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On the Multiplicative Logarithmic Strain Space Formulation

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A new constitutive approach to finite-deformation formulations of elasto-plasticity (with isotropic thermal expansion) is presented, which is well-suited to a broad range of metals used for industrial applications. The constitutive equations are formulated within the logarithmic strain space. The coupling of elasticity, plasticity and thermal expansion is based on a multiplicative approach of commutative-symmetric stretch tensor products with symmetrizing rotation tensors in the middle of two symmetric stretch tensors. It is essential for finite-deformation formulations to refer to an appropriate reference configuration κ_0 in order to model material orthotropy correctly (and not to mix up material / physical anisotropy with deformation-induced / geometrical anisotropy). Furthermore, it is essential to define proper stretch and strain tensors which are geometrical interpretable, i.e. which only depend on a current κ and a reference κ_0 configuration (and not on the geometrical deformation path between these configurations). Therefore, all stretch and strain tensors are defined with respect to the same reference configuration κ_0, the total stretches and strains as well as the partial ones: the elastic, plastic and thermal stretches and strains. And these reference configurations should be the appropriate ones, where—in the case of fiber-reinforced (metal) bodies—the fibers are placed orthogonal to each other or where Walter Noll's symmetry group considerations characterize the physical nature of materials with their most specificity as: orthotropic, transversal isotropic or fully isotropic.

1 Preliminaries, Notation and Functions of Symmetric Tensors

Finite-deformation models of simple materials are based on the tensor field of the deformation gradient

$$\mathsf{F} = {}^{\partial \mathbf{x}}\!/_{\partial \mathbf{X}} = {}^{\partial x_i}\!/_{\partial X_j}\, \mathbf{E}_i \otimes \mathbf{E}_j = F_{ij}\, \mathbf{E}_i \otimes \mathbf{E}_j = \mathsf{R}.\overbrace{\sqrt{\underbrace{\mathsf{F}^T.\mathsf{F}}_{\mathsf{C}}}}^{\mathsf{U}} = \overbrace{\sqrt{\underbrace{\mathsf{F}.\mathsf{F}^T}_{\mathsf{b}\,=\,\mathsf{R}.\mathsf{C}.\mathsf{R}^T}}}^{\mathsf{v}\,=\,\mathsf{R}.\mathsf{U}.\mathsf{R}^T}.\mathsf{R} \tag{1}$$

which is given by the partial derivative of the current position vector $\mathbf{x}$ with respect to the reference position vector $\mathbf{X}$, where $\otimes$ denotes the dyadic product and a "dot" the single contraction in tensor products. The deformation gradient F is a positive definite, second-order tensor (with nine internal degrees of freedom, given e.g. by its nine tensor components F_{ij} with respect to an $\mathbf{E}_i \otimes \mathbf{E}_j$ base). From the polar decomposition theorem, the deformation gradient (1) may multiplicatively be split into a (proper orthonormal $\mathsf{R}^{-1} = \mathsf{R}^T$) material rotation tensor (with three internal degrees of freedom)

$$\mathsf{R} = \mathsf{F}.\underbrace{\left(\mathsf{F}^T.\mathsf{F}\right)^{-\frac{1}{2}}}_{\sqrt{\mathsf{F}^{-1}.\mathsf{F}^{-T}}} = \underbrace{\left(\mathsf{F}.\mathsf{F}^T\right)^{-\frac{1}{2}}}_{\sqrt{\mathsf{F}^{-T}.\mathsf{F}^{-1}}}.\mathsf{F} = \sqrt{\mathsf{F}.\mathsf{F}^T}.\mathsf{F}^{-T} = \mathsf{F}^{-T}.\sqrt{\mathsf{F}^T.\mathsf{F}} = \mathsf{R}^{-T} \tag{2}$$

and positive definite, symmetric right

$$\mathsf{U} = \sqrt{\mathsf{C}} = \sqrt{\mathsf{F}^T.\mathsf{F}} = U_{ij}\, \mathbf{E}_i \otimes \mathbf{E}_j = \mathsf{U}^T = U_{ij}\, \mathbf{E}_j \otimes \mathbf{E}_i = U_{ji}\, \mathbf{E}_i \otimes \mathbf{E}_j \tag{3}$$

or left

$$\mathsf{v} = \sqrt{\mathsf{b}} = \sqrt{\mathsf{F}.\mathsf{F}^T} = \mathsf{v}^T \tag{4}$$

stretch tensors (with six internal degrees of freedom), where F^T, F^{-1} and F^{-T} denote the transpose, the inverse and the transposed inverse of a second-order tensor F, respectively. The symmetric right (3) and left (4) stretch tensors are given by the square roots of the symmetric right

$$\mathsf{C} = \mathsf{F}^T.\mathsf{F} = \mathsf{U}^2 = \mathsf{C}^T \tag{5}$$

and left

$$\mathsf{b} = \mathsf{F}.\mathsf{F}^T = \mathsf{v}^2 = \mathsf{b}^T \tag{6}$$

Cauchy-Green deformation tensors (Cauchy, 1827; Green, 1839). The left Cauchy-Green deformation tensor (6) is denoted with a small "b" instead of a capital "B" and the left stretch tensor (4) with a small "v" instead of a capital "V" in order to write *Eulerian tensors* like $\mathbf{x}$, b or v (which are altered under superposed rigid body motions) *with small letters* and *Lagrangean tensors* like $\mathbf{X}$, U or C (which are invariant under superposed rigid body motions) *with capital letters*.

Functions of symmetric tensors, like the square roots in equations (1)–(4), are most simply expressed in their spectral representation

$$f(\mathsf{C}) = f(\hat{C}_a\,\hat{\mathbf{E}}_a\otimes\hat{\mathbf{E}}_a) = f(\hat{C}_a)\,\hat{\mathbf{E}}_a\otimes\hat{\mathbf{E}}_a \tag{7}$$

and the function is just defined on the tensor's principal values (which are scalars). All principal quantities are marked with a "hat", and the summation convention is applied even to multiple repeated indices. The six internal degrees of freedom of a symmetric tensor can be viewed as: three principal values and three principal directions (represented by the orientation of the orthonormal principal base vectors). The material R-forward rotation of a Lagrangean tensor (function) is therefore given by the corresponding Eulerian tensor (function)

$$f(\mathsf{b}) = f(\mathsf{R}.\mathsf{C}.\mathsf{R}^T) = \mathsf{R}.f(\mathsf{C}).\mathsf{R}^T = f(\hat{C}_a)\,\underbrace{\mathsf{R}.\hat{\mathbf{E}}_a}_{\hat{\mathbf{e}}_a}\otimes\underbrace{\hat{\mathbf{E}}_a.\mathsf{R}^T}_{\hat{\mathbf{e}}_a} \tag{8}$$

where the Lagrangean principal base vectors $\hat{\mathbf{E}}_a$ are just material R-forward rotated to the Eulerian ones

$$\hat{\mathbf{e}}_a = \mathsf{R}.\hat{\mathbf{E}}_a = \hat{\mathbf{E}}_a.\mathsf{R}^T\,. \tag{9}$$

The (functions of) corresponding Lagrangean (7) and Eulerian (8) tensors have the same principal values. Only the orientation of their principal axes is given by the material R-forward (9) or R^T-backward rotation

$$\hat{\mathbf{E}}_a = \mathsf{R}^T.\hat{\mathbf{e}}_a = \hat{\mathbf{e}}_a.\mathsf{R} \tag{10}$$

of their principal base vectors.

2 Total and Partial Stretch or Strain Tensor Definitions and their Need for Geometrical Interpretability

Exclusively the Cauchy-Green deformation tensors C and b^{-1} and similar quadratic types like Green's strain (Green, 1839)

$$^{2}\mathsf{E} = \tfrac{1}{2}(\mathsf{C} - \mathrm{I}) \tag{11}$$

or Almansi's strain (Almansi, 1911)

$$^{-2}\mathsf{e} = \tfrac{1}{2}(\mathrm{I} - \mathsf{b}^{-1})\,, \tag{12}$$

where I denotes the second-order unit / identity tensor, may be projected onto arbitrary unit vectors

$$\mathbf{N} = \frac{d\mathbf{X}}{\|d\mathbf{X}\|} = \frac{\mathsf{F}^{-1}.d\mathbf{x}}{\|\mathsf{F}^{-1}.d\mathbf{x}\|} \qquad \text{and} \qquad \mathbf{n} = \frac{d\mathbf{x}}{\|d\mathbf{x}\|} = \frac{\mathsf{F}.d\mathbf{x}}{\|\mathsf{F}.d\mathbf{x}\|} \tag{13}$$

by applying $d\mathbf{x}{=}\mathsf{F}.d\mathbf{X}{=}d\mathbf{X}.\mathsf{F}^T$ from (1) and its inverse $d\mathbf{X}{=}\mathsf{F}^{-1}.d\mathbf{x}{=}d\mathbf{x}.\mathsf{F}^{-T}$ to the squared quotients

$$\left(\frac{\|d\mathbf{x}\|}{\|d\mathbf{X}\|}\right)^2 = \frac{d\mathbf{x}.d\mathbf{x}}{d\mathbf{X}.d\mathbf{X}} = \frac{d\mathbf{X}.(\mathsf{F}^T.\mathsf{F}).d\mathbf{X}}{\|d\mathbf{X}\|^2} = \mathbf{N}.\mathsf{C}.\mathbf{N} \tag{14}$$

and

$$\left(\frac{\|d\mathbf{X}\|}{\|d\mathbf{x}\|}\right)^2 = \frac{d\mathbf{X}.d\mathbf{X}}{d\mathbf{x}.d\mathbf{x}} = \frac{d\mathbf{x}.(\mathsf{F}^{-T}.\mathsf{F}^{-1}).d\mathbf{x}}{\|d\mathbf{x}\|^2} = \mathbf{n}.\mathsf{b}^{-1}.\mathbf{n} \tag{15}$$

of the lengths / (2-)norms $\|d\mathbf{x}\|$ and $\|d\mathbf{X}\|$ where the $d\mathbf{x}$ and $d\mathbf{X}$, respectively, denote the infinitesimal continuum vicinities of the material points $\mathbf{x}$ and $\mathbf{X}$ with respect to the current κ and reference κ_0 configurations. Strain tensor projections like (14) and (15) are not valid in general and especially not for stretch U and v or logarithmic strain ε and ${}^0\mathsf{e}$ tensor definitions. Most of the general symmetric stretch and strain tensors can be interpreted geometrically only along their principal directions by taking into account the function definition of their principal values.

The six internal degrees of freedom of a symmetric tensor can be viewed (in its spectral representation) as three principal values and three principal directions for

- the total stretches

$$\mathsf{U} = \frac{\ell_a}{L_a}\,\hat{\mathbf{E}}_a\otimes\hat{\mathbf{E}}_a = \sqrt{\hat{C}_a}\,\hat{\mathbf{E}}_a\otimes\hat{\mathbf{E}}_a \qquad \text{and} \qquad \mathsf{v} = \mathsf{R}.\mathsf{U}.\mathsf{R}^T = \frac{\ell_a}{L_a}\,\hat{\mathbf{e}}_a\otimes\hat{\mathbf{e}}_a \tag{16}$$

- the logarithmic total strains

$$\varepsilon = \ln(\mathsf{U}) = \ln\!\left(\frac{\ell_a}{L_a}\right)\hat{\mathbf{E}}_a\otimes\hat{\mathbf{E}}_a \qquad \text{and} \qquad {}^0\mathsf{e} = \mathsf{R}.\varepsilon.\mathsf{R}^T = \ln\!\left(\frac{\ell_a}{L_a}\right)\hat{\mathbf{e}}_a\otimes\hat{\mathbf{e}}_a \tag{17}$$

as well as for the partial stretches or the logarithmic partial strains

- the elastic stretches

$${}^e\mathsf{U} = \frac{{}^e\ell_b}{L_b}\,{}^e\hat{\mathbf{E}}_b\otimes{}^e\hat{\mathbf{E}}_b = \sqrt{{}^e\hat{C}_b}\,{}^e\hat{\mathbf{E}}_b\otimes{}^e\hat{\mathbf{E}}_b \qquad \text{and} \qquad {}^e\mathsf{v} = \mathsf{R}.{}^e\mathsf{U}.\mathsf{R}^T = \frac{{}^e\ell_b}{L_b}\,{}^e\hat{\mathbf{e}}_b\otimes{}^e\hat{\mathbf{e}}_b \tag{18}$$

- the logarithmic elastic strains

$${}^e\varepsilon = \ln({}^e\mathsf{U}) = \ln\!\left(\frac{{}^e\ell_b}{L_b}\right){}^e\hat{\mathbf{E}}_b\otimes{}^e\hat{\mathbf{E}}_b \qquad \text{and} \qquad \mathsf{R}.{}^e\varepsilon.\mathsf{R}^T = \ln\!\left(\frac{{}^e\ell_b}{L_b}\right){}^e\hat{\mathbf{e}}_b\otimes{}^e\hat{\mathbf{e}}_b \tag{19}$$

- the plastic stretches

$${}^p\mathsf{U} = \frac{{}^p\ell_c}{L_c}\,{}^p\hat{\mathbf{E}}_c\otimes{}^p\hat{\mathbf{E}}_c = \sqrt{{}^p\hat{C}_c}\,{}^p\hat{\mathbf{E}}_c\otimes{}^p\hat{\mathbf{E}}_c \qquad \text{and} \qquad {}^p\mathsf{v} = \mathsf{R}.{}^p\mathsf{U}.\mathsf{R}^T = \frac{{}^p\ell_c}{L_c}\,{}^p\hat{\mathbf{e}}_c\otimes{}^p\hat{\mathbf{e}}_c \tag{20}$$

- the logarithmic plastic strains

$${}^p\varepsilon = \ln({}^p\mathsf{U}) = \int_0^t {}^p\overset{\bullet}{\varepsilon}\,d\bar{t} = \ln\!\left(\frac{{}^p\ell_c}{L_c}\right){}^p\hat{\mathbf{E}}_c\otimes{}^p\hat{\mathbf{E}}_c \qquad \text{and} \qquad \mathsf{R}.{}^p\varepsilon.\mathsf{R}^T = \ln\!\left(\frac{{}^p\ell_c}{L_c}\right){}^p\hat{\mathbf{e}}_c\otimes{}^p\hat{\mathbf{e}}_c \tag{21}$$

- the thermal stretches

$${}^\vartheta\mathsf{U} = \frac{{}^\vartheta\ell_d}{L_d}\,{}^\vartheta\hat{\mathbf{E}}_d\otimes{}^\vartheta\hat{\mathbf{E}}_d = \hat{\Theta}_d(\vartheta)\,{}^\vartheta\hat{\mathbf{E}}_d\otimes{}^\vartheta\hat{\mathbf{E}}_d \qquad \text{and} \qquad {}^\vartheta\mathsf{v} = \mathsf{R}.{}^\vartheta\mathsf{U}.\mathsf{R}^T = \hat{\Theta}_d(\vartheta)\,{}^\vartheta\hat{\mathbf{e}}_d\otimes{}^\vartheta\hat{\mathbf{e}}_d \tag{22}$$

- the logarithmic thermal strains

$${}^\vartheta\varepsilon = \ln({}^\vartheta\mathsf{U}) = \ln\big(\hat{\Theta}_d(\vartheta)\big)\,{}^\vartheta\hat{\mathbf{E}}_d\otimes{}^\vartheta\hat{\mathbf{E}}_d \qquad \text{and} \qquad \mathsf{R}.{}^\vartheta\varepsilon.\mathsf{R}^T = \ln\big(\hat{\Theta}_d(\vartheta)\big)\,{}^\vartheta\hat{\mathbf{e}}_d\otimes{}^\vartheta\hat{\mathbf{e}}_d \tag{23}$$

  where the thermal expansion factors

$$\hat{\Theta}_d(\vartheta) = \frac{{}^\vartheta\ell_{(d)}}{L_{(d)}} \tag{24}$$

  are functions of the temperature ϑ and become $\hat{\Theta}_1(\vartheta{=}\vartheta_0){=}\hat{\Theta}_2(\vartheta{=}\vartheta_0){=}\hat{\Theta}_3(\vartheta{=}\vartheta_0){=}1$ unity at the reference temperature ϑ_0 so that the thermal stretch tensor ${}^\vartheta\mathsf{U}(\vartheta{=}\vartheta_0){=}\mathsf{I}$ merges to the second-order unit / identity tensor, the logarithmic thermal strain tensor ${}^\vartheta\varepsilon(\vartheta{=}\vartheta_0){=}\mathsf{0}$ vanishes and ${}^\vartheta\ell_1{=}L_1$, ${}^\vartheta\ell_2{=}L_2$, ${}^\vartheta\ell_3{=}L_3$ hold. The summation convention is not applied on indices enclosed in brackets, e.g. in equation (24).

