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On Estimating the Growth Rate of Perturbations in Rivlin-Ericksen Ferromagnetic Convection with Magnetic Field Dependent Viscosity

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Abstract: The unique viscoelastic properties and the consideration of MFD (magnetic field dependent) viscosity in the Viscoelastic ferrofluid model render it highly suitable for the accurate modeling of complex biological fluids, such as blood. The present research comprises the study of perturbation growth rate in a case of Rivlin-Ericksen Ferrofluid layer with viscosity depending on a vertically acting magnetic field. The complex growth rate of an arbitrary oscillatory motion with growing amplitude in Rivlin-Ericksen ferromagnetic convection for free-free boundaries has been analytically proven to be located within a semicircle in the right half of the $\sigma_r\sigma_i$-plane and $(radius)^2 = \left[RM_1/P_r\left(1 + 2(1 + \delta M_3)\pi^2 F_v\right)\right]$. Bounds are also obtained for situation involving rigid boundaries. For stress-free boundaries, the sufficient condition for PES validity is derived as $\left[RM_1 P_r/\pi^4\left(1 + (1 + \delta M_3)\pi^2 F_v\right)\right] \leq 1$, where R represents the Rayleigh number, P_r is the Prandtl number, F_v characterizes viscoelasticity of Rivlin- Ericksen ferrofluid, $M_3 > 0$ is the measure of the nonlinearity of magnetization and M_1 is the magnetic number. The mathematical derivation of these results is presented in detail. Thus the analysis presented here reveals that the oscillations in Rivlin-Ericksen ferromagnetic convection can be regulated or halted by taking into account the viscosity dependent on the magnetic field and the viscoelasticity of the fluid.

Keywords: Rivlin-Ericksen ferrofluid, Magnetic field dependent viscosity, Principle of exchange of stabilities, Perturbation growth rate

1 Introduction

Ferrofluids are a class of nanotechnology-based colloidal suspensions consisting of magnetic nanoparticles that are typically on the scale of a few nanometers to tens of nanometers in diameter and are suspended in a carrier liquid, often organic solvent or water-based solution. The magnetic nanoparticles are coated with a surfactant that prevents the particles from aggregating and settling out of the liquid carrier, resulting in a stable suspension. While Rivlin-Ericksen fluids are a class of non-Newtonian fluids that exhibit viscoelastic behavior. They are characterized by their ability to store and release energy when subjected to shear stress or deformation, resulting in a nonlinear relationship between stress and strain. Their rheological properties are governed by their internal structure rather than any external magnetic field, unlike ferrofluids where the magnetic particles tend to align themselves in the direction of the field under the influence of an external magnetic field. A thorough and authoritative summary of ferrofluid research has been presented by Rosensweig (1985).

Thus, Rivlin-Ericksen ferromagnetic fluid is a type of non-Newtonian fluid that exhibits both viscoelasticity and ferromagnetism, with a wide range of potential applications in various fields, including application in the development of smart fluids (which can respond to external stimuli such as magnetic fields, temperature, and pressure), fiber optics, in drug delivery systems, magnetic resonance imaging (MRI) contrast agents, avionics, in tissue engineering scaffolds, in field of robotics, in the field of microfluidics as a controllable fluid medium for microfluidic devices and so on [Odenbach (2008)].

The study of natural convection in a horizontal layer of fluid, where the fluid is heated from below, has been a topic of research for several decades as it holds significant implications for the control and utilization of various physical, chemical, and biological processes, and it has been studied under various conditions by several authors, including Bénard (1900), Rayleigh (1916) and Jeffreys (1926). Chandrasekhar (1981) provided a detailed analysis of Benard convection and derived mathematical equations that describe the growth of the instability and the resulting patterns of fluid flow. There is a substantial volume of literature available that pertains to the study of ferromagnetic convection. To gain a comprehensive understanding of this subject matter, Lalas and Carmi (1971), Shliomis (1972), Aniss et al. (2001), Suslov (2008), Lee and Shivakumara (2011), Prakash (2013), and Prakash et al. (2018) are among the works that can be referred to. A correction has been applied to the study of ferromagnetic convection with viscosity dependent on magnetic field by Prakash et al. (2019; 2020). Despite extensive research on thermal convection in Newtonian fluids, comparatively little emphasis has been placed on studying this phenomenon in non-Newtonian fluids. Herbert (1963) conducted an analysis, followed by Green (1968), on the issue of oscillatory convection in a viscoelastic fluid under influence of small disturbances. Sharma (1975) conducted a study on the stability of an electrically conducting Oldroyd fluid layer (1958) demonstrating that the presence of a magnetic field has a stabilizing effect on the fluid layer. A study was carried out by Bhatia and Steiner (1972) to investigate the impact of rotation on convective instability in a viscoelastic fluid layer, and their findings stands in contrast to the stabilizing effect of rotation on an ordinary viscous fluid. There are several elastic-viscous fluids that cannot be sufficiently characterized by either Maxwell's or Oldroyd's constitutive relations (Oldroyd, 1958). Rivlin-Ericksen's and Walter's

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(model B) fluids are examples of such fluids that fall outside the scope of the traditional constitutive models. Rivlin-Ericksen (1955) proposed a theoretical model to describe one of these classes of elastic-viscous fluids. Siddheshwar (2002) examined oscillatory convection in viscoelastic ferromagnetic and dielectric liquids, including Rivlin-Ericksen, Maxwell, and Oldroyd types. The study revealed that Maxwell liquids are less stable than Oldroyd liquids, while Rivlin-Ericksen liquids are relatively more stable.

The validation of the Principle of Exchange of Stabilities (PES) is fundamental in stability analysis, defining the conditions under which a system undergoes a transition from stability to instability. When applied to instability problems, this principle eliminates the unsteady terms from the linearized perturbation equations, leading to a significant mathematical simplification. Pellew and Southwell (1940), demonstrated the applicability of the PES in the classical Rayleigh-Benard instability problem. However, although the PES provides qualitative insights into instability, it does not inherently quantify the rate at which perturbations grow during the transition. On the other hand, determining the upper bounds of the growth rate contributes quantitatively, offering insights into the measure of how fast disturbances can amplify during the stability transition and aiding in predicting the system's behavior. The combined validation of the PES and the analysis of upper bounds on the growth rate furnish a comprehensive understanding of the stability characteristics of a system [Chandrasekhar (1981)]. Thus, the research contributes to fields where stability and predictability are paramount, such as in control theory, it aids engineers in designing controllers capable of managing disturbances and uncertainties. The semi-circle theorem for both Veronis and Stern thermohaline configurations describing the upper bounds for free-free, rigid-rigid, and rigid-free combinations of boundaries is established by Banerjee et al. (1981). Banerjee and Banerjee (1984) introduced a criterion for characterizing non-oscillatory motions in hydrodynamics, which was later expanded upon by Gupta et al. (1986). Jyoti Prakash (2013) and Prakash and Gupta (2013) has determined the upper bounds of the complex growth rate within a porous layer that is fully saturated with ferrofluid and also for a rotating ferrofluid layer with MFD viscosity.

