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Yet another elasto-plasticity formulation

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Abstract: An elasto-plasticity formulation is presented that requires no intermediate (stress-free) configuration, since all describing tensors are solely of proper-Eulerian or proper-Lagrangean type. This formulation—based on commutative-symmetrical elastic-plastic stretch tensor products with symmetrizing-rotation tensors in the middle—is discussed and compared with the Bilby-Kröner-Lee formulation, which defines an intermediate (stress-free) configuration that is not well-determined—as noted, e.g., by Casey & Naghdi (1980). For an Eulerian continuum description, it turns out that the symmetric elastic part of the presented formulation (with only proper-Eulerian tensors) has similarities with the elastic tensor factor ${}^{e}\mathfrak{F}$ of the Bilby-Kröner-Lee multiplicative elasto-plastic decomposition $\mathsf{F} = {}^{e}\mathfrak{F}.{}^{p}\mathfrak{F}$ of the deformation gradient F. From a Lagrangean point of view, however, the symmetric elasticity tensors of the two models differ considerably: the elastic right stretch and Cauchy-Green deformation tensor of the new formulation are proper-Lagrangean tensors, while the corresponding tensors of the Bilby-Kröner-Lee formulation are not well-determined, since they refer to an intermediate (stress-free) configuration. As finite orthotropy modeling requires a material reference configuration in which (imaginary) fibers are perpendicular to each other, finite elastic orthotropy and finite plastic orthotropy can only be modeled simultaneously based on proper-Lagrangean elastic and plastic tensors provided by commutative-symmetrical deformation tensor products and not by Bilby-Kröner-Lee formulations.

Keywords: geometric interpretation of deformation, material-convective continuum formulation, Green-Naghdi rate, commutative-symmetrical stretch tensor product, orthotropic elasto-plasticity

1 Introduction

The mapping

$$d\mathbf{x} = \mathsf{F}.d\mathbf{X} \tag{1}$$

of the infinitesimal vicinities $d\mathbf{x}$ and $d\mathbf{X}$ of the current position vector $\mathbf{x}$ and the reference position vector $\mathbf{X}$ by the deformation gradient

$$\mathsf{F} = \lambda_1 \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{E}}_1 + \lambda_2 \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{E}}_2 + \lambda_3 \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{E}}_3 = \lambda_k \underbrace{\hat{\mathbf{e}}_k \otimes \hat{\mathbf{E}}_k}_{\mathsf{R} = \mathsf{R}^{-\mathsf{T}}} \tag{2}$$

(in spectral representation) is shown graphically as material deformation in Fig. 1. A material sphere of the reference configuration (black sphere on the left in Fig. 1) is deformed into a material ellipsoid of the current configuration (blue ellipsoid on the right in Fig. 1). The principal directions/eigenvectors are marked with a *hat* and point along the semi-axes of the ellipsoid. The Eulerian and Lagrangean unit eigenvectors are $\hat{\mathbf{e}}_k = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ and $\hat{\mathbf{E}}_k = \{\hat{\mathbf{E}}_1, \hat{\mathbf{E}}_2, \hat{\mathbf{E}}_3\}$, respectively, with the proper-Eulerian tensors written in lowercase and the proper-Lagrangean tensors in uppercase. A single contraction is given by the dot product '.' operator, a dyadic product by the '⊗' operator, and the summation convention is extended to multiple indices (written with the same letter). On the right-hand side of Eq.(2), the underbraced

$$\mathsf{R} = \mathsf{R}^{-\mathsf{T}} = \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{E}}_1 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{E}}_2 + \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{E}}_3, \qquad \det(\mathsf{R}) = 1 \tag{3}$$

is the proper-orthogonal material-convective (polar) rotation tensor R from the polar decomposition of the deformation gradient $\mathsf{F} = \mathsf{v}.\mathsf{R} = \mathsf{R}.\mathsf{U}$. Since the rotation tensor (3) is proper-orthogonal, R equals its transposed inverse $\mathsf{R}^{-\mathsf{T}}$ and has a determinant equal to one. The deformation (1) of the infinitesimal vicinities $d\mathbf{x}$ and $d\mathbf{X}$ (shown in Fig. 1) can also be considered as a material mapping of a cube cut along the Lagrangean principal directions (blue cube on the left in Fig. 1) onto a cuboid/rectangular parallelepiped oriented along the Eulerian principal directions (green cuboid on the right in Fig. 1). The eigenvalues $\lambda_1 = \ell_1/L_1 > 0, \lambda_2 = \ell_2/L_2 > 0$, $\lambda_3 = \ell_3/L_3 > 0$ are the stretch ratios of corresponding lengths of the cuboid and the cube. The determinant of the deformation gradient F is of positive value $\det(\mathsf{F}) > 0$. With the transpose of the deformation gradient

$$\mathsf{F}^{\mathsf{T}} = \lambda_1 \hat{\mathbf{E}}_1 \otimes \hat{\mathbf{e}}_1 + \lambda_2 \hat{\mathbf{E}}_2 \otimes \hat{\mathbf{e}}_2 + \lambda_3 \hat{\mathbf{E}}_3 \otimes \hat{\mathbf{e}}_3, \tag{4}$$

the symmetric positive-definite left and right Cauchy-Green deformation tensors follow as

$$\mathsf{b} = \mathsf{F}.\mathsf{F}^{\mathsf{T}} = \lambda_1^2\, \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \lambda_2^2\, \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \lambda_3^2\, \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3 \quad \text{and} \quad \mathsf{C} = \mathsf{F}^{\mathsf{T}}.\mathsf{F} = \lambda_1^2\, \hat{\mathbf{E}}_1 \otimes \hat{\mathbf{E}}_1 + \lambda_2^2\, \hat{\mathbf{E}}_2 \otimes \hat{\mathbf{E}}_2 + \lambda_3^2\, \hat{\mathbf{E}}_3 \otimes \hat{\mathbf{E}}_3, \tag{5}$$

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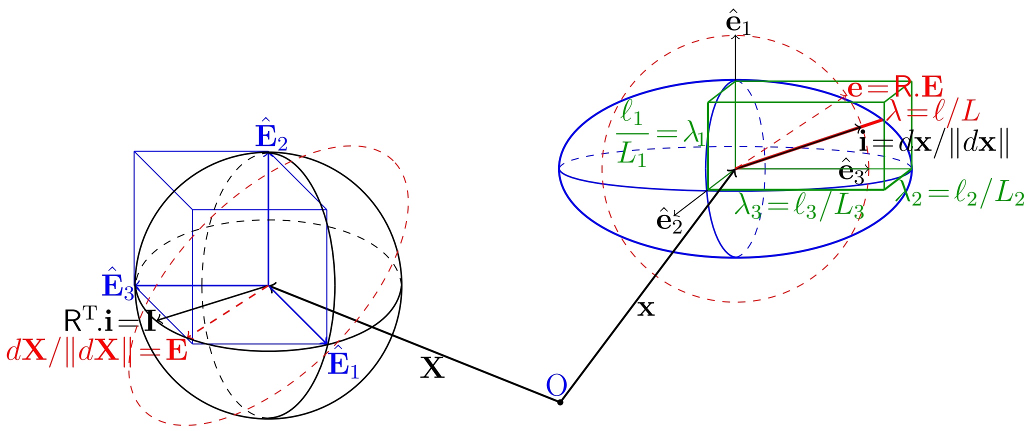

Fig. 1: Material mapping $d\mathbf{x} = \mathsf{F}.d\mathbf{X}$ of the infinitesimal vicinities $d\mathbf{x}$ and $d\mathbf{X}$ of the current position vector $\mathbf{x}$ and the reference position vector $\mathbf{X}$ by the deformation gradient F

respectively. Applying isotropic square-root tensor functions to (5), the symmetric positive-definite left and right stretch tensors

$$\mathsf{v} = \sqrt{\mathsf{b}} = \lambda_1 \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \lambda_2 \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \lambda_3 \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3 \qquad \text{and} \qquad \mathsf{U} = \sqrt{\mathsf{C}} = \lambda_1 \hat{\mathbf{E}}_1 \otimes \hat{\mathbf{E}}_1 + \lambda_2 \hat{\mathbf{E}}_2 \otimes \hat{\mathbf{E}}_2 + \lambda_3 \hat{\mathbf{E}}_3 \otimes \hat{\mathbf{E}}_3 \tag{6}$$

are obtained. From the polar decomposition of the deformation gradient $\mathsf{F} = \mathsf{v}.\mathsf{R} = \mathsf{R}.\mathsf{U}$, it follows that the right and left stretch or Cauchy-Green deformation tensors

$$\mathsf{U} = \mathsf{R}^\mathsf{T}.\mathsf{v}.\mathsf{R}, \qquad \mathsf{v} = \mathsf{R}.\mathsf{U}.\mathsf{R}^\mathsf{T} \qquad \text{or} \qquad \mathsf{C} = \mathsf{R}^\mathsf{T}.\mathsf{b}.\mathsf{R}, \qquad \mathsf{b} = \mathsf{R}.\mathsf{C}.\mathsf{R}^\mathsf{T} \tag{7}$$

can be rotated back and forth with respect to each other using the proper-orthogonal material-convective (polar) rotation tensor (3).

