$

$$
\text{PT}: \quad \begin{aligned} S_{ij}^{2D^M} &= \mathfrak{P}\left(S_{ij}^M \right) = S_{ULij}^M \\ C_{ij}^{2D^M} &= \left[\mathfrak{P}\left(\left[C_{ij}^M \right]^{-1} \right) \right]^{-1} = \left[S_{ULij}^M \right]^{-1} \end{aligned} \qquad\qquad \forall \; i, j \in \{1, 2, 3\} \tag{55}
$$

It is no surprise that the plane stress stiffness is not equal to the plane strain stiffness, which is due to subsequent disparity.

$$\mathbb{S}_{UL}^{M} \neq \left[\mathbb{C}_{UL}^{M}\right]^{-1} \tag{56}$$

The latter is a consequence of the entries in the upper right (resp. lower left block) matrices. Due to the particular ordering for the plane states, these are not zero, even if the material is isotropic. Therefore, the projection $\mathfrak{P}$ and the inversion are not commutative. Note that $\mathfrak{P}$ is more than a projector with zero eigenvalues. We actually imply that we go from a six-dimensional to a three-dimensional space, in the sense of Gurtin and Murdoch (1975). Otherwise, the inverse is not defined, or one would need the pseudoinverse. Also, the plane states allow to restrict only to three non-zero components either for $\boldsymbol{E}$ or $\boldsymbol{T}$, which would make HOOKE's law a mapping between spaces of different dimensions. The latter is not invertible, which is why we restrict to the plane components. Since the out-of-plane components depend linearly on the in-plane-components, they can be obtained as secondary or derived quantities.

$$\text{PE:} \quad \boldsymbol{T}_{\text{op}}^{M} = \begin{bmatrix} T_4^M \\ T_5^M \\ T_6^M \end{bmatrix} = \left[\mathbb{C}_{LL}^M\right] \begin{bmatrix} E_1^M \\ E_2^M \\ E_3^M \end{bmatrix} = \left[\mathbb{C}_{LL}^M\right]\left[\mathbb{C}_{UL}^M\right]^{-1} \begin{bmatrix} T_1^M \\ T_2^M \\ T_3^M \end{bmatrix} \tag{57}$$

$$\text{PT:} \quad \boldsymbol{E}_{\text{op}}^{M} = \begin{bmatrix} E_4^M \\ E_5^M \\ E_6^M \end{bmatrix} = \left[\mathbb{S}_{LL}^M\right] \begin{bmatrix} T_1^M \\ T_2^M \\ T_3^M \end{bmatrix} = \left[\mathbb{S}_{LL}^M\right]\left[\mathbb{S}_{UL}^M\right]^{-1} \begin{bmatrix} E_1^M \\ E_2^M \\ E_3^M \end{bmatrix} \tag{58}$$

Such an approach has been used for in the context of homogenization by Eidel et al. (2019) for an anisotropic material, but with the VOIGT ordering {11,22,33,23,13,12}, such that the indices 2,3,4 form the 3×3 block-matrix for plane states in the $\boldsymbol{e}_2$-$\boldsymbol{e}_3$-plane, and without normalizing the basis. They consider a PT state, hence they map all values outside this block matrix in the compliance matrix to zero (see Eq. 4 in Eidel et al. (2019)).

4.3   Properties of the 2D Stiffnesses and Compliances

The matrix-vector-notation may be useful for numeric calculations. For symbolic calculations, it is slightly more convenient to solve the three equations as presented in Sect. 3.
We have seen that in PEs the form $\boldsymbol{T} = \mathbb{C}\boldsymbol{:}\boldsymbol{E}$ and in the PTs the form $\boldsymbol{E} = \mathbb{S}\boldsymbol{:}\boldsymbol{T}$ of HOOKE's law is advantageous when going from 3D to 2D, since then one can simply restrict attention to the components of the upper left block matrix of $\mathbb{C}^M$ or $\mathbb{S}^M$, presuming the MANDEL notation with the appropriate index assignment. The simplicity of the matrix-vector notation allows to identify some properties:

