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Using a Partially-Linear Simplified Model for Gravitational Three Phase-Flow of Water, Oil and Supercritical CO_2 in Porous Media

Panters Rodríguez-Bermúdez[1⋆]

[1] Universidade Federal Fluminense (UFF)/ Ave. dos Trabalhadores 420, Vila Sta. Cecília, 27255-125, Volta Redonda,RJ, Brasil.

Abstract: In this article we use a simplified model for vertical three-phase flow in porous media of immiscible fluids like water, oil and supercritical-CO_2. The relative permeability functions of oil and CO_2 phases are assumed to be linear functions of the respective saturations while the relative permeability of the water phase is a quadratic function of water saturation. For vertical flow (gravity taken into account) and fluids having small density differences, the model intend to represent the case in which phases supercritical-CO_2 and oil are inclined to miscibility when water phase is absent for certain temperature and pressure conditions, while the flow gains some immiscibility properties as the water phase becomes present. This model leads to a 2×2 system of conservation laws which loose strict hyperbolicity at three isolated coincidence points at the edges of the saturation triangle. We use the wave curve method to solve a class of Riemann Problem.

Keywords: conservation law systems, Riemann problems, multi-phase flow, porous media, shock waves, rarefaction waves

1 Introduction

Multi-phase flow in porous media phenomena are modelled by hyperbolic conservation laws systems. They are systems of first order partial differential equations that in most of the applications are often non-linear. Initial value problems for these systems are studied, being of special interest the so-called Riemann problems Riemann (1860). The Riemann problem is a special Cauchy's initial value problem which consists of piecewise constant initial data. Due to the non-linearity of these equations, the intersection of the plane characteristic curves occurs, leading to shock waves (discontinuities) formation and the necessity of consider a weak concept of solution for these problems, see classic references such as Courant-Friedrichs Courant and Friedrichs (1948), Gelfand Gelfand (1959), Leveque LeVeque (1992, 2007), Smoller Smoller (1994), between others.

In 1942 Buckley-Leverett Buckley and Leverett modeled immiscible two-phase flow in a porous medium by a scalar conservation law, they neglected the gravity and used quadratic permeabilities for both the immiscible fluids. In 1963 Koval Koval (1963) proposed the use of linear permeabilities to model miscible two-phase flow without gravity. In 1981 Proskurowski Proskurowski (1981) extended the Buckley-Leverett model for immiscible two-phase with gravity effects taken into account.

On the other hand, for three-phase flow the model leads to a 2×2 conservation law systems. In 1956 Corey et al. Corey et al. (1956) proposed a model of quadratic permeabilities for immiscible three-phase. In 1957 Lax Lax (1957) developed a theory and entropy criteria to solve the Riemann problem locally for strictly hyperbolic conservation laws systems of order n. Liu Liu (1974, 1975) developed the "wave curve method" to obtain global Riemann solutions for strict hyperbolic systems. Isaacson, Marchesin, et. al Isaacson et al. (1992) in 1992, proved the lost of hyperbolicity at an isolated point they called "umbilic" for the Corey model without gravity, they also extended the wave curve method for that model. In 2010 and following years, Rodriguez-Bermudez and Marchesin Rodriguez-Bermudez (2010); Rodríguez-Bermúdez and Marchesin (2013, 2014) included gravity effects to immiscible three-phase flow and solved a class of Riemann problems for a Corey-type model, they also found isolated points at the boundary of the physical space of saturations where strict hyperbolicity fails. Such points were called "quasi-umbilic" points.

Other important works on three phase flow (with or without gravity) are Andrade et al. Andrade et al. (a,b), Azevedo et al. Azevedo et al. (a,b), Bell et al. Bell et al., Castañeda et al. Castañeda et al.; Castañeda and Furtado; Castañeda et al. (2022), De Souza de Souza (1992), Holden Holden (1987); Holden and Holden (1989); Holden (1990), Isaacson et al. Isaacson et al. (1988); Isaacson and Temple (1885), Keyfitz Keyfitz (1988, 1989), Matos et al. Matos et al. (2014), Medeiros Medeiros (1992), Schaeffer and Shearer Schaeffer and Shearer (1987a,b), Shearer et al. Shearer et al. (1987), Shearer Shearer (1988), Stone Stone (1970), Trangenstein Trangenstein (1989), between others.

In this work, we propose a simplified model for immiscible three-phase flow with gravity. For horizontal flow (absence of gravity), this model extends the Koval Koval (1963) model for miscible two-phase flow with an additional third immiscible fluid. For vertical flow (gravity taken into account) and fluids having small density differences, the model intend to represent the case in which phases 1 and 2 are inclined to miscibility when phase 3 is absent for certain temperature and pressure conditions, while the flow gains some immiscibility properties as phase 3 becomes present. Of course, for generic density fluids the addition of gravity effects breaks all kind of miscibility property of the model, thus in such a case the use of an immiscible Corey's-type model Corey et al. (1956) with non-linear (maybe quadratic) permeabilities would be more suitable than the model used in the present work.

We consider linear relative permeabilities for phases 1 and 2 while consider quadratic relative permeability for phase 3. For instance, phases 1 and 2 could be supercritical-CO_2 and oil, and phase 3 could be water for given temperature and pressure reservoir

⋆ E-mail address: pantersrb@id.uff.br
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conditions. We study a class o Riemann problem for this model, including two-phase flow regimes and genuine three-phase flow regimes by using the wave curve semi-analytical method to obtain the Riemann solutions for the proposed partially-linear permeability model. The novelty of the work is characterized by the combination of the proposed model with the used methodology.