To the best of our knowledge and taking into account all the previously mentioned literature regarding the determination of bounds, no such results exist for the Rivlin-Ericksen ferromagnetic fluid that offers a versatile platform for various scientific and technological applications. Most of the authors' findings on characterizing non-oscillatory motions, as well as determining bounds for the growth rate of neutral and unstable oscillatory perturbations, are not influenced by changes in viscosity, thus limiting their practical utility. Keeping in mind the importance of field dependence on viscosity, the present study attempts to establish the upper bounds on the complex growth rate of arbitrary oscillatory movements of rising or neutral amplitude in a Rivlin-Ericksen ferromagnetic fluid layer with MFD viscosity, heated from below for the case of both free- free and rigid-rigid boundaries.

2 Mathematical Formulation

A layer of Rivlin-Ericksen ferromagnetic fluid that is finitely thin, incompressible, and electrically non-conductive is considered. This fluid layer of uniform finite thickness 'd' unit, is confined by two infinitely extended horizontal planes. The system is heated underneath. Let T_0 and $T_1(< T_0)$ represent the temperatures of the lower and upper boundaries, respectively, ensuring the maintenance of a uniform temperature gradient across the layer. A uniform magnetic field $\vec{H} = (0, 0, H_0)$ and gravity field $\vec{g} = (0, 0, -g)$ are present, both acting vertically along the z-axis as shown in Fig. 1. The viscosity of the fluid layer is assumed to vary with the magnetic field, given by $\mu = \mu_1(1 + \delta \cdot \vec{F})$, where μ_1 is the fluid viscosity in the absence of an external magnetic field, and $\vec{F}$ is the magnetic induction. The isotropic nature of the viscosity variation coefficient δ is considered i.e., $\delta_1 = \delta_2 = \delta_3 = \delta$. Here, the magnetic induction $\vec{F}$ and the magnetic field $\vec{H}$ are assumed to be parallel to each other.

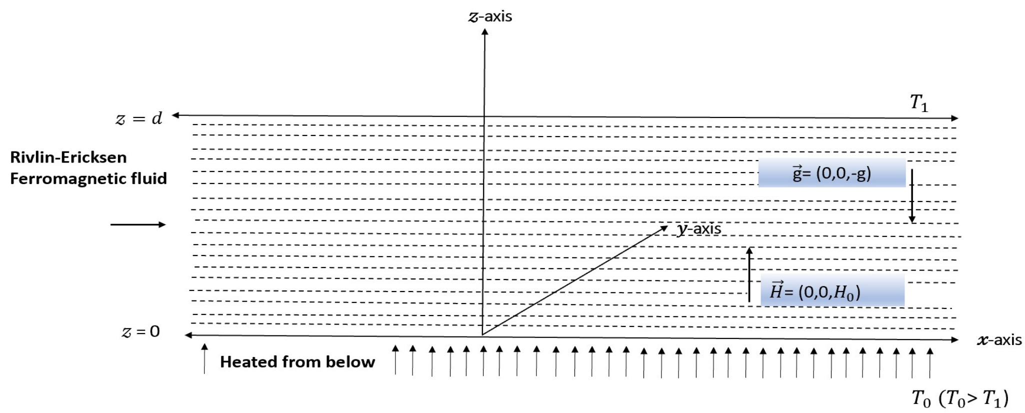

Fig. 1: Geometrical configuration

The fundamental hydrodynamic equations describing thermal convection in a ferrofluid subjected to a vertical magnetic field have been outlined in Finlayson (1970). For Viscoelastic ferrofluid with magnetic field-dependent viscosity, these equations can be reformulated using the approach proposed by Chandrasekhar (1981). Consequently, the set of governing equations for the present physical configuration, under the Boussinesq approximation, is as follows:

Expressing the fluid velocity as $\vec{q} = (u, v, w)$, the continuity equation is formulated as :

$$\nabla \cdot \vec{q} = 0, \tag{1}$$

The momentum equation for viscoelastic ferromagnetic fluid with magnetic field-dependent viscosity is given as [Siddheshwar (1999), (2002)] :

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \left[\rho_0 \frac{D\vec{q}}{Dt} + \rho\vec{g} + \nabla P - \nabla \cdot (\vec{H}\vec{F})\right] = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left[\mu \nabla^2 \vec{q}\right], \tag{2}$$

where λ_1 and λ_2 represent the relaxation and retardation time, respectively, and their values vary depending on the type of viscoelastic fluid under consideration. For Rivlin-Ericksen viscoelastic fluid $\lambda_1 \longrightarrow 0, \lambda_2 \longrightarrow \frac{\mu_2}{\mu_1}$ where μ_2 is fluid elasticity coefficient. Thus momentum equation for the present problem is formulated as :

$$\rho_0 \left[\frac{D\vec{q}}{Dt}\right] + \nabla P + \rho\vec{g} - \nabla \cdot (\vec{H}\vec{F}) = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \mu \nabla^2 \vec{q}. \tag{3}$$

Energy and state equations pertaining to the current problem are given as :

$$\left[\rho_0 C_{V,H} - \mu_0 \vec{H} \cdot \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\right] \frac{DT}{Dt} + \mu_0 T \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H} \cdot \frac{D\vec{H}}{Dt} = K_t \nabla^2 T, \tag{4}$$

$$\rho = \rho_0 \left[1 + \alpha \left(T_0 - T\right)\right]. \tag{5}$$

The aforementioned equations are formulated in a Cartesian coordinate system (x, y, z), where the origin is located at the bottom layer, and the z-axis is considered vertically upward. In these equations, P denotes pressure, T is the temperature, and μ is the field-dependent viscosity. Other parameters include $\frac{D}{Dt} = (\frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \nabla)$,$\vec{M}$, $C_{V,H}$, ρ, K_t, represents the material derivative, magnetization, specific heat at constant volume and magnetic field, density, and thermal conductivity of the fluid, respectively. Additionally, ρ_0 represents the density at reference temperature T_0 and μ_0 represents the magnetic permeability of free space.