In order to distinguish the labels of second-order partial tensors better from their power functions or transposes, all labels are written as upper left indices and the contraction in tensor products is denoted with a ”dot”.

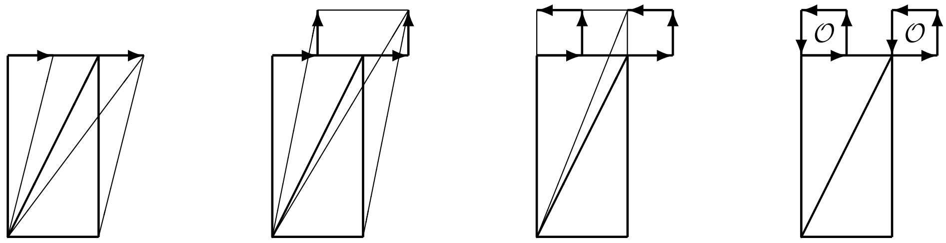

Figure 1. Closed finite-deformation cycle on the example of two triangular, constant strain Finite Elements

Any symmetric (total or partial) strain or stretch tensor must comply to the *geometrical interpretability* that its three principal values only depend on three corresponding current and reference lengths—and not on the history of their (geometrical) deformation path (Heiduschke, 1995, 1996). A logarithmic plastic strain tensor, for example, which results from the (translational and rotational materially convected) time integration of the plastic flow rule ${}^{p}\overset{\bullet}{\varepsilon}$ must obey the geometrical interpretability in order to constitute a proper strain tensor definition. In other words: if the finite-deformation path of a purely plastic material (with $\mathsf{U}={}^{p}\mathsf{U}$, ${}^{e}\mathsf{U}={}^{\vartheta}\mathsf{U}=\mathrm{I}$ and thus $\varepsilon={}^{p}\varepsilon$, ${}^{e}\varepsilon={}^{\vartheta}\varepsilon=0$) is a closed deformation cycle, as e.g. depicted in Figure 1, so that the current and reference plastic configurations are identical, then the corresponding plastic strain tensor increment (of the closed cycle) must vanish

$$\oint_{\mathcal{O}} d\varepsilon_{ij} = 0_{ij}\,. \tag{25}$$

Exclusively strain or stretch tensor definitions which obey the geometrical interpretability are of a proper type. The geometrical interpretability is a physical concept of a continuum formulation which might be used to test Finite Element procedures. Most of the general purpose Finite Element simulation tools, especially those which model finite deformations with the so-called updated Lagrangean description, fail the closed finite-deformation cycle test depicted in Figure 1. One can try it out using two triangular, constant strain elements with purely plastic material behaviour where the plastic strain tensor is summed up from the plastic-flow rule increments. (The plastic incompressibility constraint is met by the thickness of the elements which adjusts the volume constance of plasticity.) When these summed-up plastic strain tensors of closed deformation cycles do not vanish: $\oint_{\mathcal{O}} d\varepsilon_{ij} \neq 0_{ij}$, they are not geometrically interpretable, and the results of such simulation tools show a lack of any physical significance.

3 Relevance of the Reference Configuration κ_0 for the Modelling of Orthotropy or (Transversal) Isotropy

The modelling of finite-deformation anisotropy should distinguish material / physical anisotropy from deformation-induced / geometrical anisotropy. Material anisotropy must be specified by constitutive equations formulated with respect to a reference configuration κ_0 which characterizes the physical nature of the material with the most specificity as: orthotropic, transversal isotropic or isotropic. Within the deformation histories of (metal) bodies with, say, fiber reinforcement, these specific reference configurations κ_0 are given when the (metal) fibers are placed orthogonal to each other. A material only exhibits physically fully anisotropic behaviour if such a specific reference configuration κ_0 does not exists (hypothetically). From Noll's symmetry group considerations within these specific reference configurations κ_0, one can determine the physical nature of a material as: isotropic, transversal isotropic, orthotropic or fully anisotropic (Noll, 1958). For symmetric stress $\sigma=\sigma^T$ and symmetric strain (rate) tensors $@=\{{}^{e}\varepsilon,\, {}^{p}\overset{\bullet}{\varepsilon}\}=@^T$, the components of the corresponding fourth order material tensors $\mathcal{K}$ may be written in Voigt-Sayir matrix notation (Sayir, 1987) as

$$\left\{\begin{array}{r} @_{11} \\ @_{22} \\ @_{33} \\ \sqrt{2}\cdot @_{12} \\ \sqrt{2}\cdot @_{13} \\ \sqrt{2}\cdot @_{23} \end{array}\right\} = \left[\begin{array}{rrrrrr} \mathcal{K}_{1111} & \mathcal{K}_{1122} & \mathcal{K}_{1133} & \sqrt{2}\cdot\mathcal{K}_{1112} & \sqrt{2}\cdot\mathcal{K}_{1113} & \sqrt{2}\cdot\mathcal{K}_{1123} \\ \mathcal{K}_{1122} & \mathcal{K}_{2222} & \mathcal{K}_{2233} & \sqrt{2}\cdot\mathcal{K}_{2212} & \sqrt{2}\cdot\mathcal{K}_{2213} & \sqrt{2}\cdot\mathcal{K}_{2223} \\ \mathcal{K}_{1133} & \mathcal{K}_{2233} & \mathcal{K}_{3333} & \sqrt{2}\cdot\mathcal{K}_{3312} & \sqrt{2}\cdot\mathcal{K}_{3313} & \sqrt{2}\cdot\mathcal{K}_{3323} \\ \sqrt{2}\cdot\mathcal{K}_{1112} & \sqrt{2}\cdot\mathcal{K}_{2212} & \sqrt{2}\cdot\mathcal{K}_{3312} & 2\cdot\mathcal{K}_{1212} & 2\cdot\mathcal{K}_{1213} & 2\cdot\mathcal{K}_{1223} \\ \sqrt{2}\cdot\mathcal{K}_{1113} & \sqrt{2}\cdot\mathcal{K}_{2213} & \sqrt{2}\cdot\mathcal{K}_{3313} & 2\cdot\mathcal{K}_{1213} & 2\cdot\mathcal{K}_{1313} & 2\cdot\mathcal{K}_{1323} \\ \sqrt{2}\cdot\mathcal{K}_{1123} & \sqrt{2}\cdot\mathcal{K}_{2223} & \sqrt{2}\cdot\mathcal{K}_{3323} & 2\cdot\mathcal{K}_{1223} & 2\cdot\mathcal{K}_{1323} & 2\cdot\mathcal{K}_{2323} \end{array}\right] \left\{\begin{array}{r} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\cdot\sigma_{12} \\ \sqrt{2}\cdot\sigma_{13} \\ \sqrt{2}\cdot\sigma_{23} \end{array}\right\} \tag{26}$$

where the components of the symmetric tensors σ, ${}^{e}\varepsilon$ and ${}^{p}\overset{\bullet}{\varepsilon}$ are quoted as 6×1 pseudo column vectors and where the components of the fourth-order material tensors $\mathcal{K}$ obey the symmetries

$$\mathcal{K}_{ijk\ell} = \mathcal{K}_{ij\ell k} = \mathcal{K}_{jik\ell} = \mathcal{K}_{k\ell ij} \tag{27}$$

and are quoted as 6×6 pseudo matrices.

The Voigt-Sayir matrix quoted components $[[\mathcal{K}]]$ of the fourth-order material tensors $\mathcal{K}$ only exhibit their material specificity (the dots denote zeros) as orthotropic or transversal isotropic

$$\left\{\begin{array}{r} @_{11} \\ @_{22} \\ @_{33} \\ \sqrt{2}\cdot @_{12} \\ \sqrt{2}\cdot @_{13} \\ \sqrt{2}\cdot @_{23} \end{array}\right\} = \left[\begin{array}{cccccc} \mathcal{K}_{1111} & \mathcal{K}_{1122} & \mathcal{K}_{1133} & \cdot & \cdot & \cdot \\ \mathcal{K}_{1122} & \mathcal{K}_{2222} & \mathcal{K}_{2233} & \cdot & \cdot & \cdot \\ \mathcal{K}_{1133} & \mathcal{K}_{2233} & \mathcal{K}_{3333} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\cdot\mathcal{K}_{1212} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2\cdot\mathcal{K}_{1313} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\cdot\mathcal{K}_{2323} \end{array}\right] \left\{\begin{array}{r} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\cdot\sigma_{12} \\ \sqrt{2}\cdot\sigma_{13} \\ \sqrt{2}\cdot\sigma_{23} \end{array}\right\} \quad (28)$$

when they are given with respect to their principal $\mathbf{M}_i\otimes\mathbf{M}_j\otimes\mathbf{M}_k\otimes\mathbf{M}_\ell$ material base and within the specific reference configurations κ_0 (where the "fibers" are placed orthogonal).

If unspecific configurations $\tilde{\kappa}_0$ are taken as a reference for the finite-deformation formulations (where the "fibers" are not orthogonal), the material / physical and deformation-induced / geometrical anisotropies are mixed up so that the physical nature of the material is wrongly considered as fully anisotropic. Even the isotropic linearized elastic compliance tensor ${}^e\mathcal{K}$ (with two elasticy parameters) would appear as fully anisotropic (with 21 parameters) if the reference configuration is shifted by a full-fledged deformation gradient $\tilde{\mathsf{F}}_0$ from κ_0 to $\tilde{\kappa}_0$.

3.1 Full Material / Physical Anisotropy

For full material / physical anisotropy, the principal material directions (represented by the material $\mathbf{M}_k$ base vectors) in the specific reference configuration κ_0 are not orthogonal and, therefore, the tensor components may be specified with respect to arbitrary orthonormal $\mathbf{E}_k$ base vectors. Due to the symmetries (27) the $3^4{=}81$ coefficients ${}^e\mathcal{K}_{ijk\ell}$ of the fourth-order elastic compliance tensor ${}^e\mathcal{K}$ reduce to 21 material parameters / functions which specify the full anisotropic elasticity.

The plastic incompressibility condition, expressed with the logarithmic plastic strain-rate tensor

$${}^p\dot{\varepsilon}_{kk} = 0 = \dot{\lambda}\,{}^p\mathcal{K}_{kkij}\,\sigma_{ij} \qquad \forall\ \sigma_{ij} = \sigma_{ji} = \{\sigma_{11},\ \sigma_{22},\ \sigma_{33},\ \sigma_{12},\ \sigma_{13},\ \sigma_{23}\}\,, \quad (29)$$

results in six incompressibility constraints, which reduce the number of independent $\mathbf{E}_i\otimes\mathbf{E}_j\otimes\mathbf{E}_k\otimes\mathbf{E}_\ell$ coefficients to 15 material parameters / functions ${}^p\mathcal{K}_{ijk\ell}$ which specify full anisotropic, incompressible plasticity.

Full anisotropic thermal expansion, expressed with a (symmetric) logarithmic thermal strain tensor ${}^\vartheta\varepsilon$, is given by 6 components ${}^\vartheta\varepsilon_{ij}(\vartheta){=}{}^\vartheta\varepsilon_{ji}(\vartheta)$ with respect to an arbitrary orthonormal $\mathbf{E}_i\otimes\mathbf{E}_j$ base.