2 Modeling Three-phase Flow in Porous Media

We consider a partially-linear simplified permeability model for gravitational three phase-flow of phase 1 (supercritical CO_2, that we denote by g), phase 2 (oil, denoted by o) and phase 3 (water, denoted by w). We assume that porosity ϕ and absolute rock's permeability K are constant. We also assume a constant temperature, that there are no interchange of mass between the phases and that the flow occurs uniformly through the vertical direction due to gravity action. Compressibility effects are neglected and there are no source or sinks. We also assume that the density differences of the fluids are small.
Let consider the mass conservation law for each phase

$$\frac{\partial}{\partial t}\phi s_i + \frac{\partial}{\partial x}\mathbf{u}_i = 0 \quad i = g, o, w, \tag{1}$$

s_i denotes the saturation and $\mathbf{u}_i$ denotes the seepage velocity of each phase.
The Darcy's law for seepage velocity of each phase i gives:

$$\mathbf{u}_i = -K\frac{k_{r,i}}{\mu_i}\left(\frac{\partial p_i}{\partial x} - \rho_i \mathrm{g}\right) \quad i = g, o, w, \tag{2}$$

where p_i is the pressure, $k_{r,i}$ is the relative permeability function, μ_i is the viscosity (constant) and ρ_i the density (constant) for each phase i, g is the gravitational constant.
We assume that permeabilities $k_{r,i}$ are functions only of the saturations s_i and that porous medium is totally saturated, such that $s_g + s_o + s_w = 1$.
We define

$$\lambda_i = k_{r,i}/\mu_i \quad i = g, o, w; \quad \lambda = \sum_{i=w,o,g} \lambda_i, \tag{3}$$

$$f_i = \lambda_i/\lambda \quad i = g, o, w; \quad \mathbf{u} = \sum_{i=w,o,g} \mathbf{u}_i; \tag{4}$$

here λ_i is the mobility function and f_i is the fractional flow function corresponding to each phase i, λ is the total mobility and $\mathbf{u}$ is the total Darcy's seepage velocity. Notice also that $f_g + f_o + f_w = 1$.
We assume $p_i = p_j$ (*i.e.*, neglecting capillary pressures), thus substituting Darcy's law (2) in (4) and after some calculations we have

$$\mathbf{u}_i = \mathbf{u} f_i + K\lambda_i \sum_{j\neq i} f_j \rho_{ij} g, \quad i = g, o, w, \tag{5}$$

where ρ_{ij} denotes (small) density differences $\rho_i - \rho_j$ between phases i and j. Finally we substitute (5) into Eq. (1) to obtain the conservation law system for the saturations s_g, s_o and s_w:

$$\frac{\partial}{\partial t}\phi s_i + \frac{\partial F_i}{\partial x} = 0, \quad i = g, o, w, \tag{6}$$

where

$$F_i = \mathbf{u} f_i + G_i, \quad i = g, o, w, \tag{7}$$

are the components of the vectorial flux function. $(F_g, F_o, F_w)^T$ which contains the gravitational terms

$$G_g = K\lambda_g\big((1 - f_g)\rho_{go} + f_w\rho_{ow}\big)g, \tag{8}$$

$$G_o = K\lambda_o\big((1 - f_o)\rho_{ow} + f_g\rho_{wg}\big)g, \tag{9}$$

$$G_w = K\lambda_w\big((1 - f_w)\rho_{wg} + f_o\rho_{go}\big)g. \tag{10}$$

Notice that $G_g + G_o + G_w = 0$, thus $F_g + F_o + F_w = \mathbf{u}$, the total Darcy's velocity.
By adding the equations in (5) and using the relations in (3)-(4), we obtain that $\frac{\partial \mathbf{u}}{\partial x} = 0$; that indicates incompresibility of the fluids. On the other hand, we observe that the system (1) has a redundant equation, *i.e.*, any of the equations of the system can be obtained from the other two by considering incompresibility of the phases and that the porous medium is totally saturated. Therefore, we can neglect anyone of the equations in (6) and obtain a 2×2 conservation laws system to be studied. The equation to be neglected can be chosen in a convenient way for each case.

After a proper dimensionless process Rodriguez-Bermudez (2010) and neglecting the redundant equation for the phase w we obtain a model for oil and gas saturations

$$\frac{\partial s_i}{\partial t} + \frac{\partial}{\partial x}\big(\alpha f_i(s_o, s_g) + G_i(s_o, s_g)\big) = 0, \quad i = o, g, \tag{11}$$

with "gravitational flux functions" given by

$$G_o = K\lambda_o\big(\lambda_w\rho_{ow} + \lambda_g\rho_{og}\big)/\lambda, \tag{12}$$
$$G_g = K\lambda_g\big(\lambda_o\rho_{go} + \lambda_w\rho_{gw}\big)/\lambda. \tag{13}$$

where $s_w = 1 - s_o - s_g$ and α is a dimensionless parameter that represents a "convection/gravity rate" Rodriguez-Bermudez (2010); Rodríguez-Bermúdez and Marchesin (2013, 2014), which has roughly the meaning of the ratio between pressure gradient-induced and gravity-induced flows. In this paper we are interested into study only gravitational flow, so we set $\alpha = 0$ and study the following conservation law system

$$\frac{\partial s_i}{\partial t} + \frac{\partial}{\partial x}\big[G_i(s_o, s_g)\big] = 0, \quad i = o, g, \tag{14}$$

with flux components G_o, G_g given in (12)-(13).