2 Geometric interpretation of total, elastic and plastic Cauchy-Green deformation tensors

For the total deformation, the left Cauchy-Green deformation tensor $(5)_1$ can be understood as an Eulerian material ellipsoid (blue ellipsoid on the right in Fig. 1 or 2) with respect to an Eulerian material sphere (red dashed circle on the right in Fig. 1 or 2) and, the right Cauchy-Green deformation tensor $(5)_2$ can be understood as a Lagrangian material ellipsoid (red dashed ellipse on the left in Fig. 1 or 2) with respect to a Lagrangian material sphere (black sphere on the left in Fig. 1 or 2). This interpretation of the total deformation can be generalized for the elastic and plastic deformations by introducing the upper-left index '@' of an *at sign* or *monkey tail*, where *empty* (no @) means *total*, 'e' *elastic* and 'p' *plastic*. The total, elastic and plastic left stretch ${}^@\mathsf{v} = \{\mathsf{v}, {}^e\mathsf{v}, {}^p\mathsf{v}\}$ and Cauchy-Green deformation tensors ${}^@\mathsf{b} = \{\mathsf{b}, {}^e\mathsf{b}, {}^p\mathsf{b}\}$ with the Eulerian unit eigenvectors ${}^@\hat{\mathbf{e}}_k = \{\hat{\mathbf{e}}_k, {}^e\hat{\mathbf{e}}_k, {}^p\hat{\mathbf{e}}_k\}$ and the corresponding right stretch ${}^@\mathsf{U} = \{\mathsf{U}, {}^e\mathsf{U}, {}^p\mathsf{U}\}$ and Cauchy-Green deformation tensors ${}^@\mathsf{C} = \{\mathsf{C}, {}^e\mathsf{C}, {}^p\mathsf{C}\}$ with the Lagrangean unit eigenvectors ${}^@\hat{\mathbf{E}}_k = \{\hat{\mathbf{E}}_k, {}^e\hat{\mathbf{E}}_k, {}^p\hat{\mathbf{E}}_k\}$ can also be understood either as Eulerian and Lagrangean material cuboids with respect to unit cubes or as Eulerian and Lagrangean material ellipsoids with respect to unit spheres, whose edges or semi-axes are oriented along the principal directions ${}^@\hat{\mathbf{e}}_k$ and ${}^@\hat{\mathbf{E}}_k$, respectively, see Fig. 2. The total, elastic, plastic stretch ${}^@\mathsf{v}$, ${}^@\mathsf{U}$ and Cauchy-Green deformation tensors ${}^@\mathsf{b}$, ${}^@\mathsf{C}$ are symmetric and positive-definite. In particular, since the elastic and plastic Eulerian and Lagrangean unit eigenvectors

$${}^@\hat{\mathbf{e}}_k = \mathsf{R}.{}^@\hat{\mathbf{E}}_k = {}^@\hat{\mathbf{E}}_k.\mathsf{R}^\mathsf{T} \qquad \text{and} \qquad {}^@\hat{\mathbf{E}}_k = \mathsf{R}^\mathsf{T}.{}^@\hat{\mathbf{e}}_k = {}^@\hat{\mathbf{e}}_k.\mathsf{R} \tag{8}$$

are also rotated with the material-convective (polar) rotation tensor (3), the elastic and plastic stretch and Cauchy-Green deformation tensors obey

$${}^@\mathsf{v} = \mathsf{R}.{}^@\mathsf{U}.\mathsf{R}^\mathsf{T}, \qquad {}^@\mathsf{U} = \mathsf{R}^\mathsf{T}.{}^@\mathsf{v}.\mathsf{R} \qquad \text{and} \qquad {}^@\mathsf{b} = \mathsf{R}.{}^@\mathsf{C}.\mathsf{R}^\mathsf{T}, \qquad {}^@\mathsf{C} = \mathsf{R}^\mathsf{T}.{}^@\mathsf{b}.\mathsf{R} \tag{9}$$

and are therefore of proper-Eulerian or proper-Lagrangean type, cf. Heiduschke (2020). Under superposed rigid-body motions (indicated by an upper-left plus sign '+'), symmetric proper-Eulerian tensors (written in lowercase) are altered: ${}^+\mathsf{t} = \mathsf{Q}.\mathsf{t}.\mathsf{Q}^\mathsf{T}$ by the superposed rigid-body rotation Q, while symmetric proper-Lagrangean tensors (written in uppercase) are invariant: ${}^+\mathsf{T} = \mathsf{T}$.

3 Bilby-Kröner-Lee formulation

According to Casey & Naghdi (1980), the multiplicative elasto-plastic decomposition

$$\mathsf{F} = {}^e\mathfrak{F}.{}^p\mathfrak{F} = ({}^e\mathfrak{F}.\underbrace{\mathfrak{Q}^\mathsf{T}).(\mathfrak{Q}}_{\mathrm{I}}.{}^p\mathfrak{F}) \tag{10}$$

of the deformation gradient (2) into an elastic tensor factor ${}^e\mathfrak{F}$ and a plastic tensor factor ${}^p\mathfrak{F}$ is not uniquely determined with respect to an intermediate rotation $\mathfrak{Q} = \mathfrak{Q}^{-\mathsf{T}}$—since one can multiply $\mathfrak{Q}^\mathsf{T}.\mathfrak{Q} = \mathrm{I}$ (a second-order identity tensor I) between ${}^e\mathfrak{F}$ and ${}^p\mathfrak{F}$. The multiplicative elasto-plastic decomposition (10) is called the Bilby-Kröner-Lee formulation, even though Bilby et al.

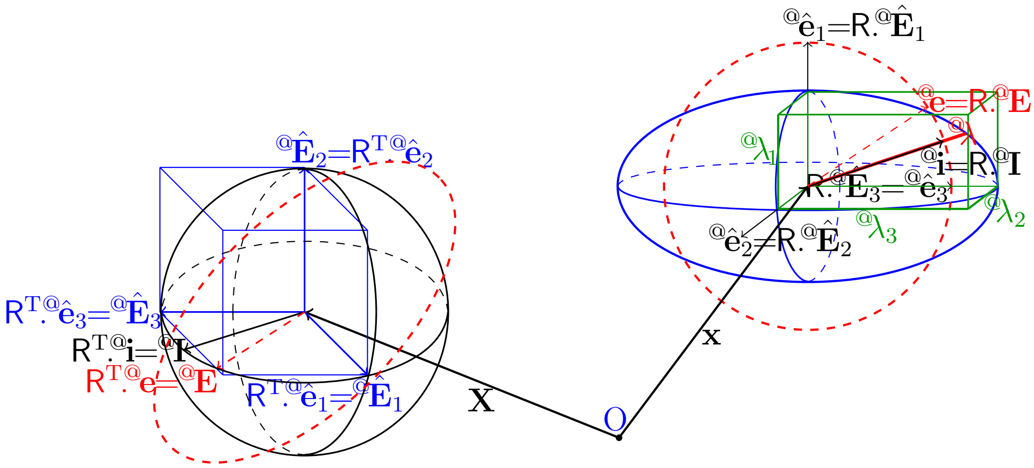

Fig. 2: Geometric interpretation of the total, elastic or plastic proper-Eulerian/proper-Lagrangean material ellipsoids and spheres or cuboids (rectangular parallelepipeds) and cubes