Furthermore, ferromagnetic fluids satisfy the Maxwell's equations. The displacement current is negligible since we are looking at a fluid that is electrically non-conducting. Thus Maxwell's equations are simplified to

$$\nabla \cdot \vec{F} = 0, \quad \nabla \times \vec{H} = 0, \tag{6}$$

where

$$\vec{F} = \mu_0 (\vec{H} + \vec{M}). \tag{7}$$

Magnetization, which is parallel to the magnetic field, is influenced by both temperature and intensity of the magnetic field as

$$\vec{M} = \frac{\vec{H}}{H} M(H,T). \tag{8}$$

Since we are considering, M_0 to denote the magnetization at a magnetic field H_0 and temperature T_0. Additionally, let K_p represent the pyromagnetic coefficient, defined as $K_p = -\left(\frac{\partial M}{\partial T}\right)_{H_0,T_0}$, and χ denote the magnetic susceptibility, defined as $\chi = \left(\frac{\partial M}{\partial H}\right)_{H_0,T_0}$, where H_0 is the uniform magnetic field of the layer when placed in an external magnetic field. The linearized magnetic equation of the state may therefore be expressed as follows:

$$M - M_0 = \chi \left(H - H_0\right) - K_p \left(T - T_0\right). \tag{9}$$

By considering the basic state components of density, velocity, temperature, pressure, magnetization, and magnetic field as $\rho_i(z)$, $\vec{q}_i$, $T_i(z)$, P_i, $\vec{M}_i(z)$ and $\vec{H}_i(z)$ respectively, the quiescent state solution is obtained as

$$\vec{q} = \vec{q}_i = 0, \quad \rho = \rho_i(z), \quad P = P_i(z),$$
$$T = T_i(z) = -\beta z + T_0, \quad \vec{H}_i = \left(H_0 - \frac{K_p \beta z}{1 + \chi}\right) \hat{k},$$
$$\vec{M}_i = \left(M_0 - \frac{K_p \beta z}{1 + \chi}\right) \hat{k}, \quad \vec{H}_i + \vec{M}_i = H_0 + M_0, \quad \beta = \frac{T_0 - T_1}{d},$$

where the subscript 'i ' denotes the initial basic or unperturbed state, and β represents the temperature gradient in the vertical direction.

To explore the instability of the equilibrium system, minor disturbances are introduced to the initial state solution, and we denote the perturbed quantities as follows:

$$\vec{q} = \vec{q}_i + \vec{q}', \quad \rho = \rho_i(z) + \rho', \quad P = P_i(z) + P', \quad T = T_i(z) + \theta, \quad \vec{H} = \vec{H}_i(z) + \vec{H}', \quad \vec{M} = \vec{M}_i(z) + \vec{M}', \tag{10}$$

where prime (′) denotes the perturbations [$\vec{q}' = (u', v', w')$, $\vec{H}' = (H_x{}', H_y{}', H_z{}')$, and $\vec{M}' = (M_x{}', M_y{}', M_z{}')$] and thus all the prime terms represent the small perturbations around basic state solution.

Now, following the conventional procedures of linear stability [Chandrasekhar (1981), Finlayson (1970)], by substituting the perturbed quantities (10) into equations (1),(3),(4), and $\nabla \cdot \mu_0(\vec{H} + \vec{M}) = 0$, we derive the linearized perturbation equations:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{11}$$

$$\rho_0 \frac{\partial u'}{\partial t} + \frac{\partial P'}{\partial x} - \mu_0 (M_0 + H_0) \frac{\partial H'_x}{\partial z} = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \mu_1 \left[1 + \delta\mu_0 (M_0 + H_0)\right] \nabla^2 u', \tag{12}$$

$$\rho_0 \frac{\partial v'}{\partial t} + \frac{\partial P'}{\partial y} - \mu_0 (M_0 + H_0) \frac{\partial H'_y}{\partial z} = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \mu_1 \left[1 + \delta\mu_0 (M_0 + H_0)\right] \nabla^2 v', \tag{13}$$

$$\rho_0 \frac{\partial w'}{\partial t} + \frac{\partial P'}{\partial z} - \mu_0 (M_0 + H_0) \frac{\partial H'_z}{\partial z} + \mu_0 (H_z + M_z) \frac{\partial}{\partial z}\left(H_0 - \frac{K_p \beta z}{1 + \chi}\right) - \rho_0 g \alpha \theta = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \mu_1 \left[1 + \delta\mu_0 (H_0 + M_0)\right] \nabla^2 w', \tag{14}$$

$$\rho_C \frac{\partial \theta}{\partial t} - \mu_0 T_0 K_p \frac{\partial}{\partial t}\left(\frac{\partial \phi'}{\partial z}\right) = K_t \nabla^2 \theta + \left(\rho_c \beta - \frac{\mu_0 T_0 K_p^2 \beta}{1 + \chi}\right) w', \tag{15}$$

$$\frac{\partial}{\partial x}(H_x + M_x) + \frac{\partial}{\partial y}(H_y + M_y) + \frac{\partial}{\partial z}(H_z + M_z) = 0, \tag{16}$$

where $\rho_C = \rho_0 C_{V,H} + \mu_0 K_p H_0$ and ϕ' is the perturbed magnetic potential.

On applying perturbation to the equations (8) and (9), we obtain:

$$H'_z + M'_z = (1 + \chi) H'_z - K_p \theta, \qquad H'_j + M'_j = \left(1 + \frac{M_0}{H_0}\right) H'_j (j = 1, 2). \tag{17}$$

The above assumption was made under the restriction that $K_p \beta d \ll (1 + \chi) H_0$. This is because the study is confined to practical conditions where the magnetization caused by temperature changes is far lower than that by an external magnetic field.

Substituting (17) and using $\vec{H}' = \nabla \phi'$ in (14)and (16) yields:

$$\rho_0 \frac{\partial w'}{\partial t} + \frac{\partial P'}{\partial z} - \mu_0 (M_0 + H_0) \frac{\partial H'_z}{\partial z} + \mu_0 K_p \beta H'_z - \frac{\mu_0 K_p^2 \beta \theta}{(1 + \chi)} - \rho_0 g \alpha \theta = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \mu_1 \left[1 + \delta\mu_0 (H_0 + M_0)\right] \nabla^2 w', \tag{18}$$

$$(1 + \chi) \frac{\partial^2 \phi'}{\partial z^2} - \left(1 + \frac{M_0}{H_0}\right) \nabla_1^2 \phi' - K_p \frac{\partial \theta}{\partial z} = 0, \tag{19}$$

where $\nabla_1^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.
Now, eliminating P' in Eqs. (12),(13),and (18) on using (11), we obtain

$$\rho_0 \frac{\partial}{\partial t}\left(\nabla^2 w'\right) - \rho_0 \alpha g \nabla_1^2 \theta - \frac{\mu_0 K_p^2 \beta}{1 + \chi} \nabla_1^2 \theta + \mu_0 K_p \beta \frac{\partial}{\partial z}(\nabla_1^2 \phi') = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \mu_1 \left[1 + \delta\mu_0 (H_0 + M_0)\right] \nabla^4 w', \tag{20}$$