2.1 Partially-Linear permeability Model

We consider a simplified permeability model such that oil-phase and CO_2-phase permeabilities are linear functions of their saturations while the water-phase permeability is a quadratic function of its saturation as follows

$$k_{r,w}(s_w) = s_w^2, \quad k_{r,i}(s_i) = s_i, \quad i = o, g. \tag{15}$$

The motivation to choose this simplified permeability model is that for horizontal flow (absence of gravity, or analogously for high values of α), this model extends the Koval Koval (1963) model for miscible two-phase flow with an additional third immiscible fluid. For vertical flow (gravity taken into account) and fluids having small density differences, this model intend to represent the case in which phases oil and supercritical-CO_2 are inclined to miscibility when water phase is absent for certain temperature and pressure conditions, while the flow gains some immiscibility properties as the water phase becomes present. Of course, for generic density fluids the addition of gravity effects breaks all kind of miscibility property of the model and we do not recommend the use of this model for such a case.

3 Riemann Problem and Elementary Waves

We denote by S the saturation vector $S = (s_o, s_g)$. The Riemann problem for the conservation law (14) is a piecewise constant initial data Cauchy problem

$$S(t=0, x) = \begin{cases} S_T & \text{if } x < 0 \\ S_B & \text{if } x > 0. \end{cases} \tag{16}$$

S_T denotes the "left" (Top) initial Riemann state while S_B denotes the "right" (Bottom) initial Riemann state. The pure gravitational problem usually have both negative and positive characteristic speeds, so in such a problem the Riemann solutions contains waves travelling "to the left" (upwards in the reservoir) and waves travelling "to the right" (downwards in the reservoir). The Riemann solutions generically are sequences of shock waves, rarefaction waves and constant-saturation states as shown in figure 1.

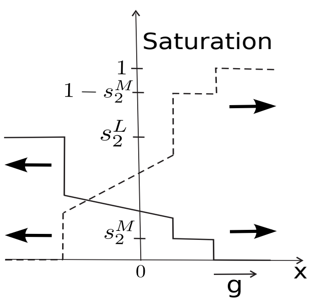
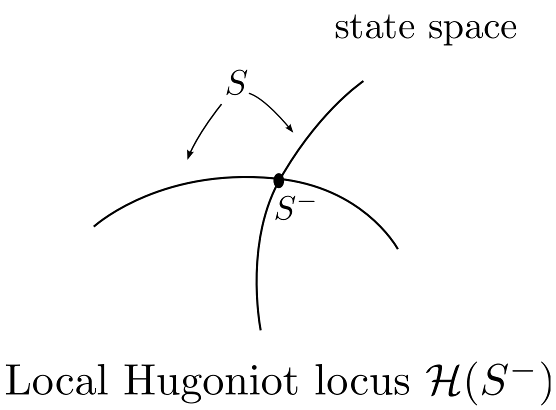
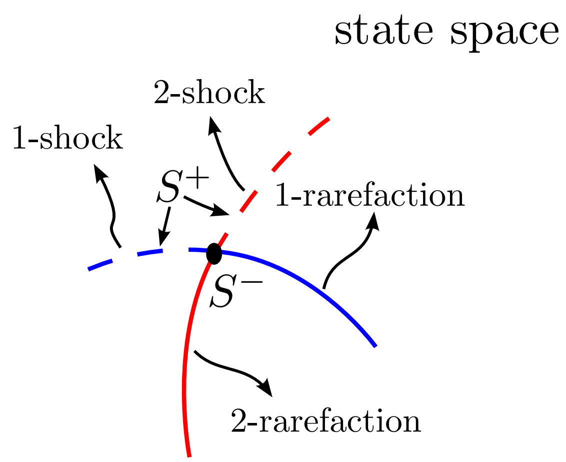


Fig. 1: Riemann solution example.

Shocks and rarefactions are also known as the "elementary waves" of the Riemann problem. A centered rarefaction wave corresponding to the family i is constructed through the integral curves of the "differential equation"



(a) Hugoniot locus example



(b) Local shocks and rarefaction curves

Fig. 2: Local Riemann Problem: Shocks and Rarefactions in space state

$$\dot{S} = r_i(S), \tag{17}$$

where $r_i(S)$ is the i-th eigenvector of the Jacobian matrix of the conservation law system, *i.e.*, $dF(S)r_i(S) = \lambda_i(S)r_i(S)$. Notice that Eq. (17) is an ODE only locally in regions where strict hyperbolicity holds. A rarefaction wave corresponds to an integral curve along which the characteristic speed λ_i is non-decreasing, with $\lambda_i(S) = x/t$.
A shock wave is a discontinuous front traveling at constant speed. The formation of shocks is a typical phenomenon of nonlinear conservation laws even for smooth initial conditions, due to the crossing of the plane characteristic curves of the hyperbolic partial differential equation. The speed of the shock satisfies an algebraic relation known as the Rankine-Hugoniot condition, which at the same time defines, for the case of systems, a curve of states S that can be connected through shocks with a given state S^-.