- the plane stiffness is positive homogeneous of degree 1 in terms of the 3D stiffness
- as a consequence of Cauchy's interlacing theorem (see, e.g., Hwang (2004)), for a plane stress state the eigenvalues of the plane stiffness tensor are smaller than the eigenvalues of the 3D stiffness tensor: $\left(\lambda^{PT} \leqslant \lambda^{3D}\right) \Longrightarrow$ stiffness is reduced
- for a plane strain state the eigenvalues of the plane compliance tensor are smaller than the eigenvalues of the 3D compliance tensor: $\left(\eta^{PE} \leqslant \eta^{3D}\right) \Longrightarrow$ stiffness is increased

5   Simplifications induced by Material Symmetries

This section is dedicated to the application of Eqs. (12) and (14). Therefore, an isotropic material is analyzed under PT and PE first and then a material with trigonal symmetry is analyzed. The application to other symmetry classes (e.g. cubic, orthotropic, monoclinic) is straight-forward.

5.1   Isotropy

In this section we want to show that Eq. (24) and Eq. (35) are the correct representations of the elasticity and compliance tetrad for the PT and the PE, respectively. Therefore, we assume an isotropic and homogeneous material under small deformations at first.

The stiffness tetrad has then two independent components C_{1111} and C_{1122}. The components of the stiffness tetrad $\mathbb{C}$ are as follows.

$$C_{1111} = C_{1111} \qquad\qquad C_{1122} = C_{1122} \qquad\qquad C_{1133} = C_{1122}$$

$$C_{2211} = C_{1122} \qquad\qquad C_{2222} = C_{1111} \qquad\qquad C_{2233} = C_{1122}$$

$$C_{3311} = C_{1122} \qquad\qquad C_{3322} = C_{1122} \qquad\qquad C_{3333} = C_{1111}$$

$$C_{2323} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad C_{1313} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad C_{1212} = \frac{1}{2}(C_{1111} - C_{1122})$$

$$C_{3232} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad C_{3131} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad C_{2121} = \frac{1}{2}(C_{1111} - C_{1122})$$

$$C_{3223} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad C_{3113} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad C_{2112} = \frac{1}{2}(C_{1111} - C_{1122})$$

$$C_{2332} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad C_{1331} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad C_{1221} = \frac{1}{2}(C_{1111} - C_{1122})$$

All other components are zero. With this definition of $\mathbb{C}$ we can calculate $\mathbb{C}^{2D}$ for a PT after Eq. (24) and insert this tetrad into Eq. (12) to determine the expressions for the stresses under a PT. This results in subsequent constitutive relations.

$$T_{11}^{PT} = \left[C_{1111} - \frac{C_{1122}^2}{C_{1111}} \right] E_{11}^{PT} + \left[C_{1122} - \frac{C_{1122}^2}{C_{1111}} \right] E_{22}^{PT}$$

$$T_{12}^{PT} = \frac{1}{2} \left[C_{1111} - C_{1122} \right] E_{12}^{PT}$$

$$T_{21}^{PT} = \frac{1}{2} \left[C_{1111} - C_{1122} \right] E_{12}^{PT} \tag{59}$$

$$T_{22}^{PT} = \left[C_{1122} - \frac{C_{1122}^2}{C_{1111}} \right] E_{11}^{PT} + \left[C_{1111} - \frac{C_{1122}^2}{C_{1111}} \right] E_{22}^{PT}$$

The term $-\frac{C_{1122}^2}{C_{1111}}$ is the correction term of the components in the normal directions for a PT to guarantee that the stresses in the third direction are zero. For this example the tetrad $\mathbb{C}^{PT}$ has the following components.

$$C_{1111}^{PT} = C_{1111} - \frac{C_{1122}^2}{C_{1111}} \qquad\qquad C_{1122}^{PT} = C_{1122} - \frac{C_{1122}^2}{C_{1111}}$$

$$C_{2211}^{PT} = C_{1122} - \frac{C_{1122}^2}{C_{1111}} \qquad\qquad C_{2222}^{PT} = C_{1111} - \frac{C_{1122}^2}{C_{1111}}$$

$$C_{1212}^{PT} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad\qquad C_{2112}^{PT} = \frac{1}{2}(C_{1111} - C_{1122})$$

$$C_{1221}^{PT} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad\qquad C_{2121}^{PT} = \frac{1}{2}(C_{1111} - C_{1122})$$

These components are not zero and afterwards, with Eqs. (19)-(21) someone can calculate the strains resulting in the third direction. These are:

$$E_{13} = 0 \qquad\qquad E_{23} = 0 \qquad\qquad E_{33} = -\frac{C_{1122}}{C_{1111}} \left(E_{11}^{PT} + E_{22}^{PT} \right)$$