In further analysis, the perturbations w', θ, and ϕ' are broken down into periodic waves that exist in two dimensions, with attention paid to disturbances identified by a specific wave number 'k'. As a result, it is hypothesized that all properties that describe the perturbation are related to x, y, and t in a distinct functional manner given as

$$(w', \theta, \phi')(x, y, z, t) = [w''(z), \theta''(z), \phi''(z)] \exp\left[i\left(k_x x + k_y y\right) + \eta t\right], \tag{21}$$

where k_x and k_y represents the wave numbers along the x and y directions respectively. The resultant wave number is given by $k = \sqrt{k_x^2 + k_y^2}$ and η is a constant that, in general, can be complex. Nondimensionalizing the variables in Eqs. (15), (19), and (20) followed by Eq. (21) and by setting

$$w_* = \frac{d}{\nu} w'', \quad z_* = \frac{z}{d}, \quad D = d\frac{d}{dz}, \quad \nu = \frac{\mu}{\rho_0}, \quad P_r = \frac{\nu \rho_C}{K_t},$$

$$\sigma = \frac{\eta d^2}{\nu}, \quad a = kd, \quad F_v = \frac{\mu_2}{\rho_0 d^2}, \quad \theta_* = \frac{K_t a R^{\frac{1}{2}}}{\rho_C \beta \nu d^2}\theta'',$$

$$R = \frac{g\alpha\beta d^4 \rho_C}{\nu K_t}, \quad \delta_* = \delta\mu_0 H_0 (1+\chi), \quad M_1 = \frac{\mu_0 K_p^2 \beta}{(1+\chi)\alpha g \rho_0},$$

$$M_2 = \frac{\mu_0 T_0 K_p^2}{(1+\chi)\rho_C}, M_3 = \frac{1 + \frac{M_0}{H_0}}{(1+\chi)}, \quad \phi_* = \frac{(1+\chi) K_t a R^{\frac{1}{2}} \phi''}{K_p \rho_C \beta \nu d^2},$$

(after removing the asterisks for convenience) we get the following non-dimensional linearized equations:

$$\left(D^2 - a^2\right)\left\{(1+\delta M_3)(1+F_v\sigma)\left(D^2 - a^2\right) - \sigma\right\} w = a R^{\frac{1}{2}} \left\{(1+M_1)\theta - M_1 D\phi\right\}, \tag{22}$$

$$\left(D^2 - a^2 - P_r\sigma\right)\theta = -M_2\sigma P_r D\phi - (1 - M_2) a R^{\frac{1}{2}} w, \tag{23}$$

$$\left(D^2 - a^2 M_3\right)\phi = D\theta, \tag{24}$$

in the subsequent analysis, M_2 is neglected due to its very small order [Finlayson (1970)]. Therefore, the Eq. (23) is simplified to:

$$\left(D^2 - a^2 - P_r\sigma\right)\theta = -a R^{\frac{1}{2}} w, \tag{25}$$

where $D = d\frac{d}{dz}$ is the differential operator, $M_1 = \frac{\mu_0 K_p^2 \beta}{(1+\chi)\alpha g \rho_0}$ is the magnetic number that describes the magnetic force-to-buoyant force ratio owing to temperature fluctuations($M_1 > 0$), $M_2 = \frac{\mu_0 T_0 K_p^2}{(1+\chi)\rho_C}$ is the magnetic parameter, $M_3 = \frac{1+\frac{M_0}{H_0}}{(1+\chi)}$ is the measure of non linearity of magnetization($M_3 > 0$) , R represents the Rayleigh number, K_t is thermal conductivity , σ is the complex growth rate, P_r(¿0) is Prandtl number and F_v characterizes the viscoelasticity of fluid.
A constant temperature is maintained at the boundaries leading to the result that any temperature fluctuations at these points are eliminated (or zero). Thus the suitable boundary conditions can be expressed as:

$$w = 0 = \theta = D\phi = D^2 w \quad \text{at} \quad z = 0 \text{ and } z = 1. \quad (\text{ stress free}) \tag{26}$$

$$w = 0 = \theta = Dw = \phi \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1. \quad (\text{ rigid-rigid}) \tag{27}$$

Equations (22), (24), and (25) together with the either of boundary conditions [(26),(27)], represent an eigenvalue problem for σ. We aim to derive condition for PES validity and to characterize σ_i when $\sigma_r \geq 0$. A given state of the system is stable, neutral, or unstable accordingly as the real part of the complex growth rate (σ_r) is negative, zero or positive, respectively. If $\sigma_r \geq 0$ implies $\sigma_i = 0$ (or equivalently, $\sigma_i \neq 0$ implies $\sigma_r < 0$) for all wave number a^2, then for neutral instability ($\sigma_r = 0$), we have $\sigma = 0$. This situation in hydrodynamic stability theory is termed as the validity of the Principle of exchange of stabilities (PES), which means that the instability sets in as stationary convection, otherwise, we shall have overstability at least when instability sets in as certain modes. Also, an arbitrary oscillatory motion of neutral or growing amplitude is described by $\sigma_r \geq 0$ and $\sigma_i \neq 0$ [Chandrasekhar (1981)].

3 Mathematical Analysis

In order to examine the validity of PES, the Pellew and Southwell (1940) method will be employed to the problem under consideration, when the fluid layer confinement is considered stress-free (dynamically free). This technique, alternatively referred to as the method of conjugate eigenfunctions, is deeply rooted in the theory of quadratic forms and has contributed significantly to fundamental findings within the field of hydrodynamic stability theory. The outlined procedure is as follows: Eq. (22) is multiplied throughout with the complex conjugation of w, denoted by w^*, followed by integration with respect to variable z over its range. This yields:

$$\int_0^1 w^*\left(D^2-a^2\right)\left\{(1+\delta M_3)(1+F_v\sigma)\left(D^2-a^2\right)-\sigma\right\}wdz=-R^{\frac{1}{2}}aM_1\int_0^1 w^*D\phi dz+R^{\frac{1}{2}}a(1+M_1)\int_0^1 w^*\theta dz. \quad (28)$$