Definition 1 (Hugoniot locus) *For a fixed S^-, the set of states S such that the pair S^-, S satisfy (18) for some σ comprises the Hugoniot locus, $\mathcal{H}(S^-)$, see an example in Figure 2a.*

$$-\sigma[S - S^-] + F(S) - F(S^-) = 0, \tag{18}$$

(18) *is called the Rankine-Hugoniot jump condition and σ is the shock speed.*

In general, solutions of the Riemann problem are not unique and entropy criteria are used to select the physically correct shocks. We use the generalized Lax's entropy conditions.

Definition 2 (Entropic Generalized Lax 1-shock and 2-shock waves) *We assume that a shock connecting states S^-, S^+ with speed σ is entropic (physically correct) if*

$$1\text{-}shock: \ \lambda_1(S^+) \leq \sigma \leq \lambda_1(S^-), \ \ and \ \ \sigma \leq \lambda_2(S^+); \ \ or \tag{19}$$

$$2\text{-}shock: \ \lambda_2(S^+) \leq \sigma \leq \lambda_2(S^-), \ \ and \ \ \lambda_1(S^-) \leq \sigma. \tag{20}$$

with at most one equality (19) *or* (20)*.*

In Figure 2b it is shown typical entropic shock and rarefaction local curves in space state.

3.1 Wave groups and Riemann solutions

We take into account only solutions satisfying the following criteria, see Fig. 3:

- The Riemann solutions consist of two wave groups separated (at most) by a constant state.
- Each wave group consists on a sequence of rarefaction waves and adjacent shock waves, or contact waves.
- Shock waves must satisfy the Generalized Lax conditions.
- No 1-wave is preceded by a 2-wave

4 Riemann Solutions along Two-Phase Regimes Containing CO_2 in the Pure Gravitational Flow

The Jacobian matrix of the conservation law system (14) with fluxes (12)-(13) is

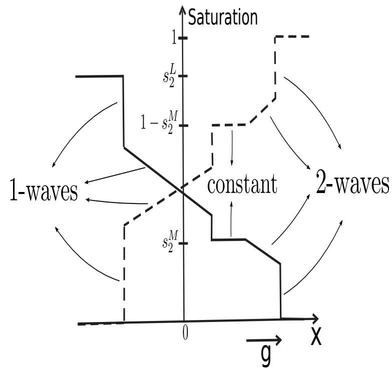
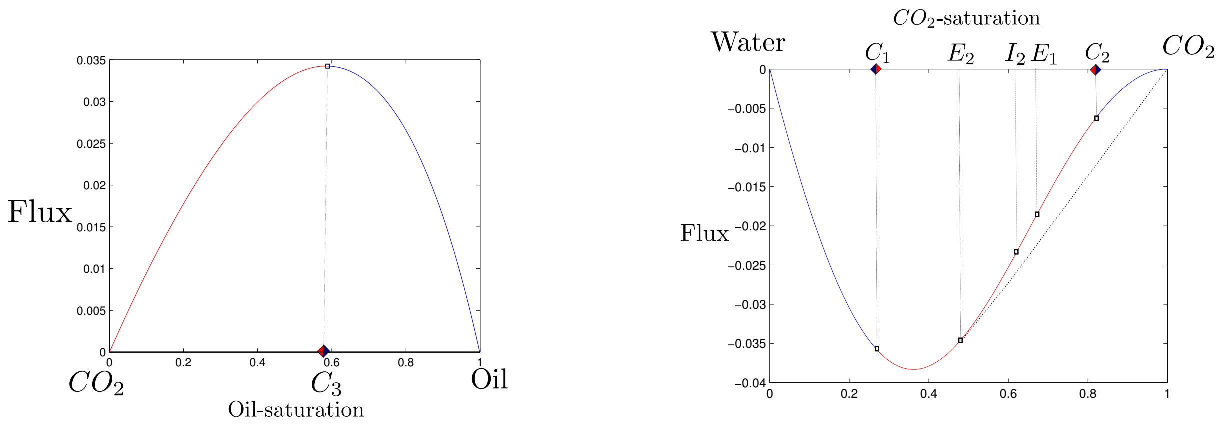


Fig. 3: typical Wave groups in Riemann solutions



(a) Flux function along CO_2-Oil regime

(b) Flux function along Water-CO_2 regime

Fig. 4: Flow functions along the two-phase regimes in the edges

$$dG(s_o, s_g) = \begin{pmatrix} \frac{\partial G_o}{\partial s_o} & \frac{\partial G_o}{\partial s_g} \\ \frac{\partial G_g}{\partial s_o} & \frac{\partial G_g}{\partial s_g}. \end{pmatrix}. \tag{21}$$

The not-ordered characteristic speeds (α and β) are the eigenvalues of the Jacobian matrix dG in (21).
Along the two-phase CO_2-Oil regime (where water phase is absent, $s_w = 0$) in the saturation triangle, we have that $s_o = 1 - s_g$ and $G_o = -G_g$ so both the equations of the conservation laws system are redundant and we can drop, for instance, the equation corresponding to the phase g. Thus the two-phase flow on this regime, for the pure gravitational flow case ($\alpha = 0$) reduces to an scalar equation in the variable s_o with flux given by

$$F(s_o) = \frac{K s_o(1 - s_o)\rho_{og}}{(s_o/\mu_o + (1 - s_o)/\mu_g)}. \tag{22}$$