Up next, the same procedure is performed for the PE. Eq. (14) will be followed by subsequent expressions.

$$E_{11}^{PE} = \frac{C_{1111}}{C_{1111}^2 - C_{1122}^2} T_{11}^{PE} - \frac{C_{1122}}{C_{1111}^2 - C_{1122}^2} T_{22}^{PE}$$

$$E_{12}^{PE} = \frac{1}{C_{1111} - C_{1122}} T_{12}^{PE}$$

$$E_{21}^{PE} = \frac{1}{C_{1111} - C_{1122}} T_{12}^{PE} \tag{60}$$

$$E_{22}^{PE} = \frac{C_{1111}}{C_{1111}^2 - C_{1122}^2} T_{22}^{PE} - \frac{C_{1122}}{C_{1111}^2 - C_{1122}^2} T_{11}^{PE}$$

Through the inversion of the compliance tetrad by applying the MANDEL sheme Brannon (2018) the corresponding stiffness tetrad $\mathbb{C}^{PE}$ is derived without a correction term for the stiffness tetrad (This result is expected, cf. Sect. 4.2.). The components of the

stiffness tetrad for the PE are:

$$C_{1111}^{PE} = C_{1111} \qquad\qquad C_{1122}^{PE} = C_{1122}$$

$$C_{2211}^{PE} = C_{1122} \qquad\qquad C_{2222}^{PE} = C_{1111}$$

$$C_{1212}^{PE} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad\qquad C_{2112}^{PE} = \frac{1}{2}(C_{1111} - C_{1122})$$

$$C_{1221}^{PE} = \frac{1}{2}(C_{1111} - C_{1122}) \qquad\qquad C_{2121}^{PE} = \frac{1}{2}(C_{1111} - C_{1122})$$

The other components are zero. And from Eqs. (30)-(32) the stresses resulting in the third directions for a PE are determined.

$$T_{13} = 0 \qquad\qquad T_{23} = 0 \qquad\qquad T_{33} = \frac{C_{1122}}{C_{1111} + C_{1122}} \left(T_{11}^{PE} + T_{22}^{PE} \right)$$

The results of this section are in accordance with the specific literature, e.g. Altenbach et al. (2018); Chaboche and Lemaitre (1990); Bertram and Glüge (2015), and serve as verification for derived equations, consequently.

5.2 Trigonal

In this section a trigonal material is analyzed under a PT and a PE. The stiffness tetrad of a trigonal material contains six independent components C_{1111}, C_{1122}, C_{1133}, C_{3333}, C_{1123} and C_{2323} which are arranged as follows.

$$C_{1111} = C_{1111} \qquad\qquad C_{1122} = C_{1122} \qquad\qquad C_{1133} = C_{1133}$$

$$C_{2211} = C_{1122} \qquad\qquad C_{2222} = C_{1111} \qquad\qquad C_{2233} = C_{1133}$$

$$C_{3311} = C_{1133} \qquad\qquad C_{3322} = C_{1133} \qquad\qquad C_{3333} = C_{3333}$$

$$C_{2323} = C_{2323} \qquad\qquad C_{1313} = C_{2323} \qquad\qquad C_{1212} = \frac{1}{2}(C_{1111} - C_{1122})$$

$$C_{3232} = C_{2323} \qquad\qquad C_{3131} = C_{2323} \qquad\qquad C_{2121} = \frac{1}{2}(C_{1111} - C_{1122})$$

$$C_{3223} = C_{2323} \qquad\qquad C_{3113} = C_{2323} \qquad\qquad C_{2112} = \frac{1}{2}(C_{1111} - C_{1122})$$

$$C_{2332} = C_{2323} \qquad\qquad C_{1331} = C_{2323} \qquad\qquad C_{1221} = \frac{1}{2}(C_{1111} - C_{1122})$$

$$C_{1123} = C_{1123} \qquad\qquad C_{2223} = -C_{1123} \qquad\qquad C_{1312} = C_{1123}$$

$$C_{1132} = C_{1123} \qquad\qquad C_{2232} = -C_{1123} \qquad\qquad C_{1321} = C_{1123}$$

$$C_{2311} = C_{1123} \qquad\qquad C_{2322} = -C_{1123} \qquad\qquad C_{1213} = C_{1123}$$

$$C_{3211} = C_{1123} \qquad\qquad C_{3222} = -C_{1123} \qquad\qquad C_{2113} = C_{1123}$$