(1957), Kröner (1959), Lee & Liu (1967) and Lee (1969) were not the first to publish this multiplicative approach. The deformation gradient F is a two-point tensor with respect to both the Eulerian configuration and the Lagrangean configuration, since in its spectral representation (2) the stretch eigenvalues λ_k are the components relative to the mixed basis dyads of the Eulerian $\hat{\mathbf{e}}_k$ and Lagrangean $\hat{\mathbf{E}}_k$ unit eigenvectors. From the spectral representation of the elastic tensor factor of the Bilby-Kröner-Lee formulation

$$ {}^e\mathfrak{F} = {}^e\lambda_1 {}^e\hat{\mathbf{e}}_1 \otimes {}^e\hat{\mathfrak{E}}_1 + {}^e\lambda_2 {}^e\hat{\mathbf{e}}_2 \otimes {}^e\hat{\mathfrak{E}}_2 + {}^e\lambda_3 {}^e\hat{\mathbf{e}}_3 \otimes {}^e\hat{\mathfrak{E}}_3 = {}^e\lambda_k \underbrace{{}^e\hat{\mathbf{e}}_k \otimes {}^e\hat{\mathfrak{E}}_k}_{{}^e\mathfrak{R} = {}^e\mathfrak{R}^{-\mathrm{T}}} \tag{11} $$

and the plastic tensor factor of the Bilby-Kröner-Lee formulation

$$ {}^p\mathfrak{F} = {}^p\lambda_1 {}^p\hat{\mathfrak{e}}_1 \otimes {}^p\hat{\mathbf{E}}_1 + {}^p\lambda_2 {}^p\hat{\mathfrak{e}}_2 \otimes {}^p\hat{\mathbf{E}}_2 + {}^p\lambda_3 {}^p\hat{\mathfrak{e}}_3 \otimes {}^p\hat{\mathbf{E}}_3 = {}^p\lambda_k \underbrace{{}^p\hat{\mathfrak{e}}_k \otimes {}^p\hat{\mathbf{E}}_k}_{{}^p\mathfrak{R} = {}^p\mathfrak{R}^{-\mathrm{T}}}, \tag{12} $$

it is obvious that ${}^e\mathfrak{F}$ and ${}^p\mathfrak{F}$ are two-point tensors that depend on an intermediate (stress-free) configuration that is not well-defined—because of the intermediate rotation $\mathfrak{Q}$ in Eq.(10). For this reason, these tensors are written in Gothic font. The elastic left stretch tensor of the Bilby-Kröner-Lee formulation

$$ {}^e\mathsf{V} = \sqrt{{}^e\mathfrak{F}.{}^e\mathfrak{F}^{\mathrm{T}}} = {}^e\lambda_1 {}^e\hat{\mathbf{e}}_1 \otimes {}^e\hat{\mathbf{e}}_1 + {}^e\lambda_2 {}^e\hat{\mathbf{e}}_2 \otimes {}^e\hat{\mathbf{e}}_2 + {}^e\lambda_3 {}^e\hat{\mathbf{e}}_3 \otimes {}^e\hat{\mathbf{e}}_3 \tag{13} $$

is of proper-Eulerian type, the plastic right stretch tensor of the Bilby-Kröner-Lee formulation

$$ {}^p\mathsf{U} = \sqrt{{}^p\mathfrak{F}^{\mathrm{T}}.{}^p\mathfrak{F}} = {}^p\lambda_1 {}^p\hat{\mathbf{E}}_1 \otimes {}^p\hat{\mathbf{E}}_1 + {}^p\lambda_2 {}^p\hat{\mathbf{E}}_2 \otimes {}^p\hat{\mathbf{E}}_2 + {}^p\lambda_3 {}^p\hat{\mathbf{E}}_3 \otimes {}^p\hat{\mathbf{E}}_3 \tag{14} $$

is of proper-Lagrangean type, but the elastic right stretch of the Bilby-Kröner-Lee formulation

$$ {}^e\mathfrak{U} = \sqrt{{}^e\mathfrak{F}^{\mathrm{T}}.{}^e\mathfrak{F}} = {}^e\lambda_1 {}^e\hat{\mathfrak{E}}_1 \otimes {}^e\hat{\mathfrak{E}}_1 + {}^e\lambda_2 {}^e\hat{\mathfrak{E}}_2 \otimes {}^e\hat{\mathfrak{E}}_2 + {}^e\lambda_3 {}^e\hat{\mathfrak{E}}_3 \otimes {}^e\hat{\mathfrak{E}}_3 \tag{15} $$

and the plastic left stretch of the Bilby-Kröner-Lee formulation

$$ {}^p\mathfrak{v} = \sqrt{{}^p\mathfrak{F}.{}^p\mathfrak{F}^{\mathrm{T}}} = {}^p\lambda_1 {}^p\hat{\mathfrak{e}}_1 \otimes {}^p\hat{\mathfrak{e}}_1 + {}^p\lambda_2 {}^p\hat{\mathfrak{e}}_2 \otimes {}^p\hat{\mathfrak{e}}_2 + {}^p\lambda_3 {}^p\hat{\mathfrak{e}}_3 \otimes {}^p\hat{\mathfrak{e}}_3 \tag{16} $$

are not of a proper type because they depend on an intermediate (stress-free) configuration which is not well-determined. The Bilby-Kröner-Lee formulation requires a plastic flow rule of type

$$ {}^p\dot{\mathfrak{F}} = \underbrace{{}^p\dot{\mathfrak{R}}.{}^p\mathfrak{R}^{\mathrm{T}}}_{{}^p\Omega = -{}^p\Omega^{\mathrm{T}}}.{}^p\mathfrak{F} + {}^p\mathfrak{R}.{}^p\dot{\mathsf{U}} = \cdots \tag{17} $$

with nine degrees of freedom specified either by nine components ${}^p\dot{\mathfrak{F}}_{iA}$ of ${}^p\dot{\mathfrak{F}} = {}^p\dot{\mathfrak{F}}_{iA}\mathbf{e}_i \otimes \mathbf{E}_A$ or by three components ${}^p\Omega_k$ of the plastic spin tensor ${}^p\Omega = -{}^p\Omega^{\mathrm{T}}$ plus six components ${}^p\dot{U}_{AB} = {}^p\dot{U}_{BA}$ of ${}^p\dot{\mathsf{U}} = {}^p\dot{\mathsf{U}}^{\mathrm{T}} = {}^p\dot{U}_{AB}\mathbf{E}_A \otimes \mathbf{E}_B$. A constraint between the total, elastic and plastic spin tensors Ω, ${}^e\Omega$ and ${}^p\Omega$, respectively, is obtained from the velocity gradient

$$ \dot{\mathsf{F}}.\mathsf{F}^{-1} = \underbrace{\dot{\mathsf{R}}.\mathsf{R}^{\mathrm{T}}}_{\Omega = -\Omega^{\mathrm{T}}} + \mathsf{R}.(\dot{\mathsf{U}}.\mathsf{U}^{-1}).\mathsf{R}^{\mathrm{T}} = \underbrace{{}^e\dot{\mathfrak{R}}.{}^e\mathfrak{R}^{\mathrm{T}}}_{{}^e\Omega = -{}^e\Omega^{\mathrm{T}}} + {}^e\mathfrak{F}.\underbrace{{}^p\dot{\mathfrak{R}}.{}^p\mathfrak{R}^{\mathrm{T}}}_{{}^p\Omega = -{}^p\Omega^{\mathrm{T}}}.{}^e\mathfrak{F}^{-1} + {}^e\mathfrak{R}.({}^e\dot{\mathfrak{U}}.{}^e\mathfrak{U}^{-1}).{}^e\mathfrak{R}^{\mathrm{T}} + ({}^e\mathfrak{F}.{}^p\mathfrak{R}).({}^p\dot{\mathsf{U}}.{}^p\mathsf{U}^{-1}).({}^e\mathfrak{F}.{}^p\mathfrak{R})^{-1} \tag{18} $$

for the Bilby-Kröner-Lee formulation.