See at figure 4-(a) the graph of the flux function (22) defined on the edge CO_2-Oil of the saturation triangle, the dimensionless parameters were taken as $K = 1$, $\rho_w = 1$, $\rho_o = 0.99$, $\rho_g = 0.98$, $\mu_w = 0.3$, $\mu_o = 0.6$, $\mu_g = 0.65$. Notice that along this two-phase edge the scalar flux function is concave and its maximum point is reached at the coincidence point C_3 where both characteristic speeds are equal to zero, *i.e.*, $F'(C_3) = 0$ and $\alpha(C_3) = \beta(C_3) = 0$. Notice also that along the edge, occurs a characteristic-family change in the coincidence point C_3, this transition is represented by a change of color (red to blue) in the graph of the flux function 4-(a).
Due to the concavity of the scalar flux function along CO_2-Oil two-phase regime, the solution of the Riemann problem (16) are very simple. Given $S_T = (s_o^T, s_g^T)$ and $S_B = (s_o^B, s_g^B)$ the initial Riemann data. If $s_o^T > s_o^B$ the Riemann solution is a rarefaction wave that can change family depending on the relative position of these states with respect to the coincidence point C_3. For the case in which $s_o^T < s_o^B$ the Riemann solution is a shock wave connecting these states. Notice that in both cases, there could be waves travelling upwards and (or) downwards in the reservoir because of the presence of both negative and positive derivatives in the scalar flux function plotted in Fig. 4-(a).
Along the Water-CO_2 two-phase regime (absence of oil) we have $s_o = 0$ and therefore $\partial G_o/\partial s_o = 0$, which leads to an upper triangular Jacobian matrix with characteristic speeds being $\alpha = \partial G_o/\partial s_o$ and $\beta = \partial G_g/\partial s_g$. By using the fluxes in (12)-(13) for the pure gravitational flow case ($\alpha = 0$), and the partially linear permeability model (15) we obtain that the characteristic speeds along the Water-CO_2 edge are

$$\alpha(s_g) = \frac{K(1/\mu_o)\left[((1-s_g)^2/\mu_w)\rho_{ow} + (s_g/\mu_g)\rho_{og}\right]}{((1-s_g)^2/\mu_w) + (s_g/\mu_g)}, \tag{23}$$

$$\beta(s_g) = \frac{K(1-s_g)\rho_{gw}}{\mu_g\mu_w}\left[\frac{((1-s_g)^3/\mu_w) - 2(s_g^2/\mu_g)}{((1-s_g)^2/\mu_w) + (s_g/\mu_g)}\right]. \tag{24}$$

On the other hand, the scalar flux function associated to the two-phase flow along the Water-CO_2 edge of the saturation triangle is the same G_g function restricted to that edge, thus the derivative of the scalar two-phase flux along this edge coincide with the eigenvalue β in (24). In figure 4-(b) is shown the graph of the scalar flux function along the two-phase regime in the Water-CO_2 edge of the saturation triangle. The coincidence points of characteristic speeds C_1 and C_2 (where $\alpha = \beta$) allow the transition between characteristic-families along this edge (transition represented by the change of colors: blue-red-blue in the flux graph). The state denoted by I_2 represents an inflection point corresponding to the fast family while the states denoted by E_1 and E_2 represent boundary-extensions of the pure state of CO_2 corresponding to the slow and fast families respectively. Notice the tangency representing a fast characteristic-shock joining the pure CO_2 state and the state E_2. For saturation values s_g of phase CO_2 such that $C_1 < s_g < C_2$ we have that $\alpha < \beta$ while for saturation values s_g such that $0 < s_g < C_1$ or $C_2 < s_g < 1$ we have $\alpha > \beta$. The Riemann solutions along the Water-CO_2 regime can be obtained through the Oleinik construction by using the convex hull of the scalar flux function (for $s_g^T < s_g^B$) and the concave hull of the scalar flux function (for $s_g^T > s_g^B$). We illustrate the solution for a simple case where the state S_T initially at the top of the reservoir consists on pure water ($s_g^T = 0$) while the state S_B initially at the bottom of the reservoir consists on pure CO_2 ($s_g^B = 1$). In such a case the Riemann solution consists on a wave sequence as follows

$$\text{pure Water} = S_T \xrightarrow{R_s} C_1 \xrightarrow{R_f} E_2 \xrightarrow{CS_f} S_B = \text{pure } CO_2, \tag{25}$$

where R_s represents a slow rarefaction wave, R_f represents a fast rarefaction wave; the change of family occurs at the characteristic coincidence point C_1. The notation CS_f represents a fast left-characteristic shock wave, whose speed coincides with the characteristic speed $\beta(E_2)$.

5 Riemann Solutions for a Genuine Three-phase Flow in the Pure Gravitational Flow

In this section, we solve a class of Riemann problems for the genuine three-phase flow (all three phases are present) modelled by the conservation laws system in (14) with fluxes (12)-(13) for the pure gravitational flow case ($\alpha = 0$), and permeabilities in (15). All calculations were performed by using the following values of the dimensionless parameters $K = 1$, $\rho_w = 1$, $\rho_o = 0.99$, $\rho_g = 0.98$, $\mu_w = 0.3$, $\mu_o = 0.6$, $\mu_g = 0.65$. We assume that initially at the top of the reservoir there is a mixture of water and oil separated by an (imaginary) interface of pure supercritical-CO_2 which is at the bottom of the reservoir, see Fig. 5.