The components of the tetrad which are not mentioned are zero. For the PT the following expressions for the stresses results from Eq. (12).

$$T_{11}^{PT} = \left[C_{1111} - \frac{C_{1133}^2}{C_{3333}} \right] E_{11}^{PT} + \left[C_{1122} - \frac{C_{1133}^2}{C_{3333}} \right] E_{22}^{PT}$$

$$T_{12}^{PT} = [C_{1111} - C_{1122}] E_{12}^{PT}$$

$$T_{21}^{PT} = [C_{1111} - C_{1122}] E_{12}^{PT} \qquad\qquad\qquad (61)$$

$$T_{22}^{PT} = \left[C_{1122} - \frac{C_{1133}^2}{C_{3333}} \right] E_{11}^{PT} + \left[C_{1111} - \frac{C_{1133}^2}{C_{3333}} \right] E_{22}^{PT}$$

The normal stresses are corrected with the term $-\frac{C_{1133}^2}{C_{3333}}$ and the shear stress is doubled. After this Eq. (24) can be used to determine the PT stiffness tetrad. This leads to a stiffness tetrad with following nonzero components.

$$C_{1111}^{PT} = C_{1111} - \frac{C_{1133}^2}{C_{3333}} \qquad\qquad\qquad\qquad C_{1122}^{PT} = C_{1122} - \frac{C_{1133}^2}{C_{3333}}$$

$$C_{2211}^{PT} = C_{1122} - \frac{C_{1133}^2}{C_{3333}} \qquad\qquad\qquad\qquad C_{2222}^{PT} = C_{1111} - \frac{C_{1133}^2}{C_{3333}}$$

$$C_{1212}^{PT} = C_{1111} - C_{1122} \qquad\qquad\qquad\qquad\qquad C_{2112}^{PT} = C_{1111} - C_{1122}$$

$$C_{1221}^{PT} = C_{1111} - C_{1122} \qquad\qquad\qquad\qquad\qquad C_{2121}^{PT} = C_{1111} - C_{1122}$$

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Finally, the strains in the third direction can be calculated. Following values for the strains result.

$$E_{13} = -\frac{C_{1123}}{C_{2323}} E_{12}^{PT}$$

$$E_{23} = -\frac{C_{1123}}{2\,C_{2323}} \left(E_{11}^{PT} - E_{22}^{PT} \right)$$

$$E_{33} = -\frac{C_{1133}}{C_{3333}} \left(E_{11}^{PT} + E_{22}^{PT} \right)$$

Due to the anisotropic material, shear strains in the third direction result next to the normal strain. As well, this analyses is done for the PE. Eq. (14) results in subsequent expressions.

$$
\begin{aligned}
E_{11}^{PE} =\ & \left[\frac{C_{3333}\left(C_{1123}^2 - C_{1111}C_{2323}\right) + C_{2323}C_{1133}^2}{\overline{C}_1} - \frac{C_{1133}^2}{\overline{C}_2} \right] T_{11}^{PE} \\
& + \left[\frac{C_{3333}\left(C_{1123}^2 + C_{1122}C_{2323}\right) - C_{2323}C_{1133}^2}{\overline{C}_1} - \frac{C_{1133}^2}{\overline{C}_2} \right] T_{22}^{PE}
\end{aligned}
$$

$$
\begin{aligned}
E_{12}^{PE} =\ & -\frac{2\,C_{2323}}{C_{1123}^2 - 2\,C_{1111}C_{2323} + 2\,C_{1122}C_{2323}} T_{12}^{PE} \\
E_{21}^{PE} =\ & -\frac{2\,C_{2323}}{C_{1123}^2 - 2\,C_{1111}C_{2323} + 2\,C_{1122}C_{2323}} T_{12}^{PE} \\
E_{22}^{PE} =\ & \left[\frac{C_{3333}\left(C_{1123}^2 + C_{1122}C_{2323}\right) - C_{2323}C_{1133}^2}{\overline{C}_1} - \frac{C_{1133}^2}{\overline{C}_2} \right] T_{11}^{PE} \\
& + \left[\frac{C_{3333}\left(C_{1123}^2 - C_{1111}C_{2323}\right) - C_{2323}C_{1133}^2}{\overline{C}_1} - \frac{C_{1133}^2}{\overline{C}_2} \right] T_{22}^{PE}
\end{aligned}
$$

$$(62)$$

Herein we have used subsequent abbreviations.