By utilizing Eqs. (24) and (25), as well as the boundary conditions outlined in Eq. (26), we can write

$$\begin{aligned}-R^{\frac{1}{2}}aM_1\int_0^1 w^*D\phi dz &= M_1\int_0^1 D\phi\left(D^2-a^2-P_r\sigma^*\right)\theta^*dz\\ &= M_1\int_0^1 D\phi D^2\theta^*dz - M_1\left(a^2+P_r\sigma^*\right)\int_0^1\theta^*D\phi dz\\ &= M_1\int_0^1 D\phi D^2\theta^*dz + M_1\left(a^2+P_r\sigma^*\right)\int_0^1\phi D\theta^*dz\\ &= M_1\int_0^1 D\phi D^2\theta^*dz + M_1\left(a^2+P_r\sigma^*\right)\int_0^1\phi\left(D^2-a^2M_3\right)\phi^*dz, \end{aligned}\quad (29)$$

and

$$aR^{\frac{1}{2}}\int_0^1 w^*\theta dz=-\int_0^1\theta\left(D^2-a^2-P_r\sigma^*\right)\theta^*dz. \quad (30)$$

Substituting Eqs. (29) and (30) in Eq. (28), we obtain

$$\begin{aligned}\int_0^1 w^*\left(D^2-a^2\right)\left\{(1+\delta M_3)(1+F_v\sigma)\left(D^2-a^2\right)-\sigma\right\}wdz =& M_1\int_0^1 D\phi D^2\theta^*dz-(1+M_1)\int_0^1\theta\left(D^2-a^2-P_r\sigma^*\right)\theta^*dz\\ &+M_1\left(a^2+P_r\sigma^*\right)\int_0^1\phi\left(D^2-a^2M_3\right)\phi^*dz. \end{aligned}\quad (31)$$

Integrating by parts the terms of Eq. (31) along with utilizing the boundary conditions (26) and the identity

$$\int_0^1\zeta^*D^{2n}\zeta dz=(-1)^n\int_0^1|D^n\zeta|^2dz,$$

where $\zeta=w(n=1,2)$ or $\zeta=\theta(n=1)$, we can express Eq. (31) in the following form:

$$\begin{aligned}(1+\delta M_3)(1+F_v\sigma)\int_0^1\left(\left|D^2w\right|^2+a^4|w|^2+2a^2|Dw|^2\right)\mathrm{d}z=&-\sigma\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz\\ &-M_1\int_0^1 D^2\phi D\theta^*dz\\ &+(1+M_1)\int_0^1\left(|D\theta|^2+a^2|\theta|^2+P_r\sigma^*|\theta|^2\right)dz\\ &-M_1\left(a^2+P_r\sigma^*\right)\int_0^1\left(|D\phi|^2+a^2M_3|\phi|^2\right)dz. \end{aligned}\quad (32)$$

By taking the complex conjugate of equation (24) and multiplying it with ϕ, the resulting equation can be integrated for variable z over its range. This yields:

$$\int_0^1\left(|D\phi|^2+a^2M_3|\phi|^2\right)dz=-\int_0^1\phi D\theta^*dz.$$

From the above equation, it is clear that $\int_0^1\phi D\theta^*dz$ necessitates to be a real value. Now multiplying Eq. (24) by $D\theta^*$ and integrating with respect to z over its vertical range, we obtain

$$\int_0^1 D^2\phi D\theta^*dz-a^2M_3\int_0^1\phi D\theta^*dz=\int_0^1|D\theta|^2dz.$$

As the right-hand side of the equation is real and $\int_0^1 \phi D\theta^* dz$ is a real quantity, it follows that $\int_0^1 D^2\phi D\theta^* dz$ in the aforementioned equation is also a real value.
In the following mathematical derivation, we equate the imaginary parts of Eq. (32) and utilize the property that $\int_0^1 D^2\phi D\theta^* dz$ is a real number, we get:

$$\begin{aligned}\sigma_{\mathrm{i}}\left(1+\delta M_3\right) F_v \int_0^1\left(|D^2 w|^2+a^4|w|^2+2 a^2|D w|^2\right) \mathrm{d}z+\sigma_{\mathrm{i}} \int_0^1|D w|^2 dz+a^2 \sigma_{\mathrm{i}} \int_0^1|w|^2 dz \\ -\sigma_{\mathrm{i}} M_1 P_r \int_0^1\left(|D \phi|^2+a^2 M_3|\phi|^2\right) dz+\sigma_{\mathrm{i}}\left(1+M_1\right) P_r \int_0^1|\theta|^2 dz=0.\end{aligned} \tag{33}$$

This clearly indicates that σ_i may or may not be equal to zero since, if σ_i is taken common throughout, the remaining terms in the product may balance to yield a result equal to zero.
Now we derive a sufficient condition for the validity of the PES in the present case.

Theorem 3.1. *If (w,θ,ϕ,σ), with $R>0$ and $\sigma_r\geq 0$, is a solution of Eqs. (22), (24), (25) together with boundary conditions (26) and $RM_1P_r/\pi^4[1+(1+\delta M_3)F_v\pi^2]\leq 1$ then $\sigma_{\mathrm{i}}=0$. In particular, PES is valid if $RM_1P_r/\pi^4[1+(1+\delta M_3)F_v\pi^2]\leq 1$.*

Proof. Suppose that $\sigma_i\neq 0$. Simplification of Eq. (33), on dividing both sides by σ_i, leads to the following equation

$$\begin{aligned}\left(1+\delta M_3\right) F_v \int_0^1\left(|D^2 w|^2+a^4|w|^2+2 a^2|D w|^2\right) \mathrm{d}z+\int_0^1|D w|^2 dz+a^2 \int_0^1|w|^2 dz = M_1 P_r \int_0^1\left(|D \phi|^2+a^2 M_3|\phi|^2\right) dz \\ -\left(1+M_1\right) P_r \int_0^1|\theta|^2 dz.\end{aligned} \tag{34}$$

Multiplication of Eq. (24) with ϕ^* (the complex conjugate of ϕ) followed by integration by parts of resulting expression along with utilizing boundary conditions (26), results in

$$\begin{aligned}\int_0^1\left(|D \phi|^2+a^2 M_3|\phi|^2\right) dz &= \int_0^1 \theta D\phi^* dz \\ &\leq \left|\int_0^1 \theta D\phi^* dz\right| \\ &\leq \int_0^1 |\theta D\phi^*|\, dz \\ &\leq \int_0^1 |\theta||D\phi| dz \\ &\leq \left(\int_0^1 |\theta|^2 dz\right)^{\frac{1}{2}}\left(\int_0^1 |D\phi|^2 dz\right)^{\frac{1}{2}} .\text{(employing the Schwartz inequality)}\end{aligned} \tag{35}$$