4 Commutative-symmetrical stretch tensor products

Applying the polar decomposition to the elastic ${}^e\mathfrak{F} = {}^e\mathsf{v}.{}^e\mathfrak{R}$ and the plastic ${}^p\mathfrak{F} = {}^p\mathfrak{R}.{}^p\mathsf{U}$ tensor factors of the multiplicative elasto-plastic decomposition (10) of the deformation gradient

$$\mathsf{F} = {}^e\mathfrak{F}.{}^p\mathfrak{F} = ({}^e\mathsf{v}.\underbrace{{}^e\mathfrak{R}).({}^p\mathfrak{R}}_{\mathfrak{R}}.{}^p\mathsf{U}) = {}^e\mathsf{v}.\mathfrak{R}.{}^p\mathsf{U}, \tag{19}$$

the proper-orthogonal elastic ${}^e\mathfrak{R}$ and plastic ${}^p\mathfrak{R}$ rotation tensors can be combined into one proper-orthogonal rotation-tensor product $\mathfrak{R} = {}^e\mathfrak{R}.{}^p\mathfrak{R}$ according to Bammann & Johnson (1987). Even though the elastic rotation tensor

$${}^e\mathfrak{R} = {}^e\hat{\mathbf{e}}_1 \otimes {}^e\hat{\mathfrak{E}}_1 + {}^e\hat{\mathbf{e}}_2 \otimes {}^e\hat{\mathfrak{E}}_2 + {}^e\hat{\mathbf{e}}_3 \otimes {}^e\hat{\mathfrak{E}}_3 = {}^e\hat{\mathbf{e}}_k \otimes {}^e\hat{\mathfrak{E}}_k \tag{20}$$

and the plastic rotation tensor

$${}^p\mathfrak{R} = {}^p\hat{\mathfrak{e}}_1 \otimes {}^p\hat{\mathbf{E}}_1 + {}^p\hat{\mathfrak{e}}_2 \otimes {}^p\hat{\mathbf{E}}_2 + {}^p\hat{\mathfrak{e}}_3 \otimes {}^p\hat{\mathbf{E}}_3 = {}^p\hat{\mathfrak{e}}_k \otimes {}^p\hat{\mathbf{E}}_k \tag{21}$$

are two-point tensors (each with one "Gothic" basis vector) each with one respect to an intermediate (stress-free) configuration, their product rotation tensor

$$\mathfrak{R} = \mathfrak{R}^{-\mathrm{T}} = {}^e\mathfrak{R}.{}^p\mathfrak{R} = {}^e\hat{\mathbf{e}}_i \underbrace{\otimes {}^e\hat{\mathfrak{E}}_i . {}^p\hat{\mathfrak{e}}_j}_{\mathfrak{R}_{ij}} \otimes {}^p\hat{\mathbf{E}}_j = \mathfrak{R}_{ij}\, {}^e\hat{\mathbf{e}}_i \otimes {}^p\hat{\mathbf{E}}_j, \qquad \det(\mathfrak{R}) = 1 \tag{22}$$

is a two-point tensor of a proper type with respect to both the Eulerian configuration and the Lagrangean configuration (without Gothic basis vectors). Substituting the elastic left stretch ${}^e\mathsf{v}$ in Eq.(19) by the R-forward rotation $\mathsf{R}.{}^e\mathsf{U}.\mathsf{R}^\mathrm{T}$ of the elastic right stretch ${}^e\mathsf{U}$, as noted on the left-hand side of

$$\mathsf{F} = {}^e\mathsf{v}.\mathfrak{R}.{}^p\mathsf{U} = (\mathsf{R}.{}^e\mathsf{U}.\underbrace{\mathsf{R}^\mathrm{T}).\mathfrak{R}}_{\mathsf{X}}.{}^p\mathsf{U} = \mathsf{R}.\underbrace{({}^e\mathsf{U}.\mathsf{X}.{}^p\mathsf{U})}_{\mathsf{U}} = {}^e\mathsf{v}.\underbrace{\mathfrak{R}.(\mathsf{R}^\mathrm{T}}_{\mathsf{x}=\mathsf{R}.\mathsf{X}.\mathsf{R}^\mathrm{T}}.{}^p\mathsf{v}.\mathsf{R}) = \underbrace{({}^e\mathsf{v}.\mathsf{x}.{}^p\mathsf{v})}_{\mathsf{v}}.\mathsf{R}, \tag{23}$$

results in the Lagrangean commutative-symmetrical elasto-plastic stretch tensor product

$$\mathsf{U} = \mathsf{U}^\mathrm{T} = {}^e\mathsf{U}.\mathsf{X}.{}^p\mathsf{U} = {}^p\mathsf{U}.\mathsf{X}^\mathrm{T}.{}^e\mathsf{U} = {}^e\mathsf{U}.\sqrt{{}^e\mathsf{U}^{-1}.{}^p\mathsf{C}.{}^e\mathsf{U}^{-1}}.{}^e\mathsf{U} = {}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.{}^e\mathsf{C}.{}^p\mathsf{U}^{-1}}.{}^p\mathsf{U} = {}^e\mathsf{C}.\mathsf{U}^{-1}.{}^p\mathsf{C} = {}^p\mathsf{C}.\mathsf{U}^{-1}.{}^e\mathsf{C} \tag{24}$$

with the Lagrangean symmetrizing-rotation tensor

$$\begin{aligned}\mathsf{X} = \mathsf{X}^{-\mathrm{T}} &= \sqrt{{}^e\mathsf{U}^{-1}.{}^p\mathsf{C}.{}^e\mathsf{U}^{-1}}.{}^e\mathsf{U}.{}^p\mathsf{U}^{-1} = \sqrt{{}^e\mathsf{U}.{}^p\mathsf{C}^{-1}.{}^e\mathsf{U}}.{}^e\mathsf{U}^{-1}.{}^p\mathsf{U} = {}^e\mathsf{U}^{-1}.{}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.{}^e\mathsf{C}.{}^p\mathsf{U}^{-1}} = {}^e\mathsf{U}.{}^p\mathsf{U}^{-1}.\sqrt{{}^p\mathsf{U}.{}^e\mathsf{C}^{-1}.{}^p\mathsf{U}}\\ &= {}^e\mathsf{U}^{-1}.\mathsf{U}.{}^p\mathsf{U}^{-1} = {}^e\mathsf{U}.\mathsf{U}^{-1}.{}^p\mathsf{U},\end{aligned} \tag{25}$$

$\det(\mathsf{X}) = 1$. And substituting analogously the plastic right stretch ${}^p\mathsf{U}$ in Eq.(19) by the R^T-back rotation $\mathsf{R}^\mathrm{T}.{}^p\mathsf{v}.\mathsf{R}$ of the plastic left stretch ${}^p\mathsf{v}$, as noted on the right-hand side of (23), results in the Eulerian commutative-symmetrical elasto-plastic stretch tensor product

$$\mathsf{v} = \mathsf{v}^\mathrm{T} = {}^e\mathsf{v}.\mathsf{x}.{}^p\mathsf{v} = {}^p\mathsf{v}.\mathsf{x}^\mathrm{T}.{}^e\mathsf{v} = {}^e\mathsf{v}.\sqrt{{}^e\mathsf{v}^{-1}.{}^p\mathsf{b}.{}^e\mathsf{v}^{-1}}.{}^e\mathsf{v} = {}^p\mathsf{v}.\sqrt{{}^p\mathsf{v}^{-1}.{}^e\mathsf{b}.{}^p\mathsf{v}^{-1}}.{}^p\mathsf{v} = {}^e\mathsf{b}.\mathsf{v}^{-1}.{}^p\mathsf{b} = {}^p\mathsf{b}.\mathsf{v}^{-1}.{}^e\mathsf{b} = \mathsf{R}.(\mathsf{U}).\mathsf{R}^\mathrm{T} \tag{26}$$

with the Eulerian symmetrizing-rotation tensor

$$\begin{aligned}\mathsf{x} = \mathsf{x}^{-\mathrm{T}} &= \sqrt{{}^e\mathsf{v}^{-1}.{}^p\mathsf{b}.{}^e\mathsf{v}^{-1}}.{}^e\mathsf{v}.{}^p\mathsf{v}^{-1} = \sqrt{{}^e\mathsf{v}.{}^p\mathsf{b}^{-1}.{}^e\mathsf{v}}.{}^e\mathsf{v}^{-1}.{}^p\mathsf{v} = {}^e\mathsf{v}^{-1}.{}^p\mathsf{v}.\sqrt{{}^p\mathsf{v}^{-1}.{}^e\mathsf{b}.{}^p\mathsf{v}^{-1}} = {}^e\mathsf{v}.{}^p\mathsf{v}^{-1}.\sqrt{{}^p\mathsf{v}.{}^e\mathsf{b}^{-1}.{}^p\mathsf{v}}\\ &= {}^e\mathsf{v}^{-1}.\mathsf{v}.{}^p\mathsf{v}^{-1} = {}^e\mathsf{v}.\mathsf{v}^{-1}.{}^p\mathsf{v} = \mathsf{R}.(\mathsf{X}).\mathsf{R}^\mathrm{T},\end{aligned} \tag{27}$$

$\det(\mathsf{x}) = 1$, cf. Heiduschke (2018a,b, 2020, 2021).