$$
\begin{aligned}
\overline{C}_1 =\ & 2\,C_{1111}\left(C_{3333}C_{1123}^2 + C_{2323}C_{1133}^2 \right) \\
& + C_{3333}\left[C_{2323}\left(C_{1122}^2 - C_{1111}^2 \right) + 2\,C_{1122}C_{1123}^2 \right] \\
& - 2\,C_{1133}^2\left(C_{2323}C_{1122} + 2\,C_{1123}^2 \right) \\
\overline{C}_2 =\ & \left(C_{1111} + C_{1122} \right)\left(C_{1111}C_{3333} + C_{1122}C_{3333} - 2\,C_{1133}^2 \right)
\end{aligned}
$$

To derive the stiffness tetrad for the PE the VOIGT sheme is applied again to the compliance tetrad and then this tetrad is inverted. The components of the stiffness tetrad for the PE which are not zero are as follows.

$$C_{1111}^{PE} = \frac{C_{1111}C_{2323} - C_{1123}^2}{C_{2323}} \qquad\qquad C_{1122}^{PE} = \frac{C_{1122}C_{2323} + C_{1123}^2}{C_{2323}}$$

$$C_{2211}^{PE} = \frac{C_{1122}C_{2323} + C_{1123}^2}{C_{2323}} \qquad\qquad C_{2222}^{PE} = \frac{C_{1111}C_{2323} - C_{1123}^2}{C_{2323}}$$

$$C_{1212}^{PE} = \frac{2\,C_{2323}\left(C_{1111} - C_{1122}\right) - C_{1123}^2}{8\,C_{2323}} \qquad C_{2112}^{PE} = \frac{2\,C_{2323}\left(C_{1111} - C_{1122}\right) - C_{1123}^2}{8\,C_{2323}}$$

$$C_{1221}^{PE} = \frac{2\,C_{2323}\left(C_{1111} - C_{1122}\right) - C_{1123}^2}{8\,C_{2323}} \qquad C_{2121}^{PE} = \frac{2\,C_{2323}\left(C_{1111} - C_{1122}\right) - C_{1123}^2}{8\,C_{2323}}$$

In the last step, the shear and normal stresses in the third direction are calculated after Eqs. (30)-(32).

$$T_{13} = \frac{C_{1123}}{C_{1111} - C_{1122}} T_{12}^{PE}$$

$$T_{23} = \frac{C_{1123}}{C_{1111} - C_{1122}} \left(T_{11}^{PE} - T_{22}^{PE} \right)$$

$$T_{33} = \frac{C_{1133}}{C_{1111} + C_{1122}} \left(T_{11}^{PE} + T_{22}^{PE} \right)$$

Again, it can be seen that through the anisotropic material, shear stresses in the third direction result.

6   Neglection of reaction stresses in the plane strain case

By a simplified calculation of the plane strain stiffness by projecting the 3D stiffness, i.e. by simply dropping the out-of-plane columns and rows, one neglects the out-of-plane reaction stresses. These can become very large when the material is nearly incompressible. Therefore, the simplified plane stress stiffness can lead to nonconservative estimates, underestimating the actual stresses. This effect can be observed in the homogenization of polymeric materials with a spherulitic microstructure. Inside the spherulites, a crystalline phase with $\nu \approx 0.3$ and an amorphous phase with $\nu \approx 0.499$ are layered. The common in plane strain enforces a considerable out of plane stress in the amourphous phase, which manifests as an apparent stiffness. The increased stiffness is also referred to as the reinforcement or contiguity factor (originally introduced by Tsai and Pagano (1968)), as the oedometric effect or as the confinement effect (Glüge et al. (2019)). If it is not taken into account, the effective stiffness of polymers is usually underestimated.