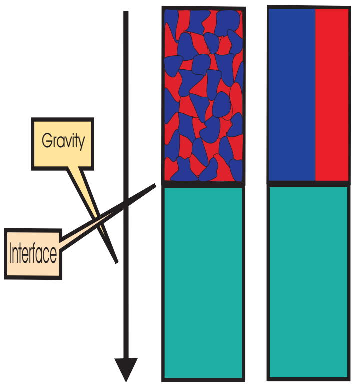


Fig. 5: Initial state in the reservoir. Top: we use blue color for water fluid and we use red color for light oil. Bottom: we use green color for supercritical-CO_2

In Fig.6-(a) are shown the integral curves (rarefaction curves in state space), in blue color are represented the slow-family integral curves while in red color are represented the fast-family integral curves. The arrows indicate the increasing of the characteristic speeds. At the boundaries of the saturation triangle are been represented (two-coloured squares) the three coincidence points where strict hyperbolicity of conservation law system fails. Next, we define relevant bifurcation curves used to obtain Riemann solutions in this paper.

Definition 3 *A state S belongs to the* ***inflection curve*** *for the family i (denoted by i-*Inflection *i=1,2) if*

$$\nabla\lambda_i(S)\cdot r_i(S) = 0. \tag{26}$$

Remark 1 *Inflection is the curve where genuine nonlinearity is lost,* i.e., *the eigenvalue does not vary monotonically along an integral curve. Rarefaction curves stop at the inflection manifold.*

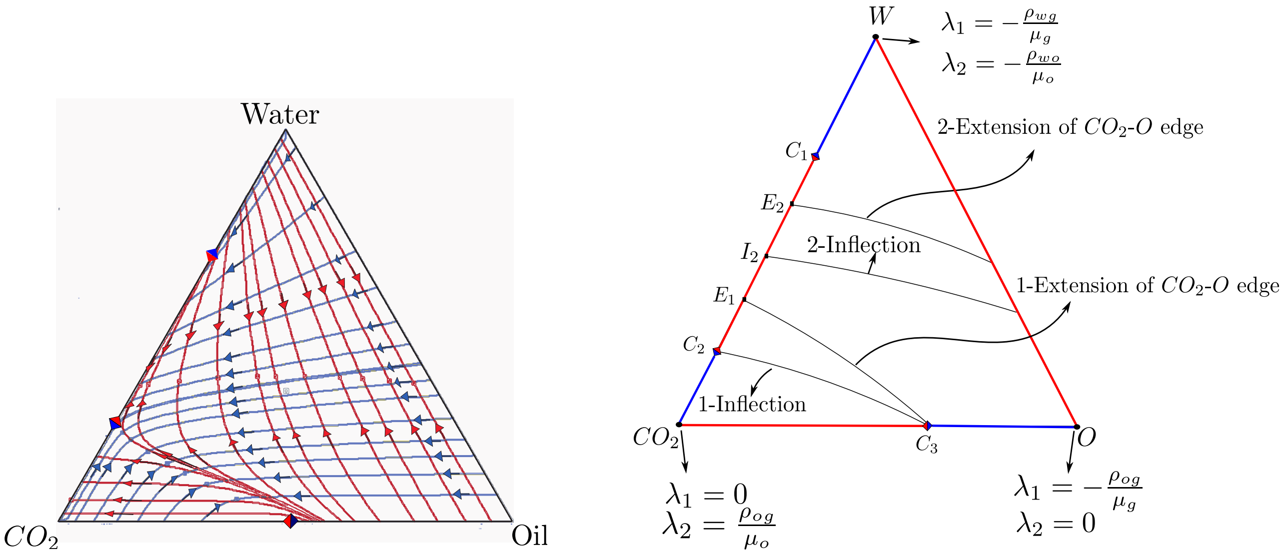


Fig. 6: (a) Rarefaction curves. (b)Bifurcations curves

Definition 4 *The state S' belongs to the* ***boundary-extension curve*** *for the family i, denoted by i*-Extension *if there exist a state S on the boundary such that*

$$S' \in \mathcal{H}(S) \quad and \quad \lambda_i(S') = \sigma(S', S). \tag{27}$$

In Fig.6-(b) C_1, C_2 and C_3 represent the coincidence points of characteristic speeds at the boundary of the triangle. The (slow and fast) inflection curves and (slow and fast) boundary-extension curves of the edge CO_2-O are also represented. The values of the ordered-characteristic speeds λ_1 (slow) and λ_2 (fast) at the vertices of the triangle (pure-phase states) were calculated and are shown in the figure. The states E_1 and E_2 which represent respectively the slow and fast extensions of the CO_2 pure state are also indicated in the figure.