Thus we obtain

$$\int_0^1 |D\phi|^2 dz \leq \left(\int_0^1 |D\phi|^2 dz\right)^{\frac{1}{2}}\left(\int_0^1 |\theta|^2 dz\right)^{\frac{1}{2}},$$

implying

$$\left(\int_0^1 |D\phi|^2 dz\right)^{\frac{1}{2}} \leq \left(\int_0^1 |\theta|^2 dz\right)^{\frac{1}{2}}.$$

Inequality (35), on using $\left(\int_0^1 |D\phi|^2 dz\right)^{\frac{1}{2}} \leq \left(\int_0^1 |\theta|^2 dz\right)^{\frac{1}{2}}$, get modified to

$$\int_0^1\left(|D \phi|^2+a^2 M_3|\phi|^2\right) dz \leq \int_0^1 |\theta|^2 dz. \tag{36}$$

Since w and θ fulfill the conditions of " $w(0)=0=w(1)$ and $\theta(0)=0=\theta(1)$", it can be inferred from the Rayleigh-Ritz inequality [Schultz (1973)] that:

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz, \tag{37}$$

and

$$\int_0^1 |D\theta|^2 dz \geq \pi^2 \int_0^1 |\theta|^2 dz. \tag{38}$$

Also,

$$\int_0^1 \left(|Dw|^2\right) dz = \left|-\int_0^1 w^* D^2 w dz\right| \leq \left|\int_0^1 w^* D^2 w dz\right| \leq \left[\int_0^1 |w|^2 dz\right]^{\frac{1}{2}} \left[\int_0^1 \left|D^2 w\right|^2 dz\right]^{\frac{1}{2}},$$

which on utilizing Eq. (37), yields

$$\int_0^1 |D^2 w|^2 dz \geq \pi^2 \int_0^1 |Dw|^2 dz. \tag{39}$$

Now on multiplying Eq. (25) with the complex conjugate of θ (i.e., θ^*) and subsequently integrating the resulting equation through several iterations via the method of integration by parts, while also incorporating the specified boundary conditions on θ (specifically, $\theta(0) = 0 = \theta(1)$), the real part of the obtained equation results to:

$$\begin{aligned}
\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + P_r\sigma_r|\theta|^2\right) dz = &\ \text{Real part of } R^{\frac{1}{2}} a \int_0^1 \theta^* w dz \\
&\leq R^{\frac{1}{2}} a \left|\int_0^1 \theta^* w dz\right| \\
&\leq R^{\frac{1}{2}} a \int_0^1 |\theta^* w|\, dz \\
&\leq R^{\frac{1}{2}} a \int_0^1 |\theta^*|\, |w| dz \\
&\leq R^{\frac{1}{2}} a \int_0^1 |\theta||w| dz \\
&\leq R^{\frac{1}{2}} a \left(\int_0^1 |\theta|^2 dz\right)^{\frac{1}{2}} \left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}}. \text{ (using Schwarz inequality)}
\end{aligned}$$

Upon combining this inequality with (38) along with utilizing $\sigma_r \geq 0$, we get:

$$\pi^2 \int_0^1 |\theta|^2 dz \leq R^{\frac{1}{2}} a \left(\int_0^1 |\theta|^2 dz\right)^{\frac{1}{2}} \left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}},$$

which implies that

$$\left(\int_0^1 |\theta|^2 dz\right)^{\frac{1}{2}} \leq \frac{R^{\frac{1}{2}} a}{\pi^2} \left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}},$$

and thus

$$\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + P_r\sigma_r|\theta|^2\right) dz \leq \frac{Ra^2}{\pi^2} \int_0^1 |w|^2 dz,$$

which upon using inequality (37), gives

$$\int_0^1 |\theta|^2 dz \leq \frac{R}{\pi^4} \int_0^1 |Dw|^2 dz. \tag{40}$$

Using inequalities(36), (39), and (40) in Eq. (34), we obtain

$$(1 + \delta M_3)\, F_v \pi^2 \int_0^1 |Dw|^2\, dz + \int_0^1 |Dw|^2 dz + {I_1}^2 \leq \frac{RM_1P_r}{\pi^4} \int_0^1 |Dw|^2 dz - (1 + M_1)\, P_r \int_0^1 |\theta|^2 dz,$$

where

$${I_1}^2 = (1 + \delta M_3) F_v a^2 \int_0^1 \left[a^2|w|^2 + 2|Dw|^2\right] dz + a^2 \int_0^1 |w|^2 dz,$$

is clearly a positive definite.
which gives,

$$\left(\left[1 + (1 + \delta M_3) F_v \pi^2\right] - \frac{RM_1P_r}{\pi^4}\right) \int_0^1 |Dw|^2 dz + {I_1}^2 + (1 + M_1)\, P_r \int_0^1 |\theta|^2 dz \leq 0.$$

It is clear from above inequality that if $RM_1P_r/\pi^4 \left[1 + (1 + \delta M_3)F_v\pi^2\right] \leq 1$, we have a contradiction, and we must have $\sigma_i = 0$. □

Thus, $\left[RM_1P_r/\pi^4\left(1+(1+\delta M_3)F_v\pi^2\right)\right]\le 1$ represents a sufficient condition for the PES to be valid in this case.

Special Case: If the viscoelastic characteristic is set to zero, the current Rivlin-Ericksen ferrofluid model simplifies to the classic ferrofluid model. In this case, the sufficient condition for PES to hold is obtained as $\left[RM_1P_r/\pi^4\right]\le 1$ [Prakash (2012), for $\delta=0$].

It is noteworthy that if we look at the complement of the sufficient condition for PES to be valid, i.e.,

$$\left[RM_1P_r/\pi^4\left(1+(1+\delta M_3)F_v\pi^2\right)\right]>1$$

must hold, we observe the potential existence of oscillatory modes with increasing or neutral amplitudes. Consequently, it becomes crucial to estimate the limits for the growth rate of these motions.
In the subsequent analysis, we have established the upper bounds of the complex growth rate of oscillatory motions ($\sigma_i\ne 0$), whether neutral or unstable ($\sigma_r\ge 0$).

Theorem 3.2. *If (w,θ,ϕ,σ), with $R>0$, $\sigma_r\ge 0$, and $\sigma_i\ne 0$,is a solution of Eqs. (22)–(25) together with boundary conditions (26) then the condition*

$$|\sigma|^2<\frac{RM_1}{P_r\left[1+2(1+\delta M_3)F_v\pi^2\right]}$$

is necessarily satisfied.