3.2 Material / Physical Orthotropy

For material / physical orthotropy, the principal material directions (represented by the material $\mathbf{M}_k$ base vectors) in the specific reference configuration κ_0 are orthogonal. Only if the fourth-order material tensor components are specified with respect to these orthonormal principal material base vectors $\mathbf{M}_i\otimes\mathbf{M}_j\otimes\mathbf{M}_k\otimes\mathbf{M}_\ell$, the Voigt-Sayir matrix quoted components $[[{}^e\mathcal{K}]]$ of the orthotropic elastic compliance tensor ${}^e\mathcal{K}$

$$\left\{\begin{array}{r} {}^e\varepsilon_{11} \\ {}^e\varepsilon_{22} \\ {}^e\varepsilon_{33} \\ \sqrt{2}\cdot{}^e\varepsilon_{12} \\ \sqrt{2}\cdot{}^e\varepsilon_{13} \\ \sqrt{2}\cdot{}^e\varepsilon_{23} \end{array}\right\} = \left[\begin{array}{cccccc} \frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{11}} & -\frac{\nu_{13}}{E_{11}} & \cdot & \cdot & \cdot \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & -\frac{\nu_{23}}{E_{22}} & \cdot & \cdot & \cdot \\ -\frac{\nu_{13}}{E_{11}} & -\frac{\nu_{23}}{E_{22}} & \frac{1}{E_{33}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\cdot\frac{1}{4\,G_{12}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{1}{4\,G_{13}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{1}{4\,G_{23}} \end{array}\right] \left\{\begin{array}{r} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\cdot\sigma_{12} \\ \sqrt{2}\cdot\sigma_{13} \\ \sqrt{2}\cdot\sigma_{23} \end{array}\right\} \quad (30)$$

and the components $\overset{\bullet}{\lambda}\,[[{}^p\mathcal{K}]]$ of the orthotropic plastic-flow tensor $\overset{\bullet}{\lambda}\,{}^p\mathcal{K}$

$$\left\{\begin{array}{r} {}^p\overset{\bullet}{\varepsilon}_{11} \\ {}^p\overset{\bullet}{\varepsilon}_{22} \\ {}^p\overset{\bullet}{\varepsilon}_{33} \\ \sqrt{2}\cdot{}^p\overset{\bullet}{\varepsilon}_{12} \\ \sqrt{2}\cdot{}^p\overset{\bullet}{\varepsilon}_{13} \\ \sqrt{2}\cdot{}^p\overset{\bullet}{\varepsilon}_{23} \end{array}\right\} = \overset{\bullet}{\lambda}\left[\begin{array}{cccccc} g+h & -h & -g & \cdot & \cdot & \cdot \\ -h & h+f & -f & \cdot & \cdot & \cdot \\ -g & -f & f+g & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\cdot\frac{n}{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{m}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{l}{2} \end{array}\right]\left\{\begin{array}{r} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\cdot\sigma_{12} \\ \sqrt{2}\cdot\sigma_{13} \\ \sqrt{2}\cdot\sigma_{23} \end{array}\right\} \tag{31}$$

exhibit the physics of orthotropy and reveal the decoupling of normal and shear components—represented by the 24 zeros (denoted as dots) within each material matrix of equations (28), (30) or (31). The 9 orthotropic elastic compliance parameters $\{E_{11}, E_{22}, E_{33}, \nu_{12}, \nu_{13}, \nu_{23}, G_{12}, G_{13}, G_{23}\}$ are defined such that $\nu_{ij}{=}\nu_{ji}$ holds for the Poisson's ratios of orthotropy (Hashin and Rosen, 1964; Hashin, 1979). The 6 orthotropic plastic-flow parameters $\{f, g, h, n, m, l\}$ with 3 plastic incompressibility constraints follow directly from the flow rule

$${}^p\overset{\bullet}{\varepsilon}_{ij} = \overset{\bullet}{\lambda}\,\frac{\partial\phi}{\partial\sigma_{ij}} \tag{32}$$

associated to Hill's orthotropic yield stress function (Hill, 1979)

$$\begin{aligned} \overline{\sigma} = \phi(\sigma_{ij}) &\equiv \sqrt{{}^p\mathcal{K}_{ijk\ell}\,\sigma_{ij}\,\sigma_{k\ell}} = \sqrt{\sigma..{}^p\mathcal{K}..\sigma} = \sqrt{\{\{\sigma\}\}^T\,[[{}^p\mathcal{K}]]\,\{\{\sigma\}\}} = \\ &= \sqrt{h(\sigma_{11}-\sigma_{22})^2 + g(\sigma_{11}-\sigma_{33})^2 + f(\sigma_{22}-\sigma_{33})^2 + \underbrace{2n\,\sigma_{12}^2 + 2m\,\sigma_{13}^2 + 2l\,\sigma_{23}^2}_{n(\sigma_{12}^2+\sigma_{21}^2)+m(\sigma_{13}^2+\sigma_{31}^2)+l(\sigma_{23}^2+\sigma_{32}^2)}} \end{aligned} \tag{33}$$

whose mathematical form is in Hill (1979) slightly modified from the original version of Hill (1948). The "double dots" denote the double contraction in tensor products: $\mathsf{A}..\mathsf{B}{=}\mathrm{tr}(\mathsf{A}^T.\mathsf{B}){=}A_{ij}\,B_{ji}$ where $\mathrm{tr}(\cdot)$ is the trace operator.

The thermal stretch tensor ${}^\vartheta\mathsf{U}$ for material orthotropy is given by the 3 principal thermal stretch factors $\{\hat{\Theta}_1(\vartheta), \hat{\Theta}_2(\vartheta), \hat{\Theta}_3(\vartheta)\}$ which correspond to the principal $\mathbf{M}_{(k)}{\otimes}\mathbf{M}_{(k)}$ components $\{{}^\vartheta\hat{\varepsilon}_1{=}\ln\big(\hat{\Theta}_1(\vartheta)\big), {}^\vartheta\hat{\varepsilon}_2{=}\ln\big(\hat{\Theta}_2(\vartheta)\big), {}^\vartheta\hat{\varepsilon}_3{=}\ln\big(\hat{\Theta}_3(\vartheta)\big)\}$ of the symmetric logarithmic thermal strain ${}^\vartheta\varepsilon$ tensor.

3.3 Material / Physical Transversal Isotropy

For material / physical transversal isotropy, the principal material directions (represented by the material $\mathbf{M}_k$ base vectors) in the specific reference configuration κ_0 are orthogonal. The material $\mathbf{M}_1$ base vector points into the logitudinal "fiber" direction and the material $\mathbf{M}_2$, $\mathbf{M}_3$ base vectors span the plane of transversal isotropy which is perpendicular to $\mathbf{M}_1$. Only if the fourth-order material tensor components are specified with respect to these orthonormal principal material base vectors $\mathbf{M}_i{\otimes}\mathbf{M}_j{\otimes}\mathbf{M}_k{\otimes}\mathbf{M}_\ell$, the Voigt-Sayir matrix quoted components $[[{}^e\mathcal{K}]]$ of the transversal isotropic, elastic compliance tensor ${}^e\mathcal{K}$

$$\left\{\begin{array}{r} {}^e\varepsilon_{11} \\ {}^e\varepsilon_{22} \\ {}^e\varepsilon_{33} \\ \sqrt{2}\cdot{}^e\varepsilon_{12} \\ \sqrt{2}\cdot{}^e\varepsilon_{13} \\ \sqrt{2}\cdot{}^e\varepsilon_{23} \end{array}\right\} = \left[\begin{array}{cccccc} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & \cdot & \cdot & \cdot \\ -\frac{\nu}{E} & \frac{1}{E'} & -\frac{\nu'}{E'} & \cdot & \cdot & \cdot \\ -\frac{\nu}{E} & -\frac{\nu'}{E'} & \frac{1}{E'} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\cdot\frac{1}{4\,G} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{1}{4\,G} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{1+\nu'}{2\,E'} \end{array}\right]\left\{\begin{array}{r} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\cdot\sigma_{12} \\ \sqrt{2}\cdot\sigma_{13} \\ \sqrt{2}\cdot\sigma_{23} \end{array}\right\} \tag{34}$$

and the components $\overset{\bullet}{\lambda}\,[[{}^{p}\mathcal{K}]]$ of the transversal isotropic, plastic-flow tensor $\overset{\bullet}{\lambda}\,{}^{p}\mathcal{K}$

$$\begin{Bmatrix} {}^{p}\overset{\bullet}{\varepsilon}_{11} \\ {}^{p}\overset{\bullet}{\varepsilon}_{22} \\ {}^{p}\overset{\bullet}{\varepsilon}_{33} \\ \sqrt{2}\cdot{}^{p}\overset{\bullet}{\varepsilon}_{12} \\ \sqrt{2}\cdot{}^{p}\overset{\bullet}{\varepsilon}_{13} \\ \sqrt{2}\cdot{}^{p}\overset{\bullet}{\varepsilon}_{23} \end{Bmatrix} = \overset{\bullet}{\lambda} \begin{bmatrix} 2g & -g & -g & \cdot & \cdot & \cdot \\ -g & f+g & -f & \cdot & \cdot & \cdot \\ -g & -f & f+g & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\cdot\frac{m}{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{m}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{2f+g}{2} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\cdot\sigma_{12} \\ \sqrt{2}\cdot\sigma_{13} \\ \sqrt{2}\cdot\sigma_{23} \end{Bmatrix} \tag{35}$$

exhibit the physics of transversal isotropy and reveal the decoupling of normal and shear components. Transversal isotropy is specified by 5 elastic compliance parameters $\{E, E', \nu, \nu', G\}$ and by 3 plastic-flow parameters $\{f, g, m\}$ with 2 plastic incompressibility constraints.

The thermal stretch tensor ${}^{\vartheta}\mathsf{U}$ for transversal isotropy is given by the 2 principal thermal stretch factors $\{\hat{\Theta}(\vartheta), \hat{\Theta}'(\vartheta)\}$ which correspond to the principal $\mathbf{M}_{(k)}\otimes\mathbf{M}_{(k)}$ components in the longitudinal ${}^{\vartheta}\hat{\varepsilon}_1=\ln\big(\hat{\Theta}(\vartheta)\big)$ and the transversal ${}^{\vartheta}\hat{\varepsilon}_2={}^{\vartheta}\hat{\varepsilon}_3=\ln\big(\hat{\Theta}'(\vartheta)\big)$ principal directions of the symmetric logarithmic thermal strain ${}^{\vartheta}\varepsilon$ tensor.

3.4 Material / Physical Isotropy

For material / physical isotropy, the orthonormal principal material $\mathbf{M}_k$ base vector systems must not be distinguished from the orthonormal arbitrary $\mathbf{E}_k$ base vector systems in the specific reference configuration κ_0 since every arbitrary base vector $\mathbf{E}_k$ points into a principal material $\mathbf{M}_k$ direction. The Voigt-Sayir matrix quoted fourth-order material tensor components $[[{}^{e}\mathcal{K}]]$ of the isotropic, elastic compliance ${}^{e}\mathcal{K}$

$$\begin{Bmatrix} {}^{e}\varepsilon_{11} \\ {}^{e}\varepsilon_{22} \\ {}^{e}\varepsilon_{33} \\ \sqrt{2}\cdot{}^{e}\varepsilon_{12} \\ \sqrt{2}\cdot{}^{e}\varepsilon_{13} \\ \sqrt{2}\cdot{}^{e}\varepsilon_{23} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & \cdot & \cdot & \cdot \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & \cdot & \cdot & \cdot \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\cdot\frac{1+\nu}{2\,E} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{1+\nu}{2\,E} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{1+\nu}{2\,E} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\cdot\sigma_{12} \\ \sqrt{2}\cdot\sigma_{13} \\ \sqrt{2}\cdot\sigma_{23} \end{Bmatrix} \tag{36}$$

and $\overset{\bullet}{\lambda}\,[[{}^{p}\mathcal{K}]]$ of the isotropic, plastic-flow tensor $\overset{\bullet}{\lambda}\,{}^{p}\mathcal{K}$

$$\begin{Bmatrix} {}^{p}\overset{\bullet}{\varepsilon}_{11} \\ {}^{p}\overset{\bullet}{\varepsilon}_{22} \\ {}^{p}\overset{\bullet}{\varepsilon}_{33} \\ \sqrt{2}\cdot{}^{p}\overset{\bullet}{\varepsilon}_{12} \\ \sqrt{2}\cdot{}^{p}\overset{\bullet}{\varepsilon}_{13} \\ \sqrt{2}\cdot{}^{p}\overset{\bullet}{\varepsilon}_{23} \end{Bmatrix} = \overset{\bullet}{\lambda} \begin{bmatrix} 2h & -h & -h & \cdot & \cdot & \cdot \\ -h & 2h & -h & \cdot & \cdot & \cdot \\ -h & -h & 2h & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\cdot\frac{3h}{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{3h}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\cdot\frac{3h}{2} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\cdot\sigma_{12} \\ \sqrt{2}\cdot\sigma_{13} \\ \sqrt{2}\cdot\sigma_{23} \end{Bmatrix} \tag{37}$$

reveal the decoupling of normal and shear components. Material isotropy is specified by 2 elastic compliance parameters $\{E, \nu\}$ and by 1 incompressible plastic-flow parameter, the hardening function h, with 1 plastic incompressibility constraint which may be expressed by a plastic Poisson's ratio of 0.5 for the negative quotient of transversal over longitudinal logarithmic plastic strain.

The thermal stretch tensor ${}^{\vartheta}\mathsf{U}=\hat{\Theta}(\vartheta)\cdot\mathrm{I}$ for material isotropy is totally governed by 1 stretch factor $\hat{\Theta}(\vartheta)$ which corresponds to the isotropic definition

$${}^{\vartheta}\varepsilon = \ln\big(\hat{\Theta}(\vartheta)\big)\cdot\mathrm{I} \tag{38}$$

of the symmetric logarithmic thermal strain tensor.

3.5 Walter Noll's Symmetry Group Considerations (1958)

One can interpret Walter Noll's symmetry group considerations (Noll, 1958) with co-ordinate rotations

$$\overline{\mathcal{K}_{abcd}} = \underbrace{Q_{ai}\,Q_{bj}}_{\mathcal{Q}_{abij}}\,\underbrace{Q_{ck}\,Q_{d\ell}}_{\mathcal{Q}_{cdk\ell}}\,\mathcal{K}_{ijk\ell} = \mathcal{Q}_{abij}\,\mathcal{Q}_{cdk\ell}\,\mathcal{K}_{ijk\ell} \tag{39}$$

of fourth-order material tensors

$$\mathcal{K} = \overbrace{\underbrace{Q_{ai}\,Q_{bj}}_{\mathcal{Q}_{abij}}\,\underbrace{Q_{ck}\,Q_{d\ell}}_{\mathcal{Q}_{cdk\ell}}\,\mathcal{K}_{ijk\ell}}^{\overline{\mathcal{K}_{abcd}}}\,\mathbf{E}_a \otimes \mathbf{E}_b \otimes \mathbf{E}_c \otimes \mathbf{E}_d = \mathcal{K}_{ijk\ell}\,\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_\ell \tag{40}$$

where the second-order rotation tensor $\mathsf{Q}{=}\mathbf{M}_k{\otimes}\mathbf{E}_k$ is defined by the orientation of the orthonormal principal material base vectors

$$\mathbf{M}_k = \mathsf{Q}.\mathbf{E}_k = \mathbf{E}_k.\mathsf{Q}^T = Q_{ik}\,\mathbf{E}_i \tag{41}$$

with respect to the orthonormal arbitrary $\mathbf{E}_k$ base vectors. The $\mathbf{E}_i{\otimes}\mathbf{E}_j$ components of Q are given in matrix notation by

$$[\mathsf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \tag{42}$$

and the co-ordinate rotations (39) of fourth-order material tensors (40) read in Voigt-Sayir matrix notation

$$\overline{[[\mathcal{K}]]} = [[\mathcal{Q}]]\,[[\mathcal{K}]]\,[[\mathcal{Q}^T]] \tag{43}$$