5.1 Riemann Solutions by the wave curve method

We use the wave curve method to obtain the Riemann solutions. Wave curves in this type of problems (in which strict hyperbolic and genuine non-linearity fail) differ from wave curves appearing in Lax's Theorem in several aspects, see Liu (1974). First, they are represented in state space by three types of elementary curves, shock curves and rarefaction curves as in the classical case as well as composite curves, which represent shock waves adjacent to rarefaction waves; the shocks lies to the right of the rarefactions. Any state S' of a composite curve satisfies

$$S' \in \mathcal{H}(S) \ \text{ with } \ \lambda_i(S) = \sigma(S', S),$$

where S traverses a rarefaction segment. Second, in each wave curve there are many such elementary curves. Each elementary curve must stop whenever its wave speed attains an extremum, and the type of elementary curve that follows is determined by simple rules. Third, since Hugoniot curves possess nonlocal (*i.e.*, detached) branches, wave curves also may have disconnected parts or branching points, see Isaacson et al. (1992).

When constructing a wave curve, successions of elementary curves obey certain rules; along shock and composite curves these rules are justified by the Bethe-Wendroff theorem, see Wendroff (1972). In any case, the rules determine the qualitative behavior of the wave speed along a wave curve, to ensure the monotonicity of the wave speed along the wave group.

We are interested into solve Riemann problems with a fixed right (bottom) initial state (the pure CO_2 state with $s_g^B = 1$) while considering several left (top) initial states (S_T lying on $W-O$ edge). In order to simplify the notation, we use the same notation for both shock and composite curves as dashed lines in the wave curve diagram, while we use solid lines to represent rarefaction curves, see Fig. 7.

For left (top) initial Riemann state S_T taken on $W-O$ edge, there is an state $S_T^4 = (s_o^{T4}, s_g^{T4})$ see Fig. 7, for which occurs a bifurcation in the solution, *i.e.*, for top initial saturation state S_T such that $s_o^T < s_o^{T4}$ the water phase becomes dominant with respect to the oil phase (*i.e.*, the leading wave travelling downwards does contain water but does not contain oil); for top initial saturation state S_T such that $s_o^T > s_o^{T4}$ the oil phase becomes dominant with respect to the water phase. This "bifurcation property" was already found in other three-phase flow models and it was called in the recent literature as "universally structure" of the Riemann Solutions, see Castañeda et al., 2022). When $S_T = S_T^4$ and $S_B = CO_2$, the Riemann solution consists in a slow rarefaction wave up to the intermediate state denoted by $S_M^4 = E_1$ and an adjacent two-phase fast shock wave connecting $S_M^4 = E_1$ with the pure CO_2 state (vertex of the triangle), see the Riemann profiles corresponding to this case in Fig. 8d.

The Riemann solutions can be split in three cases, for cases I and II the water phase is dominant with respect to the oil phase, while for case III the oil is dominant with respect to the water. The difference between cases I and II is the structure of the fast waves, for case I the fast waves are a sequence of a rarefaction wave and a left characteristic adjacent shock while for case II the fast waves consist only of a shock. The initial data S_T^1 belongs to case I, S_T^3 belongs to the case II, while S_T^2 is the limit case, see figures 8a, 8b and 8c for the profiles of the solutions. For initial data like S_T^5 or S_T^6 the structure of the solution is the same: a sequence of a slow rarefaction and a left characteristic adjacent shock up to an intermediate state S_M (lying on the $CO_2 - O$ edge) following of a fast rarefaction up to the pure CO_2 state, see Fig. 8e. The general structure for the Riemann solutions with this initial data are described in (28)-(32).

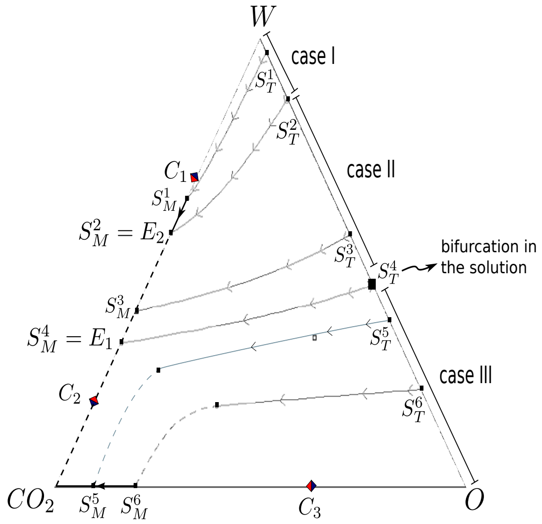


Fig. 7: Wave curves for six distinct initial data in case I, case II, case III

$$S_T = S_T^1 \xrightarrow{R_s} S_M^1 \xrightarrow{R_f} E_2 \xrightarrow{CS_f} S_B = \text{pure } CO_2, \tag{28}$$

$$S_T = S_T^2 \xrightarrow{R_s} S_M^2 = E_2 \xrightarrow{CS_f} S_B = \text{pure } CO_2, \tag{29}$$

$$S_T = S_T^3 \xrightarrow{R_s} S_M^3 \xrightarrow{S_f} S_B = \text{pure } CO_2, \tag{30}$$

$$S_T = S_T^4 \xrightarrow{R_s} S_M^4 = E_1 \xrightarrow{S_f} S_B = \text{pure } CO_2, \tag{31}$$

$$S_T = S_T^6 \xrightarrow{R_s, CS_s} S_M^6 \xrightarrow{R_f} S_B = \text{pure } CO_2. \tag{32}$$