Commutative-symmetrical stretch tensor products require only symmetric plastic flow rules ${}^p\dot{\mathsf{U}} = {}^p\dot{\mathsf{U}}^\mathrm{T} = \cdots$ with six degrees of freedom specified by the six components ${}^p\dot{U}_{AB} = {}^p\dot{U}_{BA}$ of the plastic flow tensor ${}^p\dot{\mathsf{U}} = {}^p\dot{U}_{AB}\mathbf{E}_A \otimes \mathbf{E}_B$, since in them only the material-convective (polar) rotation tensor R is used and ${}^e\mathfrak{R}$, ${}^p\mathfrak{R}$ do not occur at all.

While the plastic deformation is determined from material-convective integration of the plastic flow rule and, after this integration, can be expressed by the plastic stretch tensors ${}^p\mathsf{v}$ or ${}^p\mathsf{U}$, respectively, the elastic deformation for the commutative-symmetrical stretch tensor products (24) or (26) is given by

$${}^e\mathsf{U} = \sqrt{\mathsf{U}.{}^p\mathsf{C}^{-1}.\mathsf{U}} = \mathcal{X}_1 {}^e\hat{\mathbf{E}}_1 \otimes {}^e\hat{\mathbf{E}}_1 + \mathcal{X}_2 {}^e\hat{\mathbf{E}}_2 \otimes {}^e\hat{\mathbf{E}}_2 + \mathcal{X}_3 {}^e\hat{\mathbf{E}}_3 \otimes {}^e\hat{\mathbf{E}}_3 = \mathsf{R}^\mathrm{T}.{}^e\mathsf{v}.\mathsf{R} \tag{28}$$

or

$${}^e\mathsf{v} = \sqrt{\mathsf{v}.{}^p\mathsf{b}^{-1}.\mathsf{v}} = \mathcal{X}_1 {}^e\hat{\mathbf{e}}_1 \otimes {}^e\hat{\mathbf{e}}_1 + \mathcal{X}_2 {}^e\hat{\mathbf{e}}_2 \otimes {}^e\hat{\mathbf{e}}_2 + \mathcal{X}_3 {}^e\hat{\mathbf{e}}_3 \otimes {}^e\hat{\mathbf{e}}_3 = \sqrt{{}^e\mathfrak{F}.{}^e\mathfrak{F}^\mathrm{T}}, \tag{29}$$

respectively. Although the elastic left stretch ${}^e\mathsf{v}$ tensors (13) and (29) of both the Bilby-Kröner-Lee formulation and the commutative-symmetrical stretch tensor product are equal, the corresponding elastic right stretch tensors ${}^e\mathfrak{U}=\sqrt{{}^e\mathfrak{F}^{\mathrm{T}}.{}^e\mathfrak{F}}$ (15) of the Bilby-Kröner-Lee formulation and ${}^e\mathsf{U}$ (28) of the commutative-symmetrical stretch tensor product differ from each other. The elastic right stretch tensor (15) of the Bilby-Kröner-Lee formulation is not well-determined, cf. Casey & Naghdi (1980), since

$$ {}^e\mathfrak{U} = \sqrt{{}^e\mathfrak{F}^{\mathrm{T}}.{}^e\mathfrak{F}} = {}^e\mathfrak{R}^{\mathrm{T}}.{}^e\mathsf{v}.{}^e\mathfrak{R} = \overbrace{{}^e\mathfrak{R}^{\mathrm{T}}.\underbrace{\sqrt{\mathsf{v}.{}^p\mathsf{b}^{-1}.\mathsf{v}}}_{\mathsf{x}.\sqrt{{}^p\mathsf{v}^{-1}.\mathsf{b}.{}^p\mathsf{v}^{-1}}.\mathsf{x}^{\mathrm{T}}}.{}^e\mathfrak{R}}^{{}^p\mathfrak{R}.\mathsf{X}^{\mathrm{T}}.{}^e\mathsf{U}.\mathsf{X}.{}^p\mathfrak{R}^{\mathrm{T}} = {}^p\mathfrak{R}.\sqrt{{}^p\mathsf{U}^{-1}.\mathsf{C}.{}^p\mathsf{U}^{-1}}.{}^p\mathfrak{R}^{\mathrm{T}}} = \mathfrak{A}_1\,{}^e\hat{\mathfrak{E}}_1\otimes{}^e\hat{\mathfrak{E}}_1+\mathfrak{A}_2\,{}^e\hat{\mathfrak{E}}_2\otimes{}^e\hat{\mathfrak{E}}_2+\mathfrak{A}_3\,{}^e\hat{\mathfrak{E}}_3\otimes{}^e\hat{\mathfrak{E}}_3 \tag{30} $$

is defined with respect to an intermediate (stress-free) configuration, whereas the elastic right stretch tensor ${}^e\mathsf{U}$ (28) of the commutative-symmetrical stretch tensor product (24) is of proper-Lagrangean type.
A helpful relation can be derived from the polar decomposition of the commutative-symmetrical products with the symmetric positive-definite stretch tensor factors ${}^@\mathsf{v}=\{\mathsf{v}, {}^e\mathsf{v}, {}^p\mathsf{v}\}$ or ${}^@\mathsf{U}=\{\mathsf{U}, {}^e\mathsf{U}, {}^p\mathsf{U}\}$, as shown next: the right multiplication of, say, the left commutative-symmetrical stretch tensor product (26) with ${}^p\mathsf{v}^{-1}$ yields a non-symmetric product tensor

$$ \mathsf{A} = \mathsf{v}.{}^p\mathsf{v}^{-1} = {}^e\mathsf{v}.\mathsf{x} = \sqrt{\underbrace{\mathsf{v}.{}^p\mathsf{b}^{-1}.\mathsf{v}}_{\mathsf{A}.\mathsf{A}^{\mathrm{T}}}}.\mathsf{x} = \mathsf{x}.\sqrt{\underbrace{{}^p\mathsf{v}^{-1}.\mathsf{b}.{}^p\mathsf{v}^{-1}}_{\mathsf{A}^{\mathrm{T}}.\mathsf{A}}}\,, \tag{31} $$

from which the x-forward rotation of the left stretch tensor products follows as

$$ {}^e\mathsf{v} = \sqrt{\mathsf{v}.{}^p\mathsf{b}^{-1}.\mathsf{v}} = \mathsf{x}.\sqrt{{}^p\mathsf{v}^{-1}.\mathsf{b}.{}^p\mathsf{v}^{-1}}.\mathsf{x}^{\mathrm{T}}. \tag{32} $$

5 Proper-orthogonal arbitray basis vectors co-rotating with the material

The rotation $\mathsf{R}=\hat{\mathbf{e}}_1\otimes\hat{\mathbf{E}}_1+\hat{\mathbf{e}}_2\otimes\hat{\mathbf{E}}_2+\hat{\mathbf{e}}_3\otimes\hat{\mathbf{E}}_3$ of the Lagrangean principal axes $\hat{\mathbf{E}}_k$ to the Eulerian principal axes $\hat{\mathbf{e}}_k$ defines the proper-orthogonal material-convective (polar) rotation tensor (3). If the proper-orthogonal arbitrary Eulerian $\mathbf{e}_k$ and Lagrangean $\mathbf{E}_k$ basis vectors (without *hat*) are also material-convective co-rotational with R, then $\mathbf{e}_k=\mathsf{R}.\mathbf{E}_k=\mathbf{E}_k.\mathsf{R}^{\mathrm{T}}$ and the components of corresponding Eulerian $\mathsf{t}=t_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ and Lagrangean $\mathsf{T}=T_{ij}\mathbf{E}_i\otimes\mathbf{E}_j$ symmetric tensors are equal, $t_{ij}=T_{ij}$, because

$$ \mathsf{t} = \mathsf{R}.\mathsf{T}.\mathsf{R}^{\mathrm{T}} = T_{ij}\mathsf{R}.(\mathbf{E}_i\otimes\mathbf{E}_j).\mathsf{R}^{\mathrm{T}} = T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j = t_{ij}\mathbf{e}_i\otimes\mathbf{e}_j. \tag{33} $$