with the Voigt-Sayir matrix quoted components

$$[[\mathcal{Q}]] = \begin{bmatrix} Q_{11}\,Q_{11} & Q_{12}\,Q_{12} & Q_{13}\,Q_{13} & \sqrt{2}\,Q_{11}\,Q_{12} & \sqrt{2}\,Q_{11}\,Q_{13} & \sqrt{2}\,Q_{12}\,Q_{13} \\ Q_{21}\,Q_{21} & Q_{22}\,Q_{22} & Q_{23}\,Q_{23} & \sqrt{2}\,Q_{21}\,Q_{22} & \sqrt{2}\,Q_{21}\,Q_{23} & \sqrt{2}\,Q_{22}\,Q_{23} \\ Q_{31}\,Q_{31} & Q_{32}\,Q_{32} & Q_{33}\,Q_{33} & \sqrt{2}\,Q_{31}\,Q_{32} & \sqrt{2}\,Q_{31}\,Q_{33} & \sqrt{2}\,Q_{32}\,Q_{33} \\ \sqrt{2}\,Q_{11}\,Q_{21} & \sqrt{2}\,Q_{12}\,Q_{22} & \sqrt{2}\,Q_{13}\,Q_{23} & Q_{11}\,Q_{22}+Q_{12}\,Q_{21} & Q_{11}\,Q_{23}+Q_{13}\,Q_{21} & Q_{12}\,Q_{23}+Q_{13}\,Q_{22} \\ \sqrt{2}\,Q_{11}\,Q_{31} & \sqrt{2}\,Q_{12}\,Q_{32} & \sqrt{2}\,Q_{13}\,Q_{33} & Q_{11}\,Q_{32}+Q_{12}\,Q_{31} & Q_{11}\,Q_{33}+Q_{13}\,Q_{31} & Q_{12}\,Q_{33}+Q_{13}\,Q_{32} \\ \sqrt{2}\,Q_{21}\,Q_{31} & \sqrt{2}\,Q_{22}\,Q_{32} & \sqrt{2}\,Q_{23}\,Q_{33} & Q_{21}\,Q_{32}+Q_{22}\,Q_{31} & Q_{21}\,Q_{33}+Q_{23}\,Q_{31} & Q_{22}\,Q_{33}+Q_{23}\,Q_{32} \end{bmatrix} \tag{44}$$

of the fourth-order rotation tensor $\mathcal{Q}{=}\mathcal{Q}_{ijk\ell}\,\mathbf{E}_i{\otimes}\mathbf{E}_j{\otimes}\mathbf{E}_k{\otimes}\mathbf{E}_\ell{=}Q_{ik}\,Q_{j\ell}\,\mathbf{E}_i{\otimes}\mathbf{E}_j{\otimes}\mathbf{E}_k{\otimes}\mathbf{E}_\ell$ corresponding to (42). In general, an isotropic material tensor $\mathcal{K}$ is invariant against co-ordinate rotations (39)—expressed in Voigt-Sayir matrix notation as

$$\overline{\begin{bmatrix} A & -B & -B & \cdot & \cdot & \cdot \\ -B & A & -B & \cdot & \cdot & \cdot \\ -B & -B & A & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & A+B & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & A+B & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & A+B \end{bmatrix}} = [[\mathcal{Q}]] \begin{bmatrix} A & -B & -B & \cdot & \cdot & \cdot \\ -B & A & -B & \cdot & \cdot & \cdot \\ -B & -B & A & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & A+B & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & A+B & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & A+B \end{bmatrix} [[\mathcal{Q}^T]] \tag{45}$$

—where $\mathcal{K}$ is composed of a fourth-order unit / identity tensor with the "$A{+}B$" term on the diagonal and the "$-B$" term on the normal components of the material matrix within the upper left quadrant in Voigt-Sayir matrix notation—cf. the isotropic material matrices of (36) and (37).

A transversal isotropic material tensor $\mathcal{K}$ is invariant against co-ordinate rotations about an angle φ along the longitudinal "fiber" $\mathbf{M}_1$ direction—expressed within a principal material base vector system in Voigt-Sayir matrix notation as

$$\overline{\begin{bmatrix} A & -B & -B & \cdot & \cdot & \cdot \\ -B & D & -E & \cdot & \cdot & \cdot \\ -B & -E & D & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & G & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & G & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & D+E \end{bmatrix}} = [[{}^1\mathcal{Q}]] \begin{bmatrix} A & -B & -B & \cdot & \cdot & \cdot \\ -B & D & -E & \cdot & \cdot & \cdot \\ -B & -E & D & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & G & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & G & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & D+E \end{bmatrix} [[{}^1\mathcal{Q}^T]] \tag{46}$$

—cf. the transversal isotropic material matrices of (34) and (35)—where the $\mathbf{M}_i{\otimes}\mathbf{M}_j{\otimes}\mathbf{M}_k{\otimes}\mathbf{M}_\ell$ components of the corresponding fourth-order rotation tensor ${}^1\mathcal{Q}$ read in Voigt-Sayir notation

$$[[{}^1\mathcal{Q}]] = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & c^2 & s^2 & \cdot & \cdot & -\sqrt{2}\,c\,s \\ \cdot & s^2 & c^2 & \cdot & \cdot & \sqrt{2}\,c\,s \\ \cdot & \cdot & \cdot & c & -s & \cdot \\ \cdot & \cdot & \cdot & s & c & \cdot \\ \cdot & \sqrt{2}\,c\,s & -\sqrt{2}\,c\,s & \cdot & \cdot & c^2 - s^2 \end{bmatrix} \tag{47}$$

and are associated with the $\mathbf{M}_i \otimes \mathbf{M}_j$ components of the second-order rotation tensor ${}^1\mathsf{Q}$ in matrix notation

$$[{}^1\mathsf{Q}\,] = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & c & -s \\ \cdot & s & c \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cos(\varphi) & -\sin(\varphi) \\ \cdot & \sin(\varphi) & \cos(\varphi) \end{bmatrix} \tag{48}$$

within a principal material base vector system. The abbreviations c and s in equations (47) and (48) denote the $\cos(\varphi)$ and $\sin(\varphi)$ functions, respectively.

An orthotropic material tensor $\mathcal{K}$ is just invariant against index shifts—expressed within a principal material base vector system in Voigt-Sayir matrix notation as

$$\overline{\begin{bmatrix} F & -C & -E & \cdot & \cdot & \cdot \\ -C & A & -B & \cdot & \cdot & \cdot \\ -E & -B & D & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & H & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & I & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & G \end{bmatrix}} = [[{}^o\mathcal{Q}\,]] \begin{bmatrix} A & -B & -C & \cdot & \cdot & \cdot \\ -B & D & -E & \cdot & \cdot & \cdot \\ -C & -E & F & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & G & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & H & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & I \end{bmatrix} [[{}^o\mathcal{Q}^T]] \tag{49}$$

—cf. the orthotropic material matrices of (30) and (31)—where the Voigt-Sayir matrix quoted $\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_\ell$ components of the corresponding fourth-order rotation tensor ${}^o\mathcal{Q}$ read

$$[[{}^o\mathcal{Q}\,]] = \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix} \tag{50}$$

and are associated with the $\mathbf{M}_i \otimes \mathbf{M}_j$ components of the second-order rotation tensor ${}^o\mathsf{Q}$ in matrix notation

$$[{}^o\mathsf{Q}\,] = \begin{bmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{bmatrix} \tag{51}$$

which just reduce to the index shift matrices (50) and (51) and may be applied recursively within principal material base vector systems.

4 Commutative-symmetric Stretch Tensor Products with Symmetrizing Rotation Tensors in the Middle

For finite deformations, the orthotropy within a material description is ”lost” if a total deformation gradient F is defined as just (a chain of) tensor products ${}^e\mathsf{F} \ldots {}^p\mathsf{F}$ of partial deformation gradients (which are not commutative). After the application of a first partial deformation gradient ${}^p\mathsf{F}$, the ”orthogonal fibers” in a reference configuration κ_0 are no longer orthogonal since the reference of the follow-up deformation gradients will be finitely shifted by ${}^p\mathsf{F}$.

Within the approach presented here, *all* symmetric partial stretch and strain tensors are defined with respect to the *same* reference configuration κ_0 (of the total stretch and strain tensor definitions) which should be the one with the ”orthogonal fibers” for material orthotropy or transversal isotropy. The definition of a commutative-symmetric (partial) stretch tensor product is given by

$$\mathsf{A.X.B} = (\mathsf{A.X.B})^T = \mathsf{B.X}^T\mathsf{.A} = \mathsf{A.}\sqrt{\mathsf{A}^{-1}\mathsf{.B}^2\mathsf{.A}^{-1}}\mathsf{.A} = \mathsf{B.}\sqrt{\mathsf{B}^{-1}\mathsf{.A}^2\mathsf{.B}^{-1}}\mathsf{.B} \tag{52}$$

with a symmetrizing (proper orthonormal X^{-1}=X^T) rotation tensor

$$\begin{aligned} \mathsf{X} = \mathsf{X}^{-T} &= \sqrt{\mathsf{A}^{-1}\mathsf{.B}^2\mathsf{.A}^{-1}}\mathsf{.A.B}^{-1} = \mathsf{A.B}^{-1}\mathsf{.}\sqrt{\mathsf{B.A}^{-2}\mathsf{.B}} = \\ &= \sqrt{\mathsf{A.B}^{-2}\mathsf{.A}}\mathsf{.A}^{-1}\mathsf{.B} = \mathsf{A}^{-1}\mathsf{.B.}\sqrt{\mathsf{B}^{-1}\mathsf{.A}^2\mathsf{.B}^{-1}} \end{aligned} \tag{53}$$

in the middle of two positive definite, symmetric second-order (partial) stretch tensors A=A^T and B=B^T which is determined from the symmetry condition $\mathsf{A.X.B}$=$(\mathsf{A.X.B})^T$=$\mathsf{B.X}^T\mathsf{.A}$ of the whole product. A tensor product of symmetric tensors is, in general, not symmetric and has nine internal degrees of freedom, but the three internal degrees of freedom of the symmetrizing rotation tensor (53) reduce the internal degrees of freedom of the whole commutative-symmetric stretch tensor product (52) to six. The left and right multiplication of equation (52) with A^{-1} results in $\mathsf{X.B.A}^{-1}$=$\mathsf{A}^{-1}.\mathsf{B.X}^T$=$\sqrt{\mathsf{A}^{-1}.\mathsf{B}^2.\mathsf{A}^{-1}}$ where $\mathsf{X}^T.\mathsf{X}$=I is eliminated under the square root, and similarily the left and right multiplication with B^{-1} results in $\mathsf{B}^{-1}.\mathsf{A.X}$=$\mathsf{X}^T.\mathsf{A.B}^{-1}$=$\sqrt{\mathsf{B}^{-1}.\mathsf{A}^2.\mathsf{B}^{-1}}$ where $\mathsf{X.X}^T$=I is eliminated under the square root. The corresponding left and right back multiplication with A or B results in the r.h.s. terms $\mathsf{A}.\sqrt{\mathsf{A}^{-1}.\mathsf{B}^2.\mathsf{A}^{-1}}.\mathsf{A}$ or $\mathsf{B}.\sqrt{\mathsf{B}^{-1}.\mathsf{A}^2.\mathsf{B}^{-1}}.\mathsf{B}$ of equation (52), respectively.

If the principal axes of the symmetric tensors $\bar{\mathsf{A}}$ and $\bar{\mathsf{B}}$ do not coincide, the consecutive tensor products

$$\bar{\mathsf{A}}.\bar{\mathsf{B}} = \bar{\mathsf{X}}.\sqrt{\bar{\mathsf{B}}.\bar{\mathsf{A}}^2.\bar{\mathsf{B}}} = \sqrt{\bar{\mathsf{A}}.\bar{\mathsf{B}}^2.\bar{\mathsf{A}}}.\bar{\mathsf{X}} \qquad \text{and} \qquad \bar{\mathsf{B}}.\bar{\mathsf{A}} = \bar{\mathsf{X}}^T.\sqrt{\bar{\mathsf{A}}.\bar{\mathsf{B}}^2.\bar{\mathsf{A}}} = \sqrt{\bar{\mathsf{B}}.\bar{\mathsf{A}}^2.\bar{\mathsf{B}}}.\bar{\mathsf{X}}^T \tag{54}$$

are—in contrast to the tensor product (52)—not commutative and not symmetric: $\bar{\mathsf{A}}.\bar{\mathsf{B}} \neq \bar{\mathsf{B}}.\bar{\mathsf{A}}$. From the polar decomposition (1), the consecutive tensor products (54) are expressed on their r.h.s. as products of the symmetric square root tensors $\sqrt{\bar{\mathsf{A}}.\bar{\mathsf{B}}^2.\bar{\mathsf{A}}}$ or $\sqrt{\bar{\mathsf{B}}.\bar{\mathsf{A}}^2.\bar{\mathsf{B}}}$ and the rotation tensor

$$\begin{aligned}\bar{\mathsf{X}} = \bar{\mathsf{X}}^{-T} &= \sqrt{\bar{\mathsf{A}}.\bar{\mathsf{B}}^2.\bar{\mathsf{A}}}.\bar{\mathsf{A}}^{-1}.\bar{\mathsf{B}}^{-1} = \bar{\mathsf{A}}^{-1}.\bar{\mathsf{B}}^{-1}.\sqrt{\bar{\mathsf{B}}.\bar{\mathsf{A}}^2.\bar{\mathsf{B}}} = \\ &= \underbrace{\left(\bar{\mathsf{A}}.\bar{\mathsf{B}}^2.\bar{\mathsf{A}}\right)^{-\frac{1}{2}}}_{\sqrt{\bar{\mathsf{A}}^{-1}.\bar{\mathsf{B}}^{-2}.\bar{\mathsf{A}}^{-1}}}.\bar{\mathsf{A}}.\bar{\mathsf{B}} \;\; = \bar{\mathsf{A}}.\bar{\mathsf{B}}.\underbrace{\left(\bar{\mathsf{B}}.\bar{\mathsf{A}}^2.\bar{\mathsf{B}}\right)^{-\frac{1}{2}}}_{\sqrt{\bar{\mathsf{B}}^{-1}.\bar{\mathsf{A}}^{-2}.\bar{\mathsf{B}}^{-1}}}\end{aligned} \tag{55}$$

or its inverse $\bar{\mathsf{X}}^T$. The symmetric square root tensors

$$\sqrt{\bar{\mathsf{A}}.\bar{\mathsf{B}}^2.\bar{\mathsf{A}}} = \bar{\mathsf{X}}.\sqrt{\bar{\mathsf{B}}.\bar{\mathsf{A}}^2.\bar{\mathsf{B}}}.\bar{\mathsf{X}}^T = \bar{\mathsf{A}}.\bar{\mathsf{B}}.\bar{\mathsf{X}}^T = \bar{\mathsf{X}}.\bar{\mathsf{B}}.\bar{\mathsf{A}} \tag{56}$$

and

$$\sqrt{\bar{\mathsf{B}}.\bar{\mathsf{A}}^2.\bar{\mathsf{B}}} = \bar{\mathsf{X}}^T.\sqrt{\bar{\mathsf{A}}.\bar{\mathsf{B}}^2.\bar{\mathsf{A}}}.\bar{\mathsf{X}} = \bar{\mathsf{X}}^T.\bar{\mathsf{A}}.\bar{\mathsf{B}} = \bar{\mathsf{B}}.\bar{\mathsf{A}}.\bar{\mathsf{X}} \tag{57}$$

are the $\bar{\mathsf{X}}$-forward and the $\bar{\mathsf{X}}^T$-backward rotations of each other. Even if the principal axes of the symmetric tensors $\bar{\mathsf{A}}$ and $\bar{\mathsf{B}}$ do not coincide, the principal values of the symmetric square root tensors (56) and (57) are identical. Only the orientation of their principal axes differs according to the $\bar{\mathsf{X}}$ rotation tensor (55). Furthermore, the principal values of the commutative-symmetric stretch tensor product (52) differ, in general, from the principal values of the symmetric square root tensors (56) and (57).