6 Conclusions

A simplified model for vertical three-phase flow in porous media of immiscible fluids like water, oil and supercritical-CO_2 is being proposed. This model extends the Koval Koval (1963) model for miscible two-phase flow with an additional third immiscible fluid. For vertical flow (gravity taken into account) and fluids having small density differences, the model seams to fit well in the intention of representing the case in which phases 1 and 2 are inclined to miscibility when phase 3 is absent for certain temperature and pressure conditions, while the flow gains some immiscibility properties as phase 3 becomes present. The simplified model studied in this work leads to a 2×2 system of conservation laws which loose strict hyperbolicity at three isolated coincidence points at the edges of the saturation triangle. The wave curve method is used to solve a class of Riemann Problem, including two-phase flow regimes and genuine three-phase flow regimes. Riemann solutions were obtained and qualitatively characterized.

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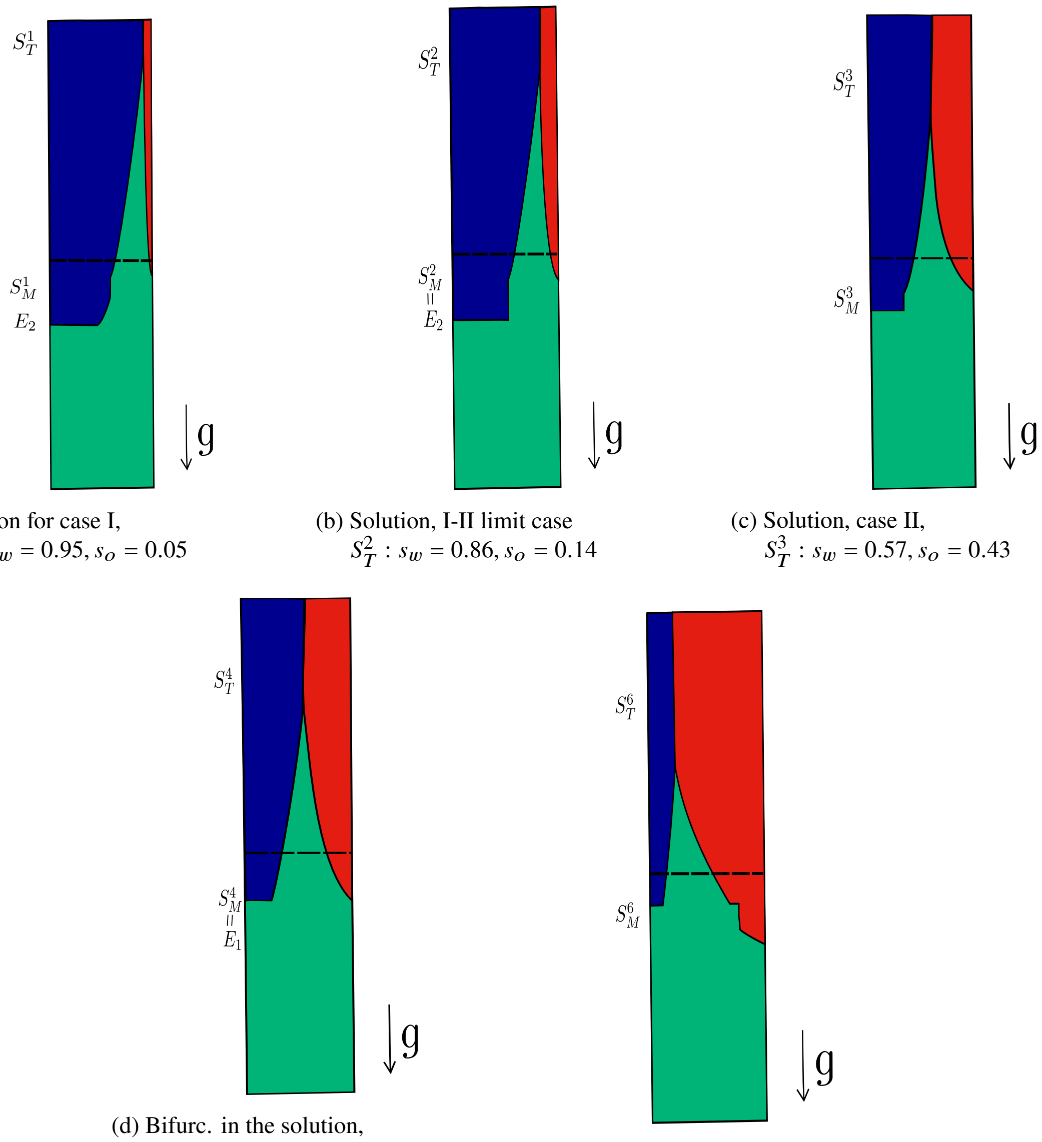

(a) Solution for case I, $S_T^1 : s_w = 0.95, s_o = 0.05$

(b) Solution, I-II limit case $S_T^2 : s_w = 0.86, s_o = 0.14$

(c) Solution, case II, $S_T^3 : s_w = 0.57, s_o = 0.43$

(d) Bifurc. in the solution, II-III limit case, $S_T^4 : s_w = 0.44, s_o = 0.56$

(e) Solution, case III, $S_T^6 : s_w = 0.22, s_o = 0.78$

Fig. 8: Saturation profiles diagram of the Riemann Solutions.

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