This is also the case for the time derivatives $\mathring{\mathsf{t}} = \mathsf{R}.\dot{\mathsf{T}}.\mathsf{R}^{\mathrm{T}}$ of corresponding Eulerian and Lagrangean symmetric tensors $\mathsf{t} = \mathsf{R}.\mathsf{T}.\mathsf{R}^{\mathrm{T}}$ when the Green-Naghdi rate $\mathring{\mathsf{t}}$ is applied as the time derivative for symmetric Eulerian tensors t. The components of $\mathring{\mathsf{t}}=\dot{t}_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ and $\dot{\mathsf{T}}=\dot{T}_{ij}\mathbf{E}_i\otimes\mathbf{E}_j$ are then also equal: $\dot{t}_{ij}=\dot{T}_{ij}$.
The time derivative of a symmetric Eulerian tensor $\mathsf{t}=t_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ is given from the product rule as

$$ \dot{\mathsf{t}} = (t_{ij}\mathbf{e}_i\otimes\mathbf{e}_j)^{\dot{}} = \underbrace{\dot{t}_{ij}\mathbf{e}_i\otimes\mathbf{e}_j}_{\mathring{\mathsf{t}}} + t_{ij}\dot{\mathbf{e}}_i\otimes\mathbf{e}_j + t_{ij}\mathbf{e}_i\otimes\dot{\mathbf{e}}_j = \mathring{\mathsf{t}} + \Omega.\mathsf{t} - \mathsf{t}.\Omega \tag{34} $$

the time derivative of the components $\dot{t}_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ plus the time derivatives of the basis unit vectors $\dot{\mathbf{e}}_k=\Omega.\mathbf{e}_k=\mathbf{e}_k.\Omega^{\mathrm{T}}=-\mathbf{e}_k.\Omega$. When the anti-symmetric spin tensor $\Omega=-\Omega^{\mathrm{T}}$ represents an arbitrary spin, the term $\dot{t}_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ represents the time derivative co-rotational with an arbitrary Ω. However, if Ω is defined in terms of (3) as the material-convective spin tensor

$$ \Omega = -\Omega^{\mathrm{T}} = \dot{\mathsf{R}}.\mathsf{R}^{\mathrm{T}}, \tag{35} $$

then the time derivative of the components $\dot{t}_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ becomes the translational and rotational material-convective time derivative $\mathring{\mathsf{t}}$ (marked with a *ring*), the so-called Green-Naghdi rate

$$ \mathring{\mathsf{t}} = \mathsf{R}.\dot{\mathsf{T}}.\mathsf{R}^{\mathrm{T}} = \mathsf{R}.\underbrace{(\mathsf{R}^{\mathrm{T}}.\mathsf{t}.\mathsf{R})^{\dot{}}}_{\dot{\mathsf{T}}=(T_{ij}\mathbf{E}_i\otimes\mathbf{E}_j)^{\dot{}}=\dot{T}_{ij}\mathbf{E}_i\otimes\mathbf{E}_j}.\mathsf{R}^{\mathrm{T}} = \dot{\mathsf{t}} - \Omega.\mathsf{t} + \mathsf{t}.\Omega \tag{36} $$

of symmetric proper-Eulerian tensors t [cf. Green & Naghdi (1965) p.273, eqs.(8.20)–(8.23)].

6 Fourth-order transformations for symmetric second-order rate tensors

When the plastic flow rule ${}^p\dot{\mathsf{C}}=\cdots$ is formulated, e.g., for the Lagrangean plastic Cauchy-Green deformation tensor rate ${}^p\dot{\mathsf{C}}$, it can be easily expressed by the Lagrangean plastic stretch tensor ${}^p\mathsf{U}$ and its rate ${}^p\dot{\mathsf{U}}$ according to ${}^p\dot{\mathsf{C}}={}^p\dot{\mathsf{U}}.{}^p\mathsf{U}+{}^p\mathsf{U}.{}^p\dot{\mathsf{U}}$.
But what if the plastic flow rule ${}^p\dot{\mathsf{U}}=\cdots$ is formulated for the Lagrangean plastic stretch tensor rate ${}^p\dot{\mathsf{U}}$? How to express ${}^p\dot{\mathsf{U}}$ as a function of ${}^p\dot{\mathsf{C}}$ and ${}^p\mathsf{C}$?
Here, transformations with fourth-order tensors, e.g., $\mathfrak{G}=\mathfrak{G}_{ijk\ell}\mathbf{E}_i\otimes\mathbf{E}_j\otimes\mathbf{E}_k\otimes\mathbf{E}_\ell=\mathfrak{G}_{jik\ell}\mathbf{E}_i\otimes\mathbf{E}_j\otimes\mathbf{E}_k\otimes\mathbf{E}_\ell=\mathfrak{G}_{ij\ell k}\mathbf{E}_i\otimes\mathbf{E}_j\otimes\mathbf{E}_k\otimes\mathbf{E}_\ell$ for the symmetric second-order rate tensors ${}^p\dot{\mathsf{U}}={}^p\dot{U}_{ij}\mathbf{E}_i\otimes\mathbf{E}_j={}^p\dot{U}_{ji}\mathbf{E}_i\otimes\mathbf{E}_j$ and ${}^p\dot{\mathsf{C}}={}^p\dot{C}_{k\ell}\mathbf{E}_k\otimes\mathbf{E}_\ell={}^p\dot{C}_{\ell k}\mathbf{E}_k\otimes\mathbf{E}_\ell$ are useful, which map two symmetric second-order rate tensors onto each other, back: ${}^p\dot{U}_{ij}=\mathfrak{G}^{-1}_{ijk\ell}\,{}^p\dot{C}_{k\ell}$ and forth: ${}^p\dot{C}_{ij}=\mathfrak{G}_{ijk\ell}\,{}^p\dot{U}_{k\ell}$.

For the transformation ${}^p\dot{C}_{ij} = \mathfrak{G}_{ijk\ell}\,{}^p\dot{U}_{k\ell}$, the tensor components relative to an arbitrary basis vector system $\mathbf{E}_k$ read

$$\begin{Bmatrix} {}^p\dot{C}_{xx} \\ {}^p\dot{C}_{yy} \\ {}^p\dot{C}_{zz} \\ \sqrt{2}\cdot{}^p\dot{C}_{xy} \\ \sqrt{2}\cdot{}^p\dot{C}_{xz} \\ \sqrt{2}\cdot{}^p\dot{C}_{yz} \end{Bmatrix} = \begin{bmatrix} \mathfrak{G}_{xxxx} & \mathfrak{G}_{xxyy} & \mathfrak{G}_{xxzz} & \sqrt{2}\cdot\mathfrak{G}_{xxxy} & \sqrt{2}\cdot\mathfrak{G}_{xxxz} & \sqrt{2}\cdot\mathfrak{G}_{xxyz} \\ \mathfrak{G}_{yyxx} & \mathfrak{G}_{yyyy} & \mathfrak{G}_{yyzz} & \sqrt{2}\cdot\mathfrak{G}_{yyxy} & \sqrt{2}\cdot\mathfrak{G}_{yyxz} & \sqrt{2}\cdot\mathfrak{G}_{yyyz} \\ \mathfrak{G}_{zzxx} & \mathfrak{G}_{zzyy} & \mathfrak{G}_{zzzz} & \sqrt{2}\cdot\mathfrak{G}_{zzxy} & \sqrt{2}\cdot\mathfrak{G}_{zzxz} & \sqrt{2}\cdot\mathfrak{G}_{zzyz} \\ \sqrt{2}\cdot\mathfrak{G}_{xyxx} & \sqrt{2}\cdot\mathfrak{G}_{xyyy} & \sqrt{2}\cdot\mathfrak{G}_{xyzz} & 2\cdot\mathfrak{G}_{xyxy} & 2\cdot\mathfrak{G}_{xyxz} & 2\cdot\mathfrak{G}_{xyyz} \\ \sqrt{2}\cdot\mathfrak{G}_{xzxx} & \sqrt{2}\cdot\mathfrak{G}_{xzyy} & \sqrt{2}\cdot\mathfrak{G}_{xzzz} & 2\cdot\mathfrak{G}_{xzxy} & 2\cdot\mathfrak{G}_{xzxz} & 2\cdot\mathfrak{G}_{xzyz} \\ \sqrt{2}\cdot\mathfrak{G}_{yzxx} & \sqrt{2}\cdot\mathfrak{G}_{yzyy} & \sqrt{2}\cdot\mathfrak{G}_{yzzz} & 2\cdot\mathfrak{G}_{yzxy} & 2\cdot\mathfrak{G}_{yzxz} & 2\cdot\mathfrak{G}_{yzyz} \end{bmatrix} \begin{Bmatrix} {}^p\dot{U}_{xx} \\ {}^p\dot{U}_{yy} \\ {}^p\dot{U}_{zz} \\ \sqrt{2}\cdot{}^p\dot{U}_{xy} \\ \sqrt{2}\cdot{}^p\dot{U}_{xz} \\ \sqrt{2}\cdot{}^p\dot{U}_{yz} \end{Bmatrix} \tag{37}$$