4.1 Generalized Lagrangean and Eulerian Strain Tensor Definitions

The commutative-symmetric (partial) stretch tensor product (52) is also well defined for power functions of positive definite, symmetric stretch tensors A^q and B^q with real valued exponents q

$$\mathsf{A}^q.\sqrt{\mathsf{A}^{-q}.\mathsf{B}^{2q}.\mathsf{A}^{-q}}.\mathsf{A}^q = \mathsf{B}^q.\sqrt{\mathsf{B}^{-q}.\mathsf{A}^{2q}.\mathsf{B}^{-q}}.\mathsf{B}^q \neq \mathsf{B}^q.\left(\mathsf{B}^{-q}.\mathsf{A}^{2q}.\mathsf{B}^{-q}\right)^{q/2}.\mathsf{B}^q \tag{58}$$

and it may be applied to the definition of (multiplicatively coupled total and partial) generalized stretch and strain tensors.

In the following, just the definitions of total Lagrangean and Eulerian strain tensors are listed. The definitions of the correponding partial Lagrangean and Eulerian strain tensors must be of the same generalized type as the total ones in order to model the constitutive equations correctly. When, within a thermo-elasto-plastic constitutive model, one focuses on pure elasticity (with no plasticity and no thermal expansion), the elastic generalized strain tensor must be equal to the total generalized strain (and the plastic and thermal strains vanish); the same argument applies for pure plasticity and for pure thermal expansion. Therefore, the total and the partial generalized strain tensors should always be of the same definition type.

A very general class of Lagrangean strain tensors (Böck and Holzapfel, 2004; Darijani and Naghdabadi, 2010) is defined by

$${}^{(q\neq -r)}\mathsf{E} = \tfrac{1}{q+r}\left(\mathsf{C}^{\frac{q}{2}} - \mathsf{C}^{-\frac{r}{2}}\right) = \tfrac{1}{q+r}\left(\mathsf{U}^q - \mathsf{U}^{-r}\right) \tag{59}$$

which merges for $r\to 0$ to the classical Lagrangean generalized strain tensors (Doyle and Ericksen, 1956; Seth, 1964; Hill, 1968)

$$^{q}\mathsf{E} = \lim_{r\to 0} {}^{(q\neq -r)}\mathsf{E} \; = \tfrac{1}{q}\big(\mathsf{C}^{\frac{q}{2}} - \mathrm{I}\big) = \tfrac{1}{q}\big(\mathsf{U}^{q} - \mathrm{I}\big)\,. \tag{60}$$

The classical Lagrangean generalized strain tensors (60) merge for $q\to 0$ to the Lagrangean logarithmic strain tensor (Hencky, 1928)

$$\varepsilon = {}^{0}\mathsf{E} = \lim_{q\to 0} {}^{q}\mathsf{E} \; = \tfrac{1}{2}\ln(\mathsf{C}) = \ln(\mathsf{U})\,. \tag{61}$$

Special cases of classical Lagrangean generalized strains (60) are the engineering strain tensor for $q{=}1$

$$^{1}\mathsf{E} = \sqrt{\mathsf{C}} - \mathrm{I} = \mathsf{U} - \mathrm{I} \tag{62}$$

and Green's strain tensor (Green, 1839) for $q{=}2$

$$^{2}\mathsf{E} = \tfrac{1}{2}\big(\mathsf{C} - \mathrm{I}\big) = \tfrac{1}{2}\big(\mathsf{U}^{2} - \mathrm{I}\big)\,. \tag{63}$$

The material R-forward rotations (8) of the tensors (58)–(61) with $\mathsf{a}{=}\mathsf{R}.\mathsf{A}.\mathsf{R}^{T}$ and $\mathsf{b}{=}\mathsf{R}.\mathsf{B}.\mathsf{R}^{T}$ result in the Eulerian form of the commutative-symmetric stretch (to the q-th power) tensor product

$$\mathsf{a}^{q}.\sqrt{\mathsf{a}^{-q}.\mathsf{b}^{2q}.\mathsf{a}^{-q}}.\mathsf{a}^{q} \; = \mathsf{b}^{q}.\sqrt{\mathsf{b}^{-q}.\mathsf{a}^{2q}.\mathsf{b}^{-q}}.\mathsf{b}^{q} \tag{64}$$

and the Eulerian classes of generalized strain tensors: the very general Eulerian strain tensors

$$^{(q\neq -r)}\mathsf{e} = \mathsf{R}.{}^{(q\neq -r)}\mathsf{E}.\mathsf{R}^{T} = \tfrac{1}{q+r}\big(\mathsf{b}^{\frac{q}{2}} - \mathsf{b}^{-\frac{r}{2}}\big) = \tfrac{1}{q+r}\big(\mathsf{v}^{q} - \mathsf{v}^{-r}\big)\,, \tag{65}$$

the classical Eulerian generalized strain tensors

$$^{q}\mathsf{e} = \mathsf{R}.{}^{q}\mathsf{E}.\mathsf{R}^{T} = \tfrac{1}{q}\big(\mathsf{b}^{\frac{q}{2}} - \mathrm{I}\big) = \tfrac{1}{q}\big(\mathsf{v}^{q} - \mathrm{I}\big) \tag{66}$$

and the Eulerian logarithmic strain tensor for $q\to 0$

$$^{0}\mathsf{e} = \mathsf{R}.\varepsilon.\mathsf{R}^{T} = \lim_{q\to 0} {}^{q}\mathsf{e} \; = \tfrac{1}{2}\ln(\mathsf{b}) = \ln(\mathsf{v})\,. \tag{67}$$

A special case of a classical Eulerian generalized strain (66) is Almansi's strain tensor (Almansi, 1911) for $q{=}{-2}$

$$^{-2}\mathsf{e} = \tfrac{1}{2}\big(\mathrm{I} - \mathsf{b}^{-1}\big) = \tfrac{1}{-2}\big(\mathsf{v}^{-2} - \mathrm{I}\big)\,. \tag{68}$$

4.2 The Multiplicative Elasto-plastic Decomposition of the Deformation Gradient

The common approach to finite elasto-plasticity is the multiplicative decomposition

$$\mathsf{F} = {}^{e}\tilde{\mathsf{F}}.{}^{p}\tilde{\mathsf{F}} \tag{69}$$

of a positive definite deformation gradient F into an elastic ${}^{e}\tilde{\mathsf{F}}$ and plastic ${}^{p}\tilde{\mathsf{F}}$ contribution. In the Finite Element literature, the multiplicative decomposition (69) is often attributed to Lee and Liu (1967) or Lee (1969) even if there is work published earlier, e.g. Kröner (1960). In his book, Bertram (2012) lists more than ten other "original" references to the multiplicative decomposition (69), and Bruhns (2015) further discusses them. The polar decompositions of the partial (elastic and plastic) deformation gradients

$$^{e}\tilde{\mathsf{F}} = {}^{e}\mathsf{v}.{}^{e}\tilde{\mathsf{R}} = {}^{e}\tilde{\mathsf{R}}.{}^{e}\tilde{\mathsf{U}} \qquad \text{and} \qquad {}^{p}\tilde{\mathsf{F}} = {}^{p}\tilde{\mathsf{u}}.{}^{p}\tilde{\mathsf{R}} = {}^{p}\tilde{\mathsf{R}}.{}^{p}\mathsf{U} \tag{70}$$

define, on the one hand, the *well determined* Eulerian elastic ${}^{e}\mathsf{v}$ and Lagrangean plastic stretch ${}^{p}\mathsf{U}$ tensors and, on the other, *undetermined* elastic and plastic rotation and stretch tensors

$$^{e}\tilde{\mathsf{R}}\,, \qquad {}^{p}\tilde{\mathsf{R}}\,, \qquad {}^{e}\tilde{\mathsf{U}} = {}^{e}\tilde{\mathsf{R}}^{T}.{}^{e}\mathsf{v}.{}^{e}\tilde{\mathsf{R}} \qquad \text{and} \qquad {}^{p}\tilde{\mathsf{u}} = {}^{p}\tilde{\mathsf{R}}.{}^{p}\mathsf{U}.{}^{p}\tilde{\mathsf{R}}^{T}\,, \tag{71}$$

which are marked with superscript tildes. The indetermination of the tensors (71) originates (Casey and Naghdi, 1980) from the possibility of multiplying ${}^{+}\tilde{\mathsf{Q}}^{T}.{}^{+}\tilde{\mathsf{Q}}$=I in between the elasto-plastic deformation gradient product

$$\mathsf{F} = \underbrace{{}^{e}\tilde{\mathsf{F}}.({}^{+}\tilde{\mathsf{Q}}^{T}}_{{}^{e}\tilde{\tilde{\mathsf{F}}}}.\underbrace{{}^{+}\tilde{\mathsf{Q}}).{}^{p}\tilde{\mathsf{F}}}_{{}^{p}\tilde{\tilde{\mathsf{F}}}} = ({}^{e}\mathsf{v}.\overbrace{\underbrace{{}^{e}\tilde{\mathsf{R}}).{}^{+}\tilde{\mathsf{Q}}^{T}}_{{}^{e}\tilde{\tilde{\mathsf{R}}}}.\underbrace{{}^{+}\tilde{\mathsf{Q}}.({}^{p}\tilde{\mathsf{R}}}_{{}^{p}\tilde{\tilde{\mathsf{R}}}}}^{{}^{x}\mathsf{R}}.{}^{p}\mathsf{U}) = \mathsf{R}.\underbrace{{}^{e}\mathsf{U}.\overbrace{\mathsf{R}^{T}.{}^{x}\mathsf{R}}^{\mathsf{X}}.{}^{p}\mathsf{U}}_{\mathsf{U}} = \underbrace{{}^{e}\mathsf{v}.\overbrace{\mathsf{R}.\mathsf{X}.\mathsf{R}^{T}}^{\mathsf{x}}.{}^{p}\mathsf{v}}_{\mathsf{v}}.\mathsf{R} \tag{72}$$

such that ${}^{e}\tilde{\tilde{\mathsf{F}}}={}^{e}\tilde{\mathsf{F}}.{}^{+}\tilde{\mathsf{Q}}^{T}$, ${}^{p}\tilde{\tilde{\mathsf{F}}}={}^{+}\tilde{\mathsf{Q}}.{}^{p}\tilde{\mathsf{F}}$, ${}^{e}\tilde{\tilde{\mathsf{R}}}={}^{e}\tilde{\mathsf{R}}.{}^{+}\tilde{\mathsf{Q}}^{T}$ and ${}^{p}\tilde{\tilde{\mathsf{R}}}={}^{+}\tilde{\mathsf{Q}}.{}^{p}\tilde{\mathsf{R}}$, where ${}^{+}\tilde{\mathsf{Q}}$ is an undetermined rotation tensor. Introducing ${}^{e}\mathsf{v}.{}^{e}\tilde{\mathsf{R}}$ and ${}^{p}\tilde{\mathsf{R}}.{}^{p}\mathsf{U}$ from (70) into the multiplicative decomposition (69) results in a product of well defined tensors (exclusively without tildes) since the ${}^{+}\tilde{\mathsf{Q}}$ indetermination of ${}^{e}\tilde{\mathsf{R}}$ and ${}^{p}\tilde{\mathsf{R}}$ cancels out for the rotation tensor

$${}^{x}\mathsf{R} = {}^{e}\tilde{\mathsf{R}}.{}^{p}\tilde{\mathsf{R}} = {}^{e}\tilde{\mathsf{R}}.\underbrace{({}^{+}\tilde{\mathsf{Q}}^{T}.{}^{+}\tilde{\mathsf{Q}})}_{\mathrm{I}}.{}^{p}\tilde{\mathsf{R}} = \underbrace{{}^{e}\tilde{\mathsf{R}}.({}^{+}\tilde{\mathsf{Q}}^{T}}_{{}^{e}\tilde{\tilde{\mathsf{R}}}}.\underbrace{{}^{+}\tilde{\mathsf{Q}}).{}^{p}\tilde{\mathsf{R}}}_{{}^{p}\tilde{\tilde{\mathsf{R}}}} = {}^{e}\tilde{\tilde{\mathsf{R}}}.{}^{p}\tilde{\tilde{\mathsf{R}}} \tag{73}$$

in the middle of the Eulerian elastic ${}^{e}\mathsf{v}$ and Lagrangean plastic ${}^{p}\mathsf{U}$ stretch tensors (Sayir, 1987; Casey, 2016). Writing either ${}^{e}\mathsf{U}=\mathsf{R}^{T}.{}^{e}\mathsf{v}.\mathsf{R}$ as the material R^{T}-back rotation of ${}^{e}\mathsf{v}$ or ${}^{p}\mathsf{v}=\mathsf{R}.{}^{p}\mathsf{U}.\mathsf{R}^{T}$ as the material R-forward rotation (8) of ${}^{p}\mathsf{U}$ reveals on the one hand the symmetrizing rotation tensor

$$\begin{aligned}\mathsf{X} = \mathsf{X}^{-T} = \mathsf{R}^{T}.{}^{x}\mathsf{R} &= \sqrt{{}^{e}\mathsf{U}^{-1}.{}^{p}\mathsf{U}^{2}.{}^{e}\mathsf{U}^{-1}}.{}^{e}\mathsf{U}^{-1}.{}^{p}\mathsf{U} = {}^{e}\mathsf{U}.{}^{p}\mathsf{U}^{-1}.\sqrt{{}^{p}\mathsf{U}.{}^{e}\mathsf{U}^{-2}.{}^{p}\mathsf{U}} = \\ &= \sqrt{{}^{e}\mathsf{U}.{}^{p}\mathsf{U}^{-2}.{}^{e}\mathsf{U}}.{}^{e}\mathsf{U}.{}^{p}\mathsf{U}^{-1} = {}^{e}\mathsf{U}^{-1}.{}^{p}\mathsf{U}.\sqrt{{}^{p}\mathsf{U}^{-1}.{}^{e}\mathsf{U}^{2}.{}^{p}\mathsf{U}^{-1}}\end{aligned} \tag{74}$$

with the corresponding commutative-symmetric elasto-plastic stretch tensor product

$$\mathsf{U} = \mathsf{U}^{T} = {}^{e}\mathsf{U}.\mathsf{X}.{}^{p}\mathsf{U} = {}^{p}\mathsf{U}.\mathsf{X}^{T}.{}^{e}\mathsf{U} = {}^{e}\mathsf{U}.\sqrt{{}^{e}\mathsf{U}^{-1}.{}^{p}\mathsf{U}^{2}.{}^{e}\mathsf{U}^{-1}}.{}^{e}\mathsf{U} = {}^{p}\mathsf{U}.\sqrt{{}^{p}\mathsf{U}^{-1}.{}^{e}\mathsf{U}^{2}.{}^{p}\mathsf{U}^{-1}}.{}^{p}\mathsf{U} \tag{75}$$

in Lagrangean form or on the other the symmetrizing rotation tensor

$$\begin{aligned}\mathsf{x} = \mathsf{x}^{-T} = \mathsf{R}.\mathsf{X}.\mathsf{R}^{T} = {}^{x}\mathsf{R}.\mathsf{R}^{T} &= \sqrt{{}^{e}\mathsf{v}^{-1}.{}^{p}\mathsf{v}^{2}.{}^{e}\mathsf{v}^{-1}}.{}^{e}\mathsf{v}^{-1}.{}^{p}\mathsf{v} = {}^{e}\mathsf{v}.{}^{p}\mathsf{v}^{-1}.\sqrt{{}^{p}\mathsf{v}.{}^{e}\mathsf{v}^{-2}.{}^{p}\mathsf{v}} = \\ &= \sqrt{{}^{e}\mathsf{v}.{}^{p}\mathsf{v}^{-2}.{}^{e}\mathsf{v}}.{}^{e}\mathsf{v}.{}^{p}\mathsf{v}^{-1} = {}^{e}\mathsf{v}^{-1}.{}^{p}\mathsf{v}.\sqrt{{}^{p}\mathsf{v}^{-1}.{}^{e}\mathsf{v}^{2}.{}^{p}\mathsf{v}^{-1}}\end{aligned} \tag{76}$$

with the corresponding commutative-symmetric elasto-plastic stretch tensor product

$$\mathsf{v} = \mathsf{v}^{T} = {}^{e}\mathsf{v}.\mathsf{x}.{}^{p}\mathsf{v} = {}^{p}\mathsf{v}.\mathsf{x}^{T}.{}^{e}\mathsf{v} = {}^{e}\mathsf{v}.\sqrt{{}^{e}\mathsf{v}^{-1}.{}^{p}\mathsf{v}^{2}.{}^{e}\mathsf{v}^{-1}}.{}^{e}\mathsf{v} = {}^{p}\mathsf{v}.\sqrt{{}^{p}\mathsf{v}^{-1}.{}^{e}\mathsf{v}^{2}.{}^{p}\mathsf{v}^{-1}}.{}^{p}\mathsf{v} \tag{77}$$

in Eulerian form, cf. (52) and (53).