To quantify the effect of the simplified plane strain stiffness we examined the eigenvalues of an apparent isotropic stiffness in both plane stress and plane strain situations. In isotropic 3D elasticity, it is well known that $\lambda_1 = 3K$ and $\lambda_2 = 2G$ hold true. Here K is the compression modulus and G is the shear modulus of the material. This means that λ_1 corresponds to the resistance to dilatations and λ_2 to the resistance to distortions. The eigenvalues of the stiffness tetrads are given in Tab. 1. Obviously the second eigenvalue is the same for all three states while the first eigenvalue of plane states differs in the following way.

$$\lambda_1^{\mathrm{PT}} = \lambda_1^{\mathrm{3D}} - C_{1122} - 2\frac{C_{1122}^2}{C_{1111}} \tag{63}$$

$$\lambda_1^{\mathrm{PE}} = \lambda_1^{\mathrm{3D}} - C_{1122} \tag{64}$$

Due to the claim of positive definiteness of all stiffness tetrads it is necessary that $C_{1111} > 0$ holds true. With this restriction and the definition of the eigenvalues (cf. Tab. 1) for all three states we can formulate the following restrictions for the component C_{1122}. Whereby, we perform a normalization with C_{1111} for a better evaluation and introduce the abbreviation $\Psi = C_{1122}/C_{1111}$.

$$3\mathrm{D:} \quad -\frac{1}{2} < \Psi < 1 \tag{65}$$

$$\mathrm{PT:} \quad -\frac{1}{2} < \Psi < 1 \tag{66}$$

$$\mathrm{PE:} \quad -1 \; < \Psi < 1 \tag{67}$$

Considering these restrictions we plot the first eigenvalue presented in Tab. 1 on the normalized stiffness coefficient Ψ, while normalizing the eigenvalues by C_{1111} as well. All three cases are visualized in Fig. 2, next to the normalized values of steel (stiff material) and a nearly incompressible material. The values are taken from Bertram and Glüge (2015). We can clearly identify a linear dependence in the 3D case and for a plane strain case, while the first eigenvalue is strongly non-linear in C_{1122} in the plane strain case.

We can identify two points of intersection between the 3D- and the PT case, which are at $\Psi = -0.5$ and $\Psi = 0$. Comparing the 3D- and the plane strain case, we find one intersection at $\Psi = 0$. Only in these intersections the plane state stiffness is obtained by dropping rows and columns of the 3D stiffness. Fig. 2 contains exemplary values of steel and a nearly incompressible material for these three states. One can see that the simplified calculation of the plane stiffnesses can only be used for stiff materials but not for a nearly incompressible materials due to the large error.

This is most clearly understood in terms of Poisson's ratio ν. For a nearly incompressible material with $\nu \to 0.5$ (equal to $\Psi \to 1$) the error is largest because the true apparent compressive modulus tends to infinity, which is not accounted for by simply dropping rows and columns from the 3D stiffness. The intersection at $\Psi = 0$ corresponds to $\nu = 0$, i.e. the case that $K = 2G/3$, and the interval $-1 < \Psi < 0$ represents auxetic materials with negative Poisson's ratios, i.e. $0 < K < 2G/3$. The second eigenvalue is not changing under the assumptions of plane states. Similar results are obtained by plotting the first eigenvalue and the eigenvalue ratio versus Poisson's ratio, cf. Figure 3. Especially, the right diagram in Fig. 3 shows the growing error when a simplified plane stiffness is obtained in case of a plane strain state. Again, for $\nu \to 0.5$ the error becomes infinite.

7   Summary

In the forgoing chapters the reduced plane stiffnesses for plane stress and plane strain states are derived, once with tensor algebra and apart from that in vector-matrix notation. Both possibilities have their eligibility:

- In the tensorial notation, the primitive geometric quantities like the normal vector of the plane stress or plane state plane, the directions and planes of material symmetry and the rotations of the symmetry group appear directly in the equations. These

Tab. 1: Isotropic eigenvalues $\lambda_\alpha \; \forall \, \alpha \in \{1, 2\}$ of $\mathbb{C}$, $\mathbb{C}^{\mathrm{PT}}$, and $\mathbb{C}^{\mathrm{PE}}$

	$\mathbb{C}$	$\mathbb{C}^{\mathrm{PT}}$	$\mathbb{C}^{\mathrm{PE}}$
λ_1	$C_{1111} + 2C_{1122}$	$C_{1111} + C_{1122} - 2\frac{C_{1122}^2}{C_{1111}}$	$C_{1111} + C_{1122}$
λ_2	$C_{1111} - C_{1122}$	$C_{1111} - C_{1122}$	$C_{1111} - C_{1122}$