in Voigt (1910) pseudo vector/matrix notation and with the (red) factors $\sqrt{2}$ and 2 of Sayir (1970). The symmetric positive-definite second-order Lagrangean tensors ${}^p\mathsf{C}$ and ${}^p\mathsf{U}$ are isotropic tensor functions of each other and therefore share the same Lagrangean principal axes $\hat{\mathbf{E}}_k$ (which are functions of time). On the one hand, the component matrices of ${}^p\mathsf{C}$ and ${}^p\mathsf{U}$

$$[{}^p\mathsf{C}] = \begin{bmatrix} {}^p\lambda_1{}^2 & 0 & 0 \\ 0 & {}^p\lambda_2{}^2 & 0 \\ 0 & 0 & {}^p\lambda_3{}^2 \end{bmatrix}, \qquad [{}^p\mathsf{U}] = \begin{bmatrix} {}^p\lambda_1 & 0 & 0 \\ 0 & {}^p\lambda_2 & 0 \\ 0 & 0 & {}^p\lambda_3 \end{bmatrix} \tag{38}$$

have a diagonal form in spectral representation. On the other hand, the component matrices of their time derivatives ${}^p\dot{\mathsf{C}}$ and ${}^p\dot{\mathsf{U}}$

$$[{}^p\dot{\mathsf{C}}] = \begin{bmatrix} 2\,{}^p\lambda_1{}^p\dot\lambda_1 & ({}^p\lambda_1{}^2-{}^p\lambda_2{}^2)\Lambda_3 & ({}^p\lambda_3{}^2-{}^p\lambda_1{}^2)\Lambda_2 \\ ({}^p\lambda_1{}^2-{}^p\lambda_2{}^2)\Lambda_3 & 2\,{}^p\lambda_2{}^p\dot\lambda_2 & ({}^p\lambda_2{}^2-{}^p\lambda_3{}^2)\Lambda_1 \\ ({}^p\lambda_3{}^2-{}^p\lambda_1{}^2)\Lambda_2 & ({}^p\lambda_2{}^2-{}^p\lambda_3{}^2)\Lambda_1 & 2\,{}^p\lambda_3{}^p\dot\lambda_3 \end{bmatrix}, \quad [{}^p\dot{\mathsf{U}}] = \begin{bmatrix} {}^p\dot\lambda_1 & ({}^p\lambda_1-{}^p\lambda_2)\Lambda_3 & ({}^p\lambda_3-{}^p\lambda_1)\Lambda_2 \\ ({}^p\lambda_1-{}^p\lambda_2)\Lambda_3 & {}^p\dot\lambda_2 & ({}^p\lambda_2-{}^p\lambda_3)\Lambda_1 \\ ({}^p\lambda_3-{}^p\lambda_1)\Lambda_2 & ({}^p\lambda_2-{}^p\lambda_3)\Lambda_1 & {}^p\dot\lambda_3 \end{bmatrix} \tag{39}$$

exhibit diagonal and off-diagonal components in their spectral representations as well. The diagonal components are obtained from the time derivatives of the eigenvalues and the off-diagonal components from the time derivatives $\dot{\hat{\mathbf{E}}}_k = \Lambda \times \hat{\mathbf{E}}_k$ of the Lagrangean unit eigenvectors $\hat{\mathbf{E}}_k$ spinning with the rotation speed vector $\Lambda = \Lambda_1\hat{\mathbf{E}}_1 + \Lambda_2\hat{\mathbf{E}}_2 + \Lambda_3\hat{\mathbf{E}}_3$ of the Lagrangean principal axes $\hat{\mathbf{E}}_k$, where '$\times$' denotes the cross product operator.

In a principal co-ordinate system $\hat{\mathbf{E}}_k$, the pseudo-matrix of the components $\hat{\mathfrak{G}}_{ijk\ell}$ of the fourth-order transformation tensor $\mathfrak{G}$ has a diagonal form in Voigt-Sayir representation, such that

$$\begin{Bmatrix} 2\,{}^p\lambda_1{}^p\dot\lambda_1 \\ 2\,{}^p\lambda_2{}^p\dot\lambda_2 \\ 2\,{}^p\lambda_3{}^p\dot\lambda_3 \\ \sqrt{2}\cdot({}^p\lambda_1^2-{}^p\lambda_2^2)\Lambda_3 \\ \sqrt{2}\cdot({}^p\lambda_3^2-{}^p\lambda_1^2)\Lambda_2 \\ \sqrt{2}\cdot({}^p\lambda_2^2-{}^p\lambda_3^2)\Lambda_1 \end{Bmatrix} = \begin{bmatrix} 2\,{}^p\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\,{}^p\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,{}^p\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\cdot\dfrac{{}^p\lambda_1+{}^p\lambda_2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\cdot\dfrac{{}^p\lambda_1+{}^p\lambda_3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\cdot\dfrac{{}^p\lambda_2+{}^p\lambda_3}{2} \end{bmatrix} \begin{Bmatrix} {}^p\dot\lambda_1 \\ {}^p\dot\lambda_2 \\ {}^p\dot\lambda_3 \\ \sqrt{2}\cdot({}^p\lambda_1-{}^p\lambda_2)\Lambda_3 \\ \sqrt{2}\cdot({}^p\lambda_3-{}^p\lambda_1)\Lambda_2 \\ \sqrt{2}\cdot({}^p\lambda_2-{}^p\lambda_3)\Lambda_1 \end{Bmatrix} \tag{40}$$

its components can directly be determined from the spectral components (39) of the two pseudo-vectors in (40). The reverse transformation ${}^p\dot{U}_{ij} = \mathfrak{G}^{-1}_{ijk\ell}\,{}^p\dot{C}_{k\ell}$ with the component matrix of $\mathfrak{G}^{-1}$ in Voigt-Sayir representation is given by

$$\begin{Bmatrix} {}^p\dot\lambda_1 \\ {}^p\dot\lambda_2 \\ {}^p\dot\lambda_3 \\ \sqrt{2}\cdot({}^p\lambda_1-{}^p\lambda_2)\Lambda_3 \\ \sqrt{2}\cdot({}^p\lambda_3-{}^p\lambda_1)\Lambda_2 \\ \sqrt{2}\cdot({}^p\lambda_2-{}^p\lambda_3)\Lambda_1 \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{2\,{}^p\lambda_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{2\,{}^p\lambda_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{2\,{}^p\lambda_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\cdot\dfrac{1/2}{{}^p\lambda_1+{}^p\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\cdot\dfrac{1/2}{{}^p\lambda_1+{}^p\lambda_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\cdot\dfrac{1/2}{{}^p\lambda_2+{}^p\lambda_3} \end{bmatrix} \begin{Bmatrix} 2\,{}^p\lambda_1{}^p\dot\lambda_1 \\ 2\,{}^p\lambda_2{}^p\dot\lambda_2 \\ 2\,{}^p\lambda_3{}^p\dot\lambda_3 \\ \sqrt{2}\cdot({}^p\lambda_1^2-{}^p\lambda_2^2)\Lambda_3 \\ \sqrt{2}\cdot({}^p\lambda_3^2-{}^p\lambda_1^2)\Lambda_2 \\ \sqrt{2}\cdot({}^p\lambda_2^2-{}^p\lambda_3^2)\Lambda_1 \end{Bmatrix} \tag{41}$$

the reciprocal diagonal components of (40) relative to a principal co-ordinate system $\hat{\mathbf{E}}_k$, cf. Heiduschke (2019).
With (41), a plastic flow rule ${}^p\dot{\mathsf{U}} = \cdots$ for the Lagrangean plastic stretch tensor rate ${}^p\dot{\mathsf{U}}$ can also be expressed in terms of ${}^p\dot{\mathsf{C}}$ and ${}^p\mathsf{C}$ (in spectral representation).