5 The Chaining of (Partial) Stretch Tensors and Commutative-symmetric Stretch Tensor Products

Within chained commutative-symmetric stretch tensor products, the parentheses (round brackets) mark specific products of positive definite, symmetric (partial) stretch tensors $\mathsf{A}=\mathsf{A}^{T}$, $\mathsf{B}=\mathsf{B}^{T}$ and $\mathsf{C}=\mathsf{C}^{T}$ with their corresponding symmetrizing rotation tensors $\mathsf{X}=\mathsf{X}^{-T}$ and $\mathsf{Y}=\mathsf{Y}^{-T}$ (in the middle), e.g.

$$(\mathsf{A}.\mathsf{X}.\mathsf{B}).\mathsf{Y}.\mathsf{C} = (\mathsf{B}.\mathsf{X}^{T}.\mathsf{A}).\mathsf{Y}.\mathsf{C} = \mathsf{C}.\mathsf{Y}^{T}.(\mathsf{B}.\mathsf{X}^{T}.\mathsf{A}) = \mathsf{C}.\mathsf{Y}^{T}.(\mathsf{A}.\mathsf{X}.\mathsf{B})\,. \tag{78}$$

The associative law for chained commutative-symmetric stretch tensor products

$$(\mathsf{A}.\mathsf{X}.\mathsf{B}).\mathsf{Y}.\mathsf{C} \neq \mathsf{A}.\mathsf{X}.[\mathsf{B}.\mathsf{Y}.\mathsf{C}] \tag{79}$$

is misleading since the (proper orthonormal) rotation tensors $\mathsf{X}=\mathsf{X}^{-T}$ and $\mathsf{Y}=\mathsf{Y}^{-T}$ only symmetrize the products of $(\mathsf{A}.\mathsf{X}.\mathsf{B})=(\mathsf{A}.\mathsf{X}.\mathsf{B})^{T}=(\mathsf{B}.\mathsf{X}^{T}.\mathsf{A})$ and $(\,\cdot\,).\mathsf{Y}.\mathsf{C}=\mathsf{C}.\mathsf{Y}^{T}.(\,\cdot\,)$ on the l.h.s. of inequality (79). When the principal axes of the symmetric tensors A, B and C do not coincide, the rotation tensors X and Y will not symmetrize the products of $\mathsf{A}.\mathsf{X}.[\,\cdot\,]\neq[\,\cdot\,]^{T}.\mathsf{X}^{T}.\mathsf{A}$ and $[\mathsf{B}.\mathsf{Y}.\mathsf{C}]\neq[\mathsf{B}.\mathsf{Y}.\mathsf{C}]^{T}=[\mathsf{C}.\mathsf{Y}^{T}.\mathsf{B}]$ on the r.h.s. of inequality (79) where the asymmetry of the tensor product is indicated with the square brackets.

Table 1: Thermo-elasto-plastic commutative-symmetric stretch tensor product sets of ${}^e\mathsf{U}$, ${}^p\mathsf{U}$, ${}^\vartheta\mathsf{U}$ and U

total right stretch tensor U	**"product set"**	**elastic right stretch tensor** ${}^e\mathsf{U}$
${}^e\mathsf{U}.{}^{e(p\vartheta)}\mathsf{Y}.\big(\underbrace{{}^p\mathsf{U}.{}^{p\vartheta}\mathsf{X}.{}^\vartheta\mathsf{U}}_{{}^{p\vartheta}\mathsf{U}}\big) = {}^{e(p\vartheta)}\mathsf{U}$	"$e(p\vartheta)$"	$\sqrt{\mathsf{U}.\big(\underbrace{{}^\vartheta\mathsf{U}^{-1}.\sqrt{{}^\vartheta\mathsf{U}.{}^p\mathsf{U}^{-2}.{}^\vartheta\mathsf{U}}.{}^\vartheta\mathsf{U}^{-1}}_{{}^p\mathsf{U}^{-1}.\sqrt{{}^p\mathsf{U}.{}^\vartheta\mathsf{U}^{-2}.{}^p\mathsf{U}}.{}^p\mathsf{U}^{-1}}\big)^2.\mathsf{U}}$
${}^p\mathsf{U}.{}^{p(\vartheta e)}\mathsf{Y}.\big(\underbrace{{}^\vartheta\mathsf{U}.{}^{\vartheta e}\mathsf{X}.{}^e\mathsf{U}}_{{}^{\vartheta e}\mathsf{U}}\big) = {}^{p(\vartheta e)}\mathsf{U}$	"$p(\vartheta e)$"	$\sqrt{\sqrt{\mathsf{U}.{}^p\mathsf{U}^{-2}.\mathsf{U}}.{}^\vartheta\mathsf{U}^{-2}.\sqrt{\mathsf{U}.{}^p\mathsf{U}^{-2}.\mathsf{U}}}$
${}^\vartheta\mathsf{U}.{}^{\vartheta(ep)}\mathsf{Y}.\big(\underbrace{{}^e\mathsf{U}.{}^{ep}\mathsf{X}.{}^p\mathsf{U}}_{{}^{ep}\mathsf{U}}\big) = {}^{\vartheta(ep)}\mathsf{U}$	"$\vartheta(ep)$"	$\sqrt{\sqrt{\mathsf{U}.{}^\vartheta\mathsf{U}^{-2}.\mathsf{U}}.{}^p\mathsf{U}^{-2}.\sqrt{\mathsf{U}.{}^\vartheta\mathsf{U}^{-2}.\mathsf{U}}}$
${}^{ep}\mathsf{U}.{}^\vartheta\mathsf{U}.{}^{t/\vartheta}\mathsf{Y}^T = {}^{t/\vartheta}\mathsf{Y}.{}^\vartheta\mathsf{U}.{}^{ep}\mathsf{U} =$ $= \sqrt{{}^{ep}\mathsf{U}.{}^\vartheta\mathsf{U}^2.{}^{ep}\mathsf{U}} = {}^{t/\vartheta}\mathsf{U}$	"t/ϑ"	$\sqrt{\underbrace{\mathsf{U}.\big(\mathsf{U}.{}^\vartheta\mathsf{U}^2.\mathsf{U}\big)^{-\frac{1}{2}}.\mathsf{U}}_{{}^\vartheta\mathsf{U}^{-1}.\sqrt{{}^\vartheta\mathsf{U}.\mathsf{U}^2.{}^\vartheta\mathsf{U}}.{}^\vartheta\mathsf{U}^{-1}}.{}^p\mathsf{U}^{-2}.\underbrace{\mathsf{U}.\big(\mathsf{U}.{}^\vartheta\mathsf{U}^2.\mathsf{U}\big)^{-\frac{1}{2}}.\mathsf{U}}_{{}^\vartheta\mathsf{U}^{-1}.\sqrt{{}^\vartheta\mathsf{U}.\mathsf{U}^2.{}^\vartheta\mathsf{U}}.{}^\vartheta\mathsf{U}^{-1}}}$

When the principal axes of the symmetric partial stretch tensors ${}^\vartheta\mathsf{U}$, ${}^e\mathsf{U}$ and ${}^p\mathsf{U}$ differ, there are three different sets of chained thermo-elasto-plastic commutative-symmetric stretch tensor products "$e(p\vartheta)$", "$p(\vartheta e)$" and "$\vartheta(ep)$" which are compiled in Table 1. The column on the left of Table 1 shows the total right stretch tensor U for each product set and the column on the right the corresponding elastic right stretch tensor ${}^e\mathsf{U}$. The fourth product set "t/ϑ" of Table 1 arises from

$$\underbrace{{}^e\mathsf{U}.{}^{ep}\mathsf{X}.{}^p\mathsf{U}}_{{}^{ep}\mathsf{U}} = \underbrace{\mathsf{U}.{}^{t/\vartheta}\mathsf{Y}.{}^\vartheta\mathsf{U}^{-1}}_{{}^{t/\vartheta}\mathsf{U}} = {}^\vartheta\mathsf{U}^{-1}.{}^{t/\vartheta}\mathsf{Y}^T.\mathsf{U} = {}^\vartheta\mathsf{U}^{-1}.\sqrt{{}^\vartheta\mathsf{U}.\mathsf{U}^2.{}^\vartheta\mathsf{U}}.{}^\vartheta\mathsf{U}^{-1} = \mathsf{U}.\big(\mathsf{U}.{}^\vartheta\mathsf{U}^2.\mathsf{U}\big)^{-\frac{1}{2}}.\mathsf{U}\,. \tag{80}$$

5.1 The Unique Thermo-elasto-plastic Decomposition of Stretch Tensors for Isotropic Thermal Expansion

A good portion of materials and especially metals exhibit isotropic thermal expansion behaviour, such that the thermal stretch may be specified as an isotropic / spherical tensor

$${}^\vartheta\mathsf{U} = {}^\vartheta\mathsf{v} = \Theta(\vartheta)\cdot\mathrm{I} = \frac{{}^\vartheta\ell}{L}\cdot\mathrm{I} \tag{81}$$

with a scalar thermal expansion factor $\Theta(\vartheta){=}\frac{{}^\vartheta\ell}{L}$ multiplied by the second-order unit / identity tensor I. This factor $\Theta(\vartheta)$ is a function of the current temperature ϑ, and becomes $\Theta(\vartheta_0){=}1$ unity at the reference temperature ϑ_0 (which implies ${}^\vartheta\ell{=}L$). For isotropic thermal expansion (81), the differences between the four thermo-elasto-plastic commutative-symmetric stretch tensor products of Table 1 vanish, and they merge into one unique commutative-symmetric stretch tensor product in which the total right stretch tensor is given by

$$\mathsf{U} = \Theta(\vartheta)\cdot\overbrace{\underbrace{{}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.{}^e\mathsf{U}^2.{}^p\mathsf{U}^{-1}}.{}^p\mathsf{U}}_{{}^e\mathsf{U}.\sqrt{{}^e\mathsf{U}^{-1}.{}^p\mathsf{U}^2.{}^e\mathsf{U}^{-1}}.{}^e\mathsf{U}}}^{{}^{ep}\mathsf{U}} = {}^{e(p\vartheta)}\mathsf{U} = {}^{p(\vartheta e)}\mathsf{U} = {}^{\vartheta(ep)}\mathsf{U} = {}^{t/\vartheta}\mathsf{U}\,, \tag{82}$$

the total left stretch tensor by

$$\mathsf{v} = \Theta(\vartheta)\cdot{}^p\mathsf{v}.\sqrt{{}^p\mathsf{v}^{-1}.{}^e\mathsf{v}^2.{}^p\mathsf{v}^{-1}}.{}^p\mathsf{v} = \Theta(\vartheta)\cdot{}^e\mathsf{v}.\sqrt{{}^e\mathsf{v}^{-1}.{}^p\mathsf{v}^2.{}^e\mathsf{v}^{-1}}.{}^e\mathsf{v}\,, \tag{83}$$

the elastic right and left stretch tensors by

$${}^e\mathsf{U} = \frac{\sqrt{\mathsf{U}.{}^p\mathsf{U}^{-2}.\mathsf{U}}}{\Theta(\vartheta)} = \frac{\sqrt[3]{|\mathsf{F}|}}{\Theta(\vartheta)}\cdot\sqrt{\mathsf{U}^*.{}^p\mathsf{U}^{-2}.\mathsf{U}^*} \qquad \text{and} \qquad {}^e\mathsf{v} = \frac{\sqrt{\mathsf{v}.{}^p\mathsf{v}^{-2}.\mathsf{v}}}{\Theta(\vartheta)} = \frac{\sqrt[3]{|\mathsf{F}|}}{\Theta(\vartheta)}\cdot\sqrt{\mathsf{v}^*.{}^p\mathsf{v}^{-2}.\mathsf{v}^*} \tag{84}$$

and the corresponding unique multiplicative polar decomposition by

$$\mathsf{F} = \mathsf{R}.\overbrace{\underbrace{{}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.{}^e\mathsf{U}^2.{}^p\mathsf{U}^{-1}}.{}^p\mathsf{U}}_{{}^e\mathsf{U}.\sqrt{{}^e\mathsf{U}^{-1}.{}^p\mathsf{U}^2.{}^e\mathsf{U}^{-1}}.{}^e\mathsf{U}} \cdot \Theta(\vartheta)}^{\mathsf{U}} = \underbrace{\Theta(\vartheta) \cdot \overbrace{{}^p\mathsf{v}.\sqrt{{}^p\mathsf{v}^{-1}.{}^e\mathsf{v}^2.{}^p\mathsf{v}^{-1}}.{}^p\mathsf{v}}^{{}^e\mathsf{v}.\sqrt{{}^e\mathsf{v}^{-1}.{}^p\mathsf{v}^2.{}^e\mathsf{v}^{-1}}.{}^e\mathsf{v}}}_{\mathsf{v}}.\mathsf{R}\,. \tag{85}$$