Proof. After multiplying it by respective complex conjugate, integration of Eq. (25) with respect to z over its vertical range through a series of suitable iterations along with utilizing the boundary conditions (26), results in:

$$\int_0^1\left(\left|D^2\theta\right|^2+2a^2|D\theta|^2+a^4|\theta|^2\right)dz+2P_r\sigma_r\int_0^1\left(|D\theta|^2+a^2|\theta|^2\right)dz+P_r^2|\sigma|^2\int_0^1|\theta|^2dz=Ra^2\int_0^1|w|^2dz. \tag{41}$$

Since $\sigma_r\ge 0$, we obtain from Eq. (41), that

$$\int_0^1|\theta|^2dz\le\frac{Ra^2}{P_r^2|\sigma|^2}\int_0^1|w|^2dz. \tag{42}$$

From Eq. (34), we have

$$(1+\delta M_3)\,F_v\int_0^1\left(|D^2w|^2+a^4|w|^2+2a^2|Dw|^2\right)\mathrm{d}z+\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz=M_1P_r\int_0^1\left(|D\phi|^2+a^2M_3|\phi|^2\right)dz$$
$$-(1+M_1)\,P_r\int_0^1|\theta|^2dz.$$

Now by employing the inequalities (36) and (42), we can derive the following expression from Eq. (34).

$$2(1+\delta M_3)F_va^2\int_0^1|Dw|^2dz+a^2\int_0^1|w|^2dz+{I_2}^2+(1+M_1)\,P_r\int_0^1|\theta|^2dz\le\frac{Ra^2M_1}{P_r|\sigma|^2}\int_0^1|w|^2dz,$$

and on using $\int_0^1|Dw|^2dz\ge\pi^2\int_0^1|w|^2dz$, we get

$$2(1+\delta M_3)F_va^2\pi^2\int_0^1|w|^2dz+a^2\int_0^1|w|^2dz+{I_2}^2+(1+M_1)\,P_r\int_0^1|\theta|^2dz\le\frac{Ra^2M_1}{P_r|\sigma|^2}\int_0^1|w|^2dz,$$

where ${I_2}^2=(1+\delta M_3)F_v\int_0^1[|D^2w|^2+a^4|w|^2]dz+\int_0^1|Dw|^2dz$, is a positive definite.
The above equation can be rearranged as:

$$\left[1+2(1+\delta M_3)F_v\pi^2-\frac{RM_1}{P_r|\sigma|^2}\right]a^2\int_0^1|w|^2dz+{I_2}^2+(1+M_1)\,P_r\int_0^1|\theta|^2dz\le 0. \tag{43}$$

It follows from inequality (43) that

$$|\sigma|^2<\frac{RM_1}{P_r\left[1+2(1+\delta M_3)F_v\pi^2\right]}.$$

With this result, the theorem's proof concluded.

□

An equivalent statement of the above theorem is: for the case of free-free boundaries in the Rivlin-Ericksen ferromagnetic fluid layer, the complex growth rate of an arbitrary oscillatory motion with growing amplitude lies within a semicircle in the right half of the $\sigma_r\sigma_i-$ plane. The semicircle has its center at the origin and the square of its radius is given by $RM_1/P_r\left[1+2(1+\delta M_3)F_v\pi^2\right]$.

Now we will determine the upper bounds of complex growth rates of oscillatory motions of growing amplitude in Rivlin-Ericksen ferromagnetic convection, considering the case of rigid-rigid boundaries (27), as follows:

Theorem 3.3. *If (w,θ,ϕ,σ), with $R>0$, $\sigma_r\geq 0$, and $\sigma_i\neq 0$, is a solution of the Eqs. (22)–(25) along with the boundary conditions (27) then*

$$|\sigma|^2\sigma_i^2<\left(\frac{RM_1}{P_r\left[1+2(1+\delta M_3)F_v\pi^2\right]}\right)^2.$$

Proof. On integrating Eq. (22), after multiplying it by w^*, with respect to z, we obtain:

$$\int_0^1 w^*\left(D^2-a^2\right)\left\{(1+\delta M_3)\left(1+F_v\sigma\right)\left(D^2-a^2\right)-\sigma\right\}wdz=-R^{\frac{1}{2}}aM_1\int_0^1 w^*D\phi dz+R^{\frac{1}{2}}a\left(1+M_1\right)\int_0^1 w^*\theta dz. \quad (44)$$

The second term on R.H.S. of above equation on utilizing Eq. (25) and boundary conditions (27) gives

$$R^{\frac{1}{2}}a\left(1+M_1\right)\int_0^1 w^*\theta dz=-\left(1+M_1\right)\int_0^1\theta\left(D^2-a^2-P_r\sigma^*\right)\theta^*dz. \quad (45)$$

Thus on using (45), Eq. (44) modified to:

$$\int_0^1 w^*\left(D^2-a^2\right)\left\{(1+\delta M_3)\left(1+F_v\sigma\right)\left(D^2-a^2\right)-\sigma\right\}wdz=-\left(1+M_1\right)\int_0^1\theta\left(D^2-a^2-P_r\sigma^*\right)\theta^*dz$$
$$-R^{1/2}aM_1\int_0^1 w^*D\phi dz. \quad (46)$$

By iteratively applying integration by parts to the various terms in Eq. (44) and incorporating the boundary conditions (27), along with the following identity:

$$\int_0^1\zeta^*D^{2n}\zeta dz=(-1)^n\int_0^1\left|D^n\zeta\right|^2dz,$$

where $\zeta=w$ with $n=1,2$, or $\zeta=\theta$ with $n=1$, we obtain

$$(1+\delta M_3)(1+F_v\sigma)\int_0^1\left(|D^2w|^2+a^4|w|^2+2a^2|Dw|^2\right)dz+\sigma\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz$$
$$=-R^{\frac{1}{2}}\mathrm{a}M_1\int_0^1 w^*D\phi dz$$
$$+\left(1+M_1\right)\int_0^1\left(|D\theta|^2+a^2|\theta|^2+P_r\sigma^*|\theta|^2\right)dz. \quad (47)$$

Extracting the imaginary parts on both sides of above Eq. (47) and then dividing the equation thus obtained throughout by $\sigma_i(\neq 0)$, yields:

$$(1+\delta M_3)F_v\int_0^1\left(|D^2w|^2+a^4|w|^2+2a^2|Dw|^2\right)dz+\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz$$
$$=-\frac{R^{1/2}aM_1}{\sigma_i}\,\mathrm{Im}\left[\int_0^1 w^*D\phi dz\right]$$
$$-\left(1+M_1\right)P_r\int_0^1|\theta|^2dz. \quad (48)$$