7 Material orthotropy at finite deformation

The components of elastic ${}^e\varepsilon_{ij}$ and plastic ${}^p\dot\varepsilon_{ij}$ responses are combined by $@_{ij} = \{{}^e\varepsilon_{ij}, {}^p\dot\varepsilon_{ij}/\dot\lambda\} = @_{ji}$ in order to reflect hyperelasticity ${}^e\varepsilon_{ij} = \mathfrak{R}_{ijk\ell}\,\sigma_{k\ell}$ and non-linear plastic flow ${}^p\dot\varepsilon_{ij} = \dot\lambda\,{}^p\mathfrak{R}_{ijk\ell}\,\sigma_{k\ell}$, where ${}^@\mathfrak{R}_{ijk\ell} = {}^@\mathfrak{R}_{ijk\ell}({}^e\varepsilon_{ab}, {}^p\varepsilon_{cd})$. The orthotropy of materials at finite deformation can only be formulated from a material reference configuration in which (imaginary) fibers are perpendicular to each other. And the characteristic orthotropic decoupling of normal and shear components

$$\begin{Bmatrix} @_{11} \\ @_{22} \\ @_{33} \\ \sqrt{2}\cdot@_{12} \\ \sqrt{2}\cdot@_{13} \\ \sqrt{2}\cdot@_{23} \end{Bmatrix} = \begin{bmatrix} \mathfrak{R}_{1111} & \mathfrak{R}_{1122} & \mathfrak{R}_{1133} & 0 & 0 & 0 \\ \mathfrak{R}_{1122} & \mathfrak{R}_{2222} & \mathfrak{R}_{2233} & 0 & 0 & 0 \\ \mathfrak{R}_{1133} & \mathfrak{R}_{2233} & \mathfrak{R}_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\cdot\mathfrak{R}_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\cdot\mathfrak{R}_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\cdot\mathfrak{R}_{2323} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\cdot\sigma_{12} \\ \sqrt{2}\cdot\sigma_{13} \\ \sqrt{2}\cdot\sigma_{23} \end{Bmatrix} \tag{42}$$

occurs only relative to a proper-orthogonal material basis vector systems ${}^{@}\mathbf{M}_k = \{\mathbf{M}_k, {}^{@}\mathbf{M}_k, {}^{@}\mathbf{M}_k\}$, in which the Lagrangean basis vectors ${}^{@}\mathbf{M}_k$ point in the perpendicular fiber directions of the material reference configuration. For other arbitrary Lagrangean basis vector systems $\mathbf{E}_k \neq {}^{@}\mathbf{M}_k$ (including the principal systems ${}^{@}\hat{\mathbf{E}}_k$), the linearized component matrix (of elastic compliance or plastic flow) is fully occupied

$$\begin{Bmatrix} @_{xx} \\ @_{yy} \\ @_{zz} \\ \sqrt{2}\cdot @_{xy} \\ \sqrt{2}\cdot @_{xz} \\ \sqrt{2}\cdot @_{yz} \end{Bmatrix} = \begin{bmatrix} \mathfrak{R}_{xxxx} & \mathfrak{R}_{xxyy} & \mathfrak{R}_{xxzz} & \sqrt{2}\cdot\mathfrak{R}_{xxxy} & \sqrt{2}\cdot\mathfrak{R}_{xxxz} & \sqrt{2}\cdot\mathfrak{R}_{xxyz} \\ \mathfrak{R}_{xxyy} & \mathfrak{R}_{yyyy} & \mathfrak{R}_{yyzz} & \sqrt{2}\cdot\mathfrak{R}_{yyxy} & \sqrt{2}\cdot\mathfrak{R}_{yyxz} & \sqrt{2}\cdot\mathfrak{R}_{yyyz} \\ \mathfrak{R}_{xxzz} & \mathfrak{R}_{yyzz} & \mathfrak{R}_{zzzz} & \sqrt{2}\cdot\mathfrak{R}_{zzxy} & \sqrt{2}\cdot\mathfrak{R}_{zzxz} & \sqrt{2}\cdot\mathfrak{R}_{zzyz} \\ \sqrt{2}\cdot\mathfrak{R}_{xxxy} & \sqrt{2}\cdot\mathfrak{R}_{yyxy} & \sqrt{2}\cdot\mathfrak{R}_{zzxy} & 2\cdot\mathfrak{R}_{xyxy} & 2\cdot\mathfrak{R}_{xyxz} & 2\cdot\mathfrak{R}_{xyyz} \\ \sqrt{2}\cdot\mathfrak{R}_{xxxz} & \sqrt{2}\cdot\mathfrak{R}_{yyxz} & \sqrt{2}\cdot\mathfrak{R}_{zzxz} & 2\cdot\mathfrak{R}_{xyxz} & 2\cdot\mathfrak{R}_{xzxz} & 2\cdot\mathfrak{R}_{xzyz} \\ \sqrt{2}\cdot\mathfrak{R}_{xxyz} & \sqrt{2}\cdot\mathfrak{R}_{yyyz} & \sqrt{2}\cdot\mathfrak{R}_{zzyz} & 2\cdot\mathfrak{R}_{xyyz} & 2\cdot\mathfrak{R}_{xzyz} & 2\cdot\mathfrak{R}_{yzyz} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sqrt{2}\cdot\sigma_{xy} \\ \sqrt{2}\cdot\sigma_{xz} \\ \sqrt{2}\cdot\sigma_{yz} \end{Bmatrix} \qquad (43)$$

and rotated away from the material basis vector system ${}^{@}\mathbf{M}_k$. The linearized component matrix (43) looks like full anisotropy with 21 coefficients instead of the 9 coefficients of orthotropy.
Orthotropic material formulated from a material reference configuration where the (imaginary) fibers are *not* perpendicular to each other also results in a linearized component matrix that looks like (43), since material orthotropy and finite-deformation-induced anisotropy are then in a mix and interfere.
Therefore, the modeling of an orthotropic elastic-plastic material with perpendicular elastic and plastic fibers in the reference configuration can only be realized by means of Lagrangean commutative-symmetrical stretch tensor products and not by means of Bilby-Kröner-Lee formulations (10). On the one hand, the elastic ${}^{e}\mathsf{U}$ and plastic ${}^{p}\mathsf{U}$ right stretch tensors of a Lagrangean commutative-symmetrical stretch tensor product (24) can refer both to the same reference configuration with perpendicular elastic and plastic fibers, since U, ${}^{e}\mathsf{U}$ and ${}^{p}\mathsf{U}$ can appear on the rightmost side of the right polar decomposition

$$\mathsf{F} = \mathsf{R}.\mathsf{U} = \mathsf{R}.{}^{p}\mathsf{U}.\mathsf{X}^{\mathrm{T}}.{}^{e}\mathsf{U} = \mathsf{R}.{}^{e}\mathsf{U}.\mathsf{X}.{}^{p}\mathsf{U} \qquad (44)$$

of the deformation gradient. Hence, orthotropic elasto-plasticity can be modeled simultaneously with commutative-symmetrical stretch tensor products. On the other hand, the elastic fibers perpendicular in the reference configuration are deformed by the plastic tensor factor ${}^{p}\mathfrak{F}$ of the Bilby-Kröner-Lee formulation (10) to skew fibers in the intermediate (stress-free) configuration, so that the elastic tensor factor ${}^{e}\mathfrak{F}$ of the Bilby-Kröner-Lee formulation refers to skewed and not perpendicular fibers. Hence, orthotropic elasticity cannot be modeled with the Bilby-Kröner-Lee formulation (10).

8 Conclusion

The Bilby-Kröner-Lee formulation (10) introduces an intermediate (stress-free) configuration that is not well-determined according to Casey & Naghdi (1980). Since the elastic (11) and plastic (12) tensor factors as well as the elastic right (15) and plastic left (16) stretch tensors of the Bilby-Kröner-Lee formulation are defined with respect to that intermediate configuration, they are also not well-determined and are therefore called to be of non-proper type. In this context, in particular, the elastic right stretch (30) of the Bilby-Kröner-Lee formulation is put into question.
In contrast, the proposed elasto-plasticity formulation is based solely on proper-Eulerian and proper-Lagrangian tensors defined by the commutative-symmetrical stretch tensor products (24), (26), (28), (29). It is therefore independent of an intermediate (stress-free) configuration and does not use tensors related to it. The elastic right stretch tensor (28) of the proposed formulation is of proper-Lagrangian type and the plastic left stretch tensor ${}^{p}\mathsf{v} = \sqrt{\mathsf{v}.{}^{e}\mathsf{b}^{-1}.\mathsf{v}}$ of proper-Eulerian type.
When elastic and plastic right-stretches are defined by the right commutative-symmetrical stretch tensor product with respect to a reference configuration, then elastic and plastic orthotropy can both be modeled simultaneously. In the Bilby-Kröner-Lee formulation, however, only plastic orthotropy can be modeled—and no elastic orthotropy. Therefore, and in particular for elasto-plastic orthotropy, a formulation based on commutative-symmetrical stretch tensor products should be employed (to decouple material orthotropy and plastic deformation-induced anisotropy and) to model finite-elastic orthotropy and finite-plastic orthotropy simultaneously.

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