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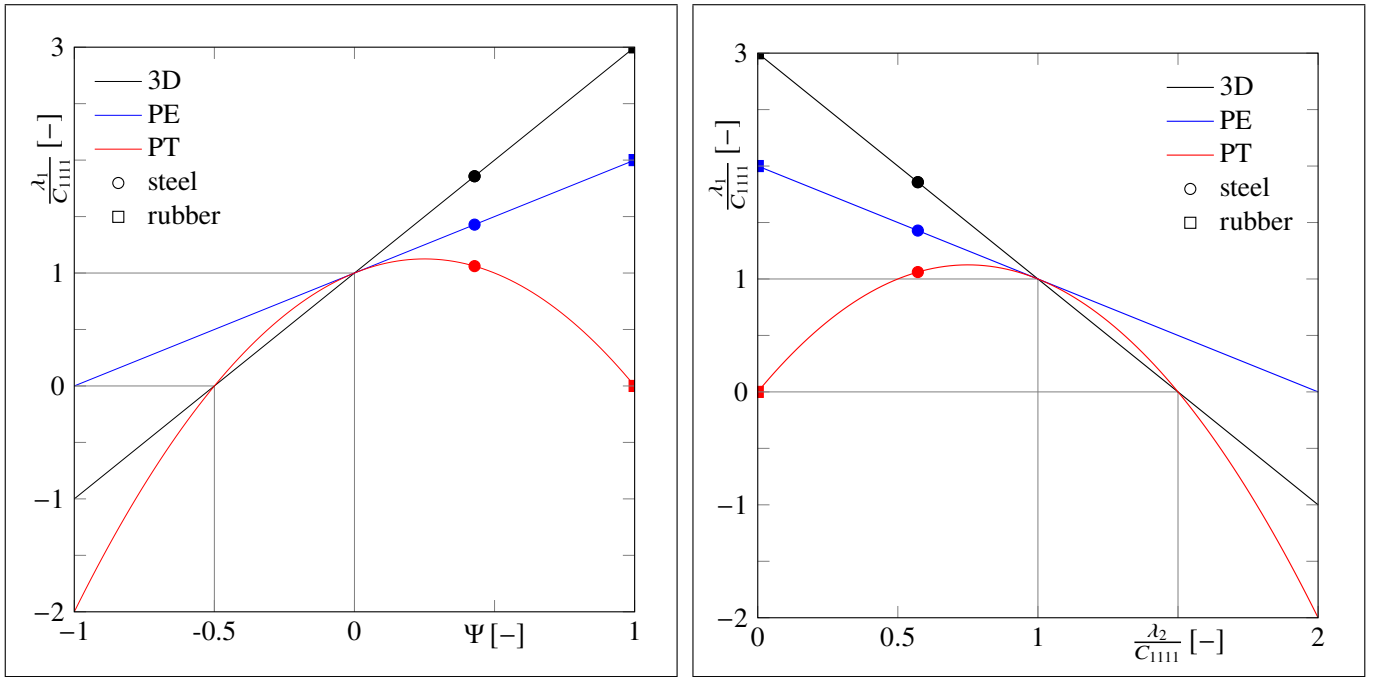Tech. Mech., Vol. 40, Is. 2, (2020), 160–174

Fig. 2: Normalized first eigenvalue versus Ψ (left) and normalized first versus normalized second eigenvalue (right) for an isotropic material.

are rather obfuscated in the 6×6 matrix notation. For example, if the plane of symmetry does not coincide with the plane stress or plane state plane, it is very hard to give the reduced stiffness directly in matrix form.

- On the other hand, many theorems of linear algebra require a vector-matrix notation *with underlying orthonormal bases*. For example, the fact that the plane stress stiffness is smaller than the plane strain stiffness in terms of eigenvalues follows directly from Cauchy's interlacing theorem, which requires a matrix representation. Also, the eigenvalues (or Kelvin-moduli) are readily calculated from a matrix representation.

Therefore one should have both representations available, be able to switch between them, and employ them accordingly. From these investigations (eigenvalue analysis) we know that the assumption of the plane stress state leads always to a smaller reduced plane stiffness and the plane strain leads always to an increased plane stiffness for all symmetry classes. This is a consequence of the reaction stresses due to the additional kinematic constraint in the plane strain case.