The *unimodular stretch tensors* (marked with a "star")

$$\mathsf{U}^* = \frac{\mathsf{U}}{\sqrt[3]{|\mathsf{U}|}} = \frac{\mathsf{U}}{\sqrt[3]{|\mathsf{F}|}} \qquad \text{and} \qquad \mathsf{v}^* = \frac{\mathsf{v}}{\sqrt[3]{|\mathsf{v}|}} = \frac{\mathsf{v}}{\sqrt[3]{|\mathsf{F}|}} \tag{86}$$

within the equations of (84) are *measures of pure finite distortion* (Flory, 1961) since determinants of unimodular tensors are unity $|\mathsf{U}^*|=|\mathsf{v}^*|=1$ by definition. For plastic incompressibility, all plastic stretch tensors ${}^p\mathsf{U}$ and ${}^p\mathsf{v}$ are unimodular $|{}^p\mathsf{U}|=|{}^p\mathsf{v}|=1$ and thus not marked with a "star" within this work. Applying the Lagrangean unimodular stretch tensors U^* (86), ${}^e\mathsf{U}^*$ and ${}^p\mathsf{U}$, the *multiplicative decoupling of finite dilatation and finite distortion* may be written as

$$\mathsf{U} = \sqrt[3]{|\mathsf{F}|} \cdot \mathsf{U}^* = \underbrace{\sqrt[3]{|{}^e\mathsf{U}|} \cdot \Theta(\vartheta)}_{\sqrt[3]{|\mathsf{F}|}} \cdot \underbrace{\overbrace{{}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.({}^e\mathsf{U}^*)^2.{}^p\mathsf{U}^{-1}}.{}^p\mathsf{U}}^{{}^e\mathsf{U}^*.\sqrt{({}^e\mathsf{U}^*)^{-1}.{}^p\mathsf{U}^2.({}^e\mathsf{U}^*)^{-1}}.{}^e\mathsf{U}^*}}_{\mathsf{U}^*} \tag{87}$$

with the finite dilatation factor

$$\sqrt[3]{|\mathsf{F}|} = \sqrt[3]{|{}^e\mathsf{U}|} \cdot \Theta(\vartheta) \tag{88}$$

of the isotropic dilatation tensor (which does not depend on the isovolumetric tensor ${}^p\mathsf{U}$) and with the unimodular total stretch tensor equal to the unimodular commutative-symmetric elasto-plastic stretch tensor product

$$\mathsf{U}^* = {}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.({}^e\mathsf{U}^*)^2.{}^p\mathsf{U}^{-1}}.{}^p\mathsf{U} = {}^e\mathsf{U}^*.\sqrt{({}^e\mathsf{U}^*)^{-1}.{}^p\mathsf{U}^2.({}^e\mathsf{U}^*)^{-1}}.{}^e\mathsf{U}^* = {}^p\mathsf{U}.\mathsf{X}^T.{}^e\mathsf{U}^* = {}^e\mathsf{U}^*.\mathsf{X}.{}^p\mathsf{U} \tag{89}$$

(which does not depend on the volumetric factor $\Theta(\vartheta)$ of finite dilatation).

5.2 Logarithmic (Hencky) Strain Tensors

The logarithm of $c \cdot \mathsf{A}$, a symmetric tensor A multiplied by a scalar c, is given by

$$\ln(c \cdot \mathsf{A}) = \ln(c \cdot \hat{A}_k)\, \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k = \ln(c) \cdot \underbrace{\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k}_{\mathrm{I}} + \ln(\hat{A}_k)\, \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k = \ln(c) \cdot \mathrm{I} + \ln(\mathsf{A}) \tag{90}$$

the sum of the isotropic tensor $\ln(c) \cdot \mathrm{I}$ plus the $\ln(\mathsf{A})$ tensor. For the logarithmic (Hencky) strain tensor definitions (61) and (67), the multiplicative decoupling of finite dilatation and finite distortion (87) becomes thus *additive*

$$\varepsilon = \ln(\mathsf{U}) = \underbrace{\frac{\ln(|\mathsf{F}|)}{3}}_{\frac{\varepsilon_{kk}}{3}} \cdot \mathrm{I} + \underbrace{\ln(\mathsf{U}^*)}_{\varepsilon'} = \underbrace{\frac{\ln(|{}^e\mathsf{U}|)}{3}}_{\frac{{}^e\varepsilon_{kk}}{3}} \cdot \mathrm{I} + \underbrace{\ln\big(\Theta(\vartheta)\big) \cdot \mathrm{I}}_{{}^\vartheta\varepsilon} + \underbrace{\overbrace{\ln({}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.({}^e\mathsf{U}^*)^2.{}^p\mathsf{U}^{-1}}.{}^p\mathsf{U})}^{{}^e\mathsf{U}^*.\sqrt{({}^e\mathsf{U}^*)^{-1}.{}^p\mathsf{U}^2.({}^e\mathsf{U}^*)^{-1}}.{}^e\mathsf{U}^*}}_{\ln(\mathsf{U}^*)}\,. \tag{91}$$

The isotropic thermal expansion (81) is given by the logarithmic thermal strain

$${}^\vartheta\varepsilon = \frac{{}^\vartheta\varepsilon_{kk}}{3} \cdot \mathrm{I} = \ln({}^\vartheta\mathsf{U}) = \ln\big(\Theta(\vartheta)\big) \cdot \mathrm{I} \tag{92}$$

which is an isotropic / spherical tensor. The finite dilatation is governed by the isotropic / spherical part of the logarithmic strain tensor

$$\varepsilon'' = \underbrace{\frac{\ln(|\mathsf{F}|)}{3}}_{\frac{\varepsilon_{kk}}{3}} \cdot \mathrm{I} = \ln\big(\sqrt[3]{|{}^e\mathsf{U}|} \cdot \Theta(\vartheta)\big) \cdot \mathrm{I} = \Big(\underbrace{\frac{\ln(|{}^e\mathsf{U}|)}{3}}_{\frac{{}^e\varepsilon_{kk}}{3}} + \underbrace{\ln\big(\Theta(\vartheta)\big)}_{\frac{{}^\vartheta\varepsilon_{kk}}{3}}\Big) \cdot \mathrm{I} \tag{93}$$

which trace obeys

$$\varepsilon_{kk} = \ln\big(|\mathsf{F}|\big) = {}^e\varepsilon_{kk} + {}^\vartheta\varepsilon_{kk} = \ln\big(|{}^e\mathsf{U}| \cdot \Theta^3(\vartheta)\big) = \ln(|{}^e\mathsf{U}|) + 3 \cdot \ln\big(\Theta(\vartheta)\big)\,. \tag{94}$$

The isovolumetric finite distortion is governed by the deviatoric part of the logarithmic strain tensor

$$\varepsilon' = \ln\big(\mathsf{U}^*\big) = \ln\big({}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.({}^e\mathsf{U}^*)^2.{}^p\mathsf{U}^{-1}}.{}^p\mathsf{U}\big) = \ln\big({}^e\mathsf{U}^*.\sqrt{({}^e\mathsf{U}^*)^{-1}.{}^p\mathsf{U}^2.({}^e\mathsf{U}^*)^{-1}}.{}^e\mathsf{U}^*\big)\,. \tag{95}$$

For the numerical modelling of constitutive equations, the elastic strain tensor is required as a function of total, plastic and thermal strains in order to formulate stress-strain (tensor) relations properly. Solving the equations (93)–(95) for the logarithmic elastic strain tensor results in its trace given by

$${}^e\varepsilon_{kk} = \ln\big(|{}^e\mathsf{U}|\big) = \varepsilon_{kk} - {}^\vartheta\varepsilon_{kk} = \ln\Big(\frac{|\mathsf{F}|}{\Theta^3(\vartheta)}\Big) = \ln(|\mathsf{F}|) - 3 \cdot \ln\Big(\Theta(\vartheta)\Big) \tag{96}$$

and its deviator given by

$${}^e\varepsilon' = \ln({}^e\mathsf{U}^*) = \tfrac{1}{2}\ln(\mathsf{U}^*.{}^p\mathsf{U}^{-2}.\mathsf{U}^*)\,. \tag{97}$$

5.3 Time Derivatives of Deformation Tensors for Finite Elasto-plasticity with Thermal Expansion Isotropy

The material time derivative of a symmetric second-order Lagrangean tensor $\mathsf{U}{=}\hat{U}_k\,\hat{\mathbf{E}}_k{\otimes}\hat{\mathbf{E}}_k$ is most simply expressed in its spectral representation as

$$\dot{\mathsf{U}} = \overset{\bullet}{\overline{\hat{U}_k\,\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k}} = \dot{\hat{U}}_k\,\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \hat{U}_k\,\dot{\hat{\mathbf{E}}}_k \otimes \hat{\mathbf{E}}_k + \hat{U}_k\,\hat{\mathbf{E}}_k \otimes \dot{\hat{\mathbf{E}}}_k = \dot{\hat{U}}_k\,\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \Lambda.\mathsf{U} - \mathsf{U}.\Lambda \tag{98}$$

the time derivatives $\dot{\hat{U}}_k$ of its principal values $\hat{U}_k$ (the matrix elements on the diagonal within a principal co-ordinate system) and the time derivatives $\dot{\hat{\mathbf{E}}}_k{=}\Lambda.\hat{\mathbf{E}}_k{=}{-}\hat{\mathbf{E}}_k.\Lambda$ of its rotating Lagrangean principal base vectors $\hat{\mathbf{E}}_k$ (the matrix off-diagonal elements within a principal co-ordinate system) where the second-order tensor $\Lambda{=}{-}\Lambda^T$ of the principal base vectors' rotation velocities is antisymmetric. Therefore, the components of Λ obey

$$\hat{\Lambda}_{ij} = -\hat{\Lambda}_{ji} = -\epsilon_{ijk}\,\hat{\lambda}_k \qquad\text{and}\qquad \hat{\lambda}_i = -\tfrac{1}{2}\epsilon_{ijk}\,\hat{\Lambda}_{jk} = \tfrac{1}{2}\epsilon_{ijk}\,\hat{\Lambda}_{kj} \tag{99}$$

and the matrix components of $\dot{\mathsf{U}}$ (98) within a principal co-ordinate system read

$$[\dot{\mathsf{U}}] = \begin{bmatrix} \dot{\hat{U}}_1 & \hat{\lambda}_3\big(\hat{U}_1 - \hat{U}_2\big) & \hat{\lambda}_2\big(\hat{U}_3 - \hat{U}_1\big) \\ \hat{\lambda}_3\big(\hat{U}_1 - \hat{U}_2\big) & \dot{\hat{U}}_2 & \hat{\lambda}_1\big(\hat{U}_2 - \hat{U}_3\big) \\ \hat{\lambda}_2\big(\hat{U}_3 - \hat{U}_1\big) & \hat{\lambda}_1\big(\hat{U}_2 - \hat{U}_3\big) & \dot{\hat{U}}_3 \end{bmatrix} \tag{100}$$

In an analogous manner, the logarithmic strain-rate tensor is given by the material time derivative

$$\dot{\varepsilon} = \overset{\bullet}{\overline{\ln(\mathsf{U})}} = \overset{\bullet}{\overline{\ln(\hat{U}_k)}}\,\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \Lambda.\ln(\mathsf{U}) - \ln(\mathsf{U}).\Lambda = \frac{\dot{\hat{U}}_k}{\hat{U}_k}\,\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \Lambda.\varepsilon - \varepsilon.\Lambda \tag{101}$$

of the logarithmic strain ε which may, due to the additive decoupling of isotropic finite dilatation and isovolumetric finite distortion (91), be split into its spherical part: primarily the trace of the logarithmic strain-rate tensor

$$\dot{\varepsilon}_{kk} = \overline{\ln(|\mathsf{F}|)}^{\bullet} = \frac{|\mathsf{F}|^{\bullet}}{|\mathsf{F}|} = \frac{\dot{\hat{U}}_k}{\hat{U}_k} = \frac{\dot{\hat{U}}_1}{\hat{U}_1} + \frac{\dot{\hat{U}}_2}{\hat{U}_2} + \frac{\dot{\hat{U}}_3}{\hat{U}_3} \tag{102}$$

and into its deviatoric part: the material time derivative of the logarithm of the unimodular stretch tensor

$$\dot{\varepsilon}' = \overline{\ln(\mathsf{U}^*)}^{\bullet} = \overline{\ln(\hat{U}_k^*)}^{\bullet}\, \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \Lambda.\ln(\mathsf{U}^*) - \ln(\mathsf{U}^*).\Lambda = \frac{\dot{\hat{U}}_k^*}{\hat{U}_k^*}\, \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \Lambda.\varepsilon' - \varepsilon'.\Lambda \tag{103}$$

where the $\hat{U}_k^* {=} \frac{\hat{U}_k}{\sqrt[3]{\hat{U}_1\,\hat{U}_2\,\hat{U}_3}}$ denote the principal values of the unimodular stretch tensor U^* (86). The isotropic / spherical part of the logarithmic elastic strain-rate tensor

$$^{e}\dot{\varepsilon} = \overline{\ln(^{e}\mathsf{U})}^{\bullet} = \frac{^{e}\dot{\hat{U}}_i}{^{e}\hat{U}_i}\, {}^{e}\hat{\mathbf{E}}_i \otimes {}^{e}\hat{\mathbf{E}}_i + {}^{e}\Lambda.\ln(^{e}\mathsf{U}) - \ln(^{e}\mathsf{U}).{}^{e}\Lambda = \frac{^{e}\dot{\hat{U}}_i}{^{e}\hat{U}_i}\, {}^{e}\hat{\mathbf{E}}_i \otimes {}^{e}\hat{\mathbf{E}}_i + {}^{e}\Lambda.{}^{e}\varepsilon - {}^{e}\varepsilon.{}^{e}\Lambda \tag{104}$$

is given by its trace

$$^{e}\dot{\varepsilon}_{ii} = \overline{\ln(|^{e}\mathsf{U}|)}^{\bullet} = \frac{|^{e}\mathsf{U}|^{\bullet}}{|^{e}\mathsf{U}|} = \frac{^{e}\dot{\hat{U}}_i}{^{e}\hat{U}_i} = \frac{^{e}\dot{\hat{U}}_1}{^{e}\hat{U}_1} + \frac{^{e}\dot{\hat{U}}_2}{^{e}\hat{U}_2} + \frac{^{e}\dot{\hat{U}}_3}{^{e}\hat{U}_3} \tag{105}$$

and its isovolumetric part by the logarithmic elastic strain-rate deviator

$$^{\mathsf{e}}\dot{\varepsilon}' = \overline{\ln(^{e}\mathsf{U}^*)}^{\bullet} = \frac{^{e}\dot{\hat{U}}_i^*}{^{e}\hat{U}_i^*}\, {}^{e}\hat{\mathbf{E}}_i \otimes {}^{e}\hat{\mathbf{E}}_i + {}^{e}\Lambda.\underbrace{\ln(^{e}\mathsf{U}^*)}_{^{e}\varepsilon'} - \underbrace{\ln(^{e}\mathsf{U}^*)}_{^{e}\varepsilon'}.{}^{e}\Lambda\,. \tag{106}$$

For plastic incompressibility, the logarithmic plastic strain-rate tensor is deviatoric by definition

$$^{p}\dot{\varepsilon} = \overline{\ln(^{p}\mathsf{U})}^{\bullet} = \frac{^{p}\dot{\hat{U}}_j}{^{p}\hat{U}_j}\, {}^{p}\hat{\mathbf{E}}_j \otimes {}^{p}\hat{\mathbf{E}}_j + {}^{p}\Lambda.\underbrace{\ln(^{p}\mathsf{U})}_{^{p}\varepsilon} - \underbrace{\ln(^{p}\mathsf{U})}_{^{p}\varepsilon}.{}^{p}\Lambda = {}^{\mathsf{p}}\dot{\varepsilon}' \tag{107}$$

since its trace, i.e. its isotropic / spherical part, vanishes

$$^{p}\dot{\varepsilon}_{jj} = \overline{\ln(|^{p}\mathsf{U}|)}^{\bullet} = \frac{|^{p}\mathsf{U}|^{\bullet}}{|^{p}\mathsf{U}|} = \frac{^{p}\dot{\hat{U}}_j}{^{p}\hat{U}_j} = \frac{^{p}\dot{\hat{U}}_1}{^{p}\hat{U}_1} + \frac{^{p}\dot{\hat{U}}_2}{^{p}\hat{U}_2} + \frac{^{p}\dot{\hat{U}}_3}{^{p}\hat{U}_3} = 0\,. \tag{108}$$

And for isotropic thermal expansion (92), the logarithmic thermal strain-rate tensor

$$^{\vartheta}\dot{\varepsilon} = \overline{\ln(^{\vartheta}\mathsf{U})}^{\bullet} = \overline{\ln\big(\Theta(\vartheta)\big)}^{\bullet} \cdot \mathrm{I} = \frac{d\Theta(\vartheta)}{d\vartheta}\, \dot{\vartheta} \cdot \mathrm{I} \tag{109}$$

is purely isotropic / spherical so that the logarithmic thermal strain-rate deviator $^{\vartheta}\dot{\varepsilon}'{=}0$ vanishes.