Now

$$
\begin{aligned}
-\frac{R^{\frac{1}{2}}aM_1}{\sigma_i}\mathrm{Im}\left[\int_0^1 w^* D\phi dz\right] &\leq \left|-\frac{R^{\frac{1}{2}}aM_1}{\sigma_i}\ \mathrm{Im}\ \left[\int_0^1 w^* D\phi dz\right]\right| \\
&\leq R^{\frac{1}{2}}aM_1\left|\frac{1}{\sigma_i}\int_0^1 w^* D\phi dz\right| \\
&\leq \frac{R^{\frac{1}{2}}aM_1}{|\sigma_i|}\left|\int_0^1 w^* D\phi dz\right| \\
&\leq \frac{R^{\frac{1}{2}}aM_1}{|\sigma_i|}\int_0^1 |w^* D\phi|\, dz \\
&\leq \frac{R^{\frac{1}{2}}aM_1}{|\sigma_i|}\int_0^1 |w||D\phi| dz \\
&\leq \frac{R^{\frac{1}{2}}aM_1}{|\sigma_i|}\left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}}\left(\int_0^1 |D\phi|^2 dz\right)^{\frac{1}{2}}.\text{(on using Schwartz inequality)}
\end{aligned}
\tag{49}
$$

On combinig inequalities (36) and (42), we have

$$
\left(\int_0^1 |D\phi|^2 dz\right)^{\frac{1}{2}} \leq \frac{R^{1/2}a}{P_r|\sigma|}\left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}}. \tag{50}
$$

Inequality (49), on using the inequality (50), simplified to

$$
-\frac{R^{1/2}\mathrm{a}M_1}{\sigma_i}\ \mathrm{Im}\ \left[\int_0^1 w^* D\phi dz\right] \quad \leq \frac{Ra^2 M_1}{P_r|\sigma|\,|\sigma_i|}\int_0^1 |w|^2 dz. \tag{51}
$$

Thus on utilizing inequality (51) in Eq. (48), we ultimately have

$$
\begin{aligned}
(1+\delta M_3)F_v\int_0^1\left(\left|D^2w\right|^2+2a^2|Dw|^2+a^4|w|^2\right)+\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz \\
+(1+M_1)P_r\int_0^1|\theta|^2dz \leq \frac{a^2RM_1}{P_r|\sigma|\,|\sigma_i|}\int_0^1 |w|^2 dz,
\end{aligned}
$$

which implies that

$$
2(1+\delta M_3)F_v a^2\int_0^1 |Dw|^2 dz + a^2\int_0^1 |w|^2 dz + {I_2}^2 + (1+M_1)\,P_r\int_0^1 |\theta|^2 dz \leq \frac{a^2RM_1}{P_r|\sigma|\,|\sigma_i|}\int_0^1 |w|^2 dz,
$$

and on using $[\int_0^1 |Dw|^2dz \geq \pi^2 \int_0^1 |w|^2dz]$, we get

$$
2(1+\delta M_3)F_v a^2\pi^2\int_0^1 |w|^2 dz + a^2\int_0^1 |w|^2 dz + {I_2}^2 + (1+M_1)\,P_r\int_0^1 |\theta|^2 dz \leq \frac{a^2RM_1}{P_r|\sigma|\,|\sigma_i|}\int_0^1 |w|^2 dz,
$$

where ${I_2}^2 = (1+\delta M_3)F_v\int_0^1[|D^2w|^2 + a^4|w|^2]dz + \int_0^1 |Dw|^2dz$, is a positive definite.

The above equation can be rearranged as:

$$
\left[1+2(1+\delta M_3)F_v\pi^2 - \frac{RM_1}{P_r|\sigma|\,|\sigma_i|}\right]a^2\int_0^1 |w|^2dz + {I_2}^2 + (1+M_1)\,P_r\int_0^1 |\theta|^2 dz \leq 0, \tag{52}
$$

which clearly implies that

$$
|\sigma|^2\sigma_i^2 < \left(\frac{RM_1}{P_r\,[1+2(1+\delta M_3)F_v\pi^2]}\right)^2
$$

or

$$
\left(\sigma_r^2+\sigma_i^2\right)\sigma_i^2 < \left(\frac{RM_1}{P_r\,[1+2(1+\delta M_3)F_v\pi^2]}\right)^2. \tag{53}
$$

□

The theorem can be equivalently restated as: the complex growth rate (σ_r, σ_i) of any growing amplitude oscillatory motion in Rivlin-Ericksen ferromagnetic convection, subject to rigid-rigid boundaries, is situated within the region's right half specified by Eq. (53).

4 Conclusive Remarks

Linear stability theory and Normal mode analysis are employed to obtain the upper bounds on the complex growth rate of arbitrary oscillatory($\sigma_i \neq 0$) movements of rising ($\sigma_r > 0$) or neutral ($\sigma_r = 0$) amplitude in a Rivlin-Ericksen ferromagnetic fluid layer with MFD (magnetic field dependent) viscosity, heated from below, for the case of both free- free and rigid-rigid boundaries. The key findings of this study are summarized as follows:

1. A sufficient condition for the validity of the Principle of Exchange of Stabilities (PES) for the case of stress-free (dynamically free) boundaries is obtained using the Pellew and Southwell (1940) method. It is found that the onset of convection occurs through stationary mode when $\left[RM_1 P_r/\pi^4 \left(1 + (1 + \delta M_3)\pi^2 F_v\right)\right] \leq 1$.

2. For the case of stress-free boundaries in the Rivlin-Ericksen ferromagnetic fluid layer, the complex growth rate of an arbitrary oscillatory motion of growing amplitude ($\sigma_r \geq 0$) lies within a semicircle in the right half of the $\sigma_r\sigma_i-$ plane. The semicircle has its center at the origin and the square of its radius is given by $RM_1/P_r \left[1 + 2(1 + \delta M_3)F_v\pi^2\right]$.

3. The complex growth rate (σ_r, σ_i) of any growing amplitude oscillatory ($\sigma_i \neq 0$) motion in Rivlin-Ericksen ferromagnetic convection, subject to rigid-rigid boundaries, is situated within the right half of region specified as: $\left(\sigma_r^2 + \sigma_i^2\right)\sigma_i^2 < \left(RM_1/P_r \left[1 + 2(1 + \delta M_3)F_v\pi^2\right]\right)^2$.

4. The upper bounds obtained for rigid boundaries are particularly noteworthy, as the precise closed-form solutions can be difficult to obtain in such cases.

5. The derived inequalities reveal that by deliberately adjusting the fluid's viscoelasticity, coupled with modulation of MFD viscosity, one can effectively control the oscillations in thermomagnetic convection.

6. The results obtained herein are universally applicable, as they involve only non-dimensional quantities and are independent of the wave number.

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