We limited our explanations to the field of linear elasticity with small deformations. Often, when the plane stress- or plane strain state is used, the material is assumed to be isotropic. However, in many cases the materials may be anisotropic due to the
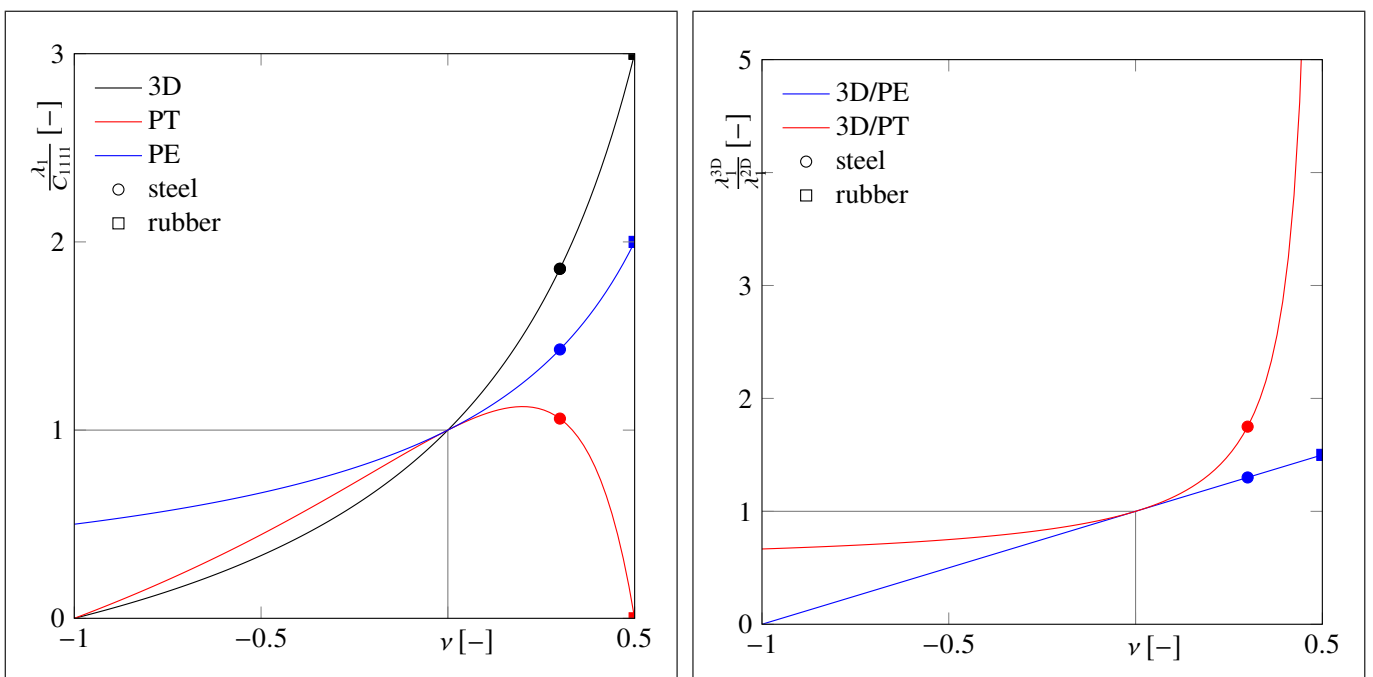

Fig. 3: Normalized first eigenvalue (left) and eigenvalue ratio (right) for an isotropic material, depending on Poisson's ratio ν

manufacturing process. Nevertheless, it appears that it is not well known how the plane stiffnesses are derived in the anisotropic case. We hope to convince the community that works on reduced-dimensional problems to use the reduced plane stiffnesses also in the anisotropic case, instead of using 3D models. The presented formulations offer the possibility to derive expressions of plane stress and plane strain elements for commercial FE codes like ABAQUS or ANSYS, or derive plate or shell equations for anisotropic materials like rolled steel sheets. However, the present elaboration does not envisage the case of time-variant problematics, i.e. when considering rheonomous material behavior, advanced constitutive relations have to be considered. These may be reduced to a plane formulation by subjecting them to an analogous treatment as presented here. Moreover, we examine the apparent increase of stiffness in case of plane strains due to the plane strain kinematic constraint, and discuss the how the error of simplified plane stiffness depends on the material's compressive behaviour. We conclude that the simplified plane stiffness that is obtained by dropping rows and columns from the 3D stiffness should not be used in the plane strain case when the compression modulus is greater than the bulk modulus. That is, when $\nu > 0.125$.

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