The material time derivative of the unimodular commutative-symmetric elasto-plastic stretch tensor product (89) follows from the product rule and with (74) as

$$
\begin{aligned}
\dot{\mathsf{U}}^* = {} & \dot{\hat{U}}_k^* \, \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \Lambda.\mathsf{U}^* - \mathsf{U}^*.\Lambda = \dot{\overline{{}^p\mathsf{U}.\mathsf{X}^T.{}^e\mathsf{U}^*}} = {}^p\dot{\mathsf{U}}.\mathsf{X}^T.{}^e\mathsf{U}^* + {}^p\mathsf{U}.\dot{\mathsf{X}}^T.{}^e\mathsf{U}^* + {}^p\mathsf{U}.\mathsf{X}^T.{}^e\dot{\mathsf{U}}^* = \\
& = \dot{\overline{{}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.({}^e\mathsf{U}^*)^2.{}^p\mathsf{U}^{-1}}.{}^p\mathsf{U}}} = {}^p\dot{\mathsf{U}}.\sqrt{{}^p\mathsf{U}^{-1}.({}^e\mathsf{U}^*)^2.{}^p\mathsf{U}^{-1}}.{}^p\mathsf{U} + \\
& \quad + {}^p\mathsf{U}.\dot{\overline{\sqrt{{}^p\mathsf{U}^{-1}.({}^e\mathsf{U}^*)^2.{}^p\mathsf{U}^{-1}}}}.{}^p\mathsf{U} + {}^p\mathsf{U}.\sqrt{{}^p\mathsf{U}^{-1}.({}^e\mathsf{U}^*)^2.{}^p\mathsf{U}^{-1}}.{}^p\dot{\mathsf{U}} = \\
& = \dot{\overline{{}^e\mathsf{U}^*.\mathsf{X}.{}^p\mathsf{U}}} = {}^e\dot{\mathsf{U}}^*.\mathsf{X}.{}^p\mathsf{U} + {}^e\mathsf{U}^*.\dot{\mathsf{X}}.{}^p\mathsf{U} + {}^e\mathsf{U}^*.\mathsf{X}.{}^p\dot{\mathsf{U}} = \\
& = \dot{\overline{{}^e\mathsf{U}^*.\sqrt{({}^e\mathsf{U}^*)^{-1}.{}^p\mathsf{U}^2.({}^e\mathsf{U}^*)^{-1}}.{}^e\mathsf{U}^*}} = {}^e\dot{\mathsf{U}}^*.\sqrt{({}^e\mathsf{U}^*)^{-1}.{}^p\mathsf{U}^2.({}^e\mathsf{U}^*)^{-1}}.{}^e\mathsf{U}^* + \\
& \quad + {}^e\mathsf{U}^*.\dot{\overline{\sqrt{({}^e\mathsf{U}^*)^{-1}.{}^p\mathsf{U}^2.({}^e\mathsf{U}^*)^{-1}}}}.{}^e\mathsf{U}^* + {}^e\mathsf{U}^*.\sqrt{({}^e\mathsf{U}^*)^{-1}.{}^p\mathsf{U}^2.({}^e\mathsf{U}^*)^{-1}}.{}^e\dot{\mathsf{U}}^* = \\
& = {}^e\dot{\mathsf{U}}^*.{}^ef^*({}^e\mathsf{U},{}^p\mathsf{U})^T + {}^ef^*({}^e\mathsf{U},{}^p\mathsf{U}).{}^e\dot{\mathsf{U}}^* + {}^p\dot{\mathsf{U}}.{}^pf^*({}^e\mathsf{U},{}^p\mathsf{U})^T + {}^pf^*({}^e\mathsf{U},{}^p\mathsf{U}).{}^p\dot{\mathsf{U}}
\end{aligned}
\tag{110}
$$

two sums of unimodular elastic ${}^e\dot{\mathsf{U}}^*$ or plastic ${}^p\dot{\mathsf{U}}$ stretch tensor rates symmetrically multiplied by corresponding unimodular elastic ${}^ef^*({}^e\mathsf{U},{}^p\mathsf{U})$ or plastic ${}^pf^*({}^e\mathsf{U},{}^p\mathsf{U})$ tensor functions of the partial stretch tensors ${}^e\mathsf{U}$ and ${}^p\mathsf{U}$.

In an analogous manner, the logarithmic strain-rate deviator is given as

$$
\begin{aligned}
\dot{\varepsilon}' = {} & \frac{\dot{\hat{U}}_k^*}{\hat{U}_k^*} \, \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k + \Lambda.\ln(\mathsf{U}^*) - \ln(\mathsf{U}^*).\Lambda = \\
= {} & \underbrace{\frac{{}^e\dot{\hat{U}}_i^*}{{}^e\hat{U}_i^*} \, {}^e\hat{\mathbf{E}}_i \otimes {}^e\hat{\mathbf{E}}_i + {}^e\Lambda.\ln({}^e\mathsf{U}^*) - \ln({}^e\mathsf{U}^*).{}^e\Lambda}_{{}^e\dot{\varepsilon}'.{}^e\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)^T + {}^e\mathsf{f}({}^e\varepsilon,{}^p\varepsilon).{}^e\dot{\varepsilon}'} + \underbrace{\frac{{}^p\dot{\hat{U}}_j}{{}^p\hat{U}_j} \, {}^p\hat{\mathbf{E}}_j \otimes {}^p\hat{\mathbf{E}}_j + {}^p\Lambda.\ln({}^p\mathsf{U}) - \ln({}^p\mathsf{U}).{}^p\Lambda}_{{}^p\dot{\varepsilon}.{}^p\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)^T + {}^p\mathsf{f}({}^e\varepsilon,{}^p\varepsilon).{}^p\dot{\varepsilon}}
\end{aligned}
\tag{111}
$$

two sums of elastic ${}^e\dot{\varepsilon}'$ or plastic ${}^p\dot{\varepsilon}$ logarithmic strain-rate deviators symmetrically multiplied by corresponding elastic ${}^e\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)$ or plastic ${}^p\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)$ tensor functions of the logarithmic partial strain tensors ${}^e\varepsilon$ and ${}^p\varepsilon$.

The elastic ${}^e\hat{\mathbf{E}}_i \otimes {}^e\hat{\mathbf{E}}_i$ and plastic ${}^p\hat{\mathbf{E}}_j \otimes {}^p\hat{\mathbf{E}}_j$ principal bases of the elastic and plastic (logarithmic) strain tensors differ in general such that all the principal values and the principal base rotation velocities of total and partial strain tensors are interdependent and coupled. The (material) time derivatives of (Lagrangean) commutative-symmetric strain tensor products with symmetrizing rotation tensors (in the middle) result from (109), (111) and the product rule in summands of partial (logarithmic) strain-rate tensors

$$
\begin{aligned}
\dot{\varepsilon} = {} & \tfrac{\dot{\varepsilon}_{\ell\ell}}{3} \cdot \mathrm{I} + \dot{\varepsilon}' = \\
= {} & \left(\tfrac{{}^e\dot{\varepsilon}_{\ell\ell}}{3} + \tfrac{d\Theta(\vartheta)}{d\vartheta}\,\dot{\vartheta}\right) \cdot \mathrm{I} + {}^e\dot{\varepsilon}'.{}^e\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)^T + {}^e\mathsf{f}({}^e\varepsilon,{}^p\varepsilon).{}^e\dot{\varepsilon}' + {}^p\dot{\varepsilon}.{}^p\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)^T + {}^p\mathsf{f}({}^e\varepsilon,{}^p\varepsilon).{}^p\dot{\varepsilon}
\end{aligned}
\tag{112}
$$

and, therefore, provide a link to the additive decomposition of total D and partial $\{{}^e\mathsf{D}, {}^p\mathsf{D}, {}^\vartheta\mathsf{D}\}$ rate of deformation tensors—as implied by the first fundamental law of thermodynamics: the total stress power

$$
\begin{aligned}
\tfrac{1}{\rho}\mathsf{T}..\mathsf{D} = {} & \tfrac{T_{kk}}{\rho}\tfrac{D_{\ell\ell}}{3} + \tfrac{1}{\rho}\mathsf{T}..\mathsf{D}' = \tfrac{T_{kk}}{\rho}\left(\tfrac{{}^eD_{\ell\ell}}{3} + \tfrac{d\Theta(\vartheta)}{d\vartheta}\,\dot{\vartheta}\right) + \tfrac{1}{\rho}\left({}^e\mathsf{T}..{}^e\mathsf{D}' + {}^p\mathsf{T}..{}^p\mathsf{D}\right) = \\
& = \tfrac{1}{\rho_0}\sigma..\dot{\varepsilon} = \tfrac{\sigma_{kk}}{\rho_0}\tfrac{\dot{\varepsilon}_{\ell\ell}}{3} + \tfrac{1}{\rho_0}\sigma..\dot{\varepsilon}' = \tfrac{\sigma_{kk}}{\rho_0}\left(\tfrac{{}^e\dot{\varepsilon}_{\ell\ell}}{3} + \tfrac{d\Theta(\vartheta)}{d\vartheta}\,\dot{\vartheta}\right) + \\
& \qquad + \tfrac{1}{\rho_0}\Big(\underbrace{{}^e\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)^T.\sigma + \sigma.{}^e\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)}_{{}^e\sigma}\Big)..{}^e\dot{\varepsilon}' + \tfrac{1}{\rho_0}\Big(\underbrace{{}^p\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)^T.\sigma + \sigma.{}^p\mathsf{f}({}^e\varepsilon,{}^p\varepsilon)}_{{}^p\sigma}\Big)..{}^p\dot{\varepsilon}\,,
\end{aligned}
\tag{113}
$$

basically the double scalar product $\mathsf{T}..\mathsf{D}{=}T_{ij}D_{ij}$ of the symmetric second-order tensors T (Cauchy stress) and D (rate of deformation), is given by the sum of the partial (elastic, plastic, thermal, ...) stress power terms, where ρ and ρ_0 denote the mass densities in the current κ and the reference κ_0 configurations, σ the symmetric second-order tensor of the stress (Hoger, 1987) which is work-conjugate to the logarithmic strain ε (rate $\dot{\varepsilon}$), where the deviators of second-order tensors $(\cdot)$ are denoted with a prime mark $(\cdot)'$ and where the traces of the strain rates $D_{\ell\ell}{=}\dot{\varepsilon}_{\ell\ell}$ are identical and where the traces of stresses obey $\frac{1}{\rho}T_{kk}{=}\frac{1}{\rho_0}\sigma_{kk}$ for the total as well as for the partial (elastic, plastic, thermal, ...) tensors.

6 Conclusion

In summary, for the constitutive modelling of finite deformations, total and partial strain tensors must be

- of the same generalized type ${}^{(q\neq-r)}\mathsf{E}$ (59)—as outlined in Subsection 4.1—since
  for pure elasticity, the elastic and the total generalized strains are identical: ${}^{e}\mathsf{E}={}^{(q\neq-r)}\mathsf{E}$;
  for pure plasticity, the plastic and the total generalized strains are identical: ${}^{p}\mathsf{E}={}^{(q\neq-r)}\mathsf{E}$;
  and for pure thermal expansion, the thermal and the total generalized strains are identical: ${}^{\vartheta}\mathsf{E}={}^{(q\neq-r)}\mathsf{E}$
- geometrical interpretable (and independent on the geometrical deformation path):
  $\oint\limits_{\mathcal{O}} d@_{ij}=0_{ij}$ for $@=\{E,{}^{e}E,{}^{p}E,{}^{\vartheta}E\}$—as outlined at the bottom of Section 2
- defined with respect to the *same* reference configuration κ_0 (the one with the orthogonal fibers)—as outlined at the beginning of Section 3

In conjunction with the commutative-symmetric stretch tensor product, the suggestions presented herein provide a new constitutive approach to the formulation of elasto-plasticity with isotropic thermal expansion under finite deformations where the Cauchy-Green elastic deformation tensors are given by

$$ {}^{e}\mathsf{C}=\frac{1}{\Theta^2(\vartheta)}\cdot\mathsf{U}.{}^{p}\mathsf{C}^{-1}.\mathsf{U} \qquad \text{and} \qquad {}^{e}\mathsf{b}=\frac{1}{\Theta^2(\vartheta)}\cdot\mathsf{F}.{}^{p}\mathsf{C}^{-1}.\mathsf{F}^{T} \tag{114} $$

and the Lagrangean logarithmic elastic strain tensor by

$$ {}^{e}\varepsilon=\ln({}^{e}\mathsf{U})=\tfrac{1}{2}\ln({}^{e}\mathsf{C})=\tfrac{1}{2}\ln(\mathsf{U}.{}^{p}\mathsf{C}^{-1}.\mathsf{U})-\ln\big(\Theta(\vartheta)\big)\cdot\mathrm{I} \tag{115} $$

with ${}^{p}\mathsf{C}^{-1}={}^{p}\mathsf{U}^{-2}$ denoting the inverse of the Lagrangean Cauchy-Green plastic deformation tensor. On the one hand, the presented constitutive equations are suitable for a wide range of metals used for industrial applications where the elastic strain tensor is an order of magnitude smaller than the second-order unit / identity tensor ${}^{e}\varepsilon\ll\mathrm{I}$ and, on the other, they are also applicable for the mechanical modelling of e.g. (human) tissue where the elastic tensor of logarithmic strain ${}^{e}\varepsilon$ may become large.

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