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Torsional Wave Frequency in Corrugated Poroelastic Layer Bonded Between Anisotropic Media

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An analytical solution is obtained for torsional surface wave propagating in pre-stressed poroelastic layer sandwiched between transversely isotropic and pre-stressed viscoelastic half-spaces with non-planer boundaries. Frequency equation has been established in closed form. Separation of variables method is adopted to obtain the dispersion relation. Substantial influence of corrugated boundary, initial stresses, porosity and inhomogeneity on the phase velocity of considered surface wave has been presented through graphs. Proposed study finds its possible applications in the fields of geophysics and seismology. Results may be extended to interpret the seismic activities near the crust-mantle boundary.

1 Introduction

The stress which is present in an elastic medium in the absence of external force is called initial stress and the medium is said to be initially stressed. The earth is an initially stressed medium possessing different layers where initial stresses exist because of dissimilarity in temperature, weight, overburden of layer, slow process of creep, gravitation, etc. These stresses have noteworthy effect on the propagation pattern of harmonic waves exhibited by earthquakes or explosions. Knowledge of propagation behavior of different seismic waves are of great significance due to their practical importance in many engineering branches including rock mechanics and Geophysical prospecting. Several authors applied the theory given by Biot (1965) for studying the propagation behavior of surface wave in initially stressed medium. Dey and Sarkar (2002) observed the effect of initial stress in porous medium when a torsional wave propagates through it. Torsional surface wave is horizontally polarized in nature and twirls the medium when it propagates through it. It is one kind of elastic wave that causes only circumferential displacement of particles which is independent of azimuthal angle. Under the influence of these waves, the medium particles twist either clockwise or anticlockwise as per the direction of motion of the waves. Some kind of surface waves are torsional in nature. The torsional surface wave exhibits peculiar characteristics due to having non-dispersive nature which unravels that the shape of the wave unaltered during progression of torsional surface wave. The propagation of torsional surface wave in different types of elastic medium received considered attention of many researchers and scientists from many decades. The phenomena of torsional surface wave in anisotropic and heterogeneous half-space subjected to initial compressive stress was studied by Chattopadhyay *et al.* (2013). Georgiadis *et al.* (2000) analyzed that the propagation of torsional surface wave also exist in linear gradient elastic half-space. Vardoulakis (1984) discussed about the dispersion of torsional surface wave in two different types of inhomogeneous elastic media. Singh and Laxman (2016) have investigated the propagation of torsional wave in a corrugated doubly layered half-space. Recently, Paswan *et al.* (2017) studied the propagation of torsional surface wave in heterogeneous fibre-reinforced layer lying over heterogeneous half-space.

Pores and solid matrix together forms the porous medium. The pores of these medium are generally occupied with liquid or gas. Porous layers are naturally found beneath the Earth's surface. Almost every naturally occurring solid material is considered to be the example of porous media. Several authors dealt with the propagation of wave in porous media including Sharma and Gogna (1991), Gardner (1962). Gupta and Gupta (2011) examined the phenomena of torsional surface wave propagation in anisotropic poroelastic half-space under the influence of gravity. The investigation of torsional surface wave in anisotropic poroelastic layer lying over inhomogeneous half-space has been carried out by Chattaraj *et al.* (2011). The interface between different substrates of the Earth are not always uniformly arranged as mountains, basins, mountain roots etc. also exist. The irregular boundaries at the interface may be of different geometrical shapes (like rectangular, triangular, parabolic, corrugated boundary, etc.). The problems with corrugated boundaries have much significance in seismology due to their closeness to the natural situation, as they leads to better comprehension and interpretation of seismic behavior at continental margins, mountain roots, etc. Propagation of seismic waves in the medium having uneven or corrugated boundaries was discussed by Chattopadhyay and De (1983), Tomar and Kaur (2007). The elastic materials with vertical axis of symmetry in which elastic properties of the materials change in vertical direction but not in a plane perpendicular to that axis are said to be transversely isotropic. These types of materials shows hexagonal symmetry. Few examples of transversely isotropic material are the crystals of hexagonal symmetry such as Cadmium, Cobalt, Beryl and Zinc etc. Wang and Zhang (1998) have considered the propagation of Love waves in an inhomogeneous porous layer lying over an inhomogeneous half-space. Coal tar, salt, sediments, etc., are some of the examples of viscoelastic material found underneath the earth surface. Kaur *et al.* (2005) studied the behavior of shear wave propagation in heterogeneous viscoelastic half-space with corrugated interface. Horizontally polarized shear wave (SH-wave) propagation in viscoelastic medium with irregularity is studied by Chattopadhyay *et al.* (2010). Kumari and Sharma (2014) investigated the propagation characteristics of torsional surface wave in a viscoelastic layer overlying an inhomogeneous elastic half-space.

This paper attempts to study the torsional waves behavior in pre-stressed poroelastic layer embraced between a transversely isotropic half-space and a pre-stressed viscoelastic half-space, with corrugated boundaries. The considered sandwich structure is much closer to the asthenosphere that forms the transition zone between the lower crust and mantle. Our main thrust is to study the wave characterization in considered structure theoretically which supports the selection of solution method to be analytic, in particular separation of variables method is adopted for the solution. The frequency equation has been obtained analytically in closed form and the noteworthy effect of the flatness parameter, initial stress, inhomogeneity parameter and porosity parameter on the phase velocity of torsional surface wave has been depicted with the help of graphs.

2 Mathematical Formulation and Solution

To model the present problem, we have considered cylindrical coordinate system in such a fashion that r-axis is along the direction of torsional wave propagation and z-axis is considered positive in vertically downward direction. As shown in Fig. 1, $M' : \varphi_1(r) - H \leq z \leq \varphi_2(r)$ represents the pre-stressed porous layer with corrugated boundaries sandwiched between a transversely isotropic half-space $M : \varphi_1(r) - H \leq z < -\infty$ and a pre-stressed viscoelastic half-space $M'' : \varphi_2(r) \leq z < \infty$; where $\varphi_1(r)$ and $\varphi_2(r)$ are continuous functions of r, representing the corrugated boundaries of porous layer; H is the average thickness of porous layer.

As functions $\varphi_l(r)$ are periodic in nature, their Fourier series expansions may be given by (Asano, 1966)

$$\varphi_l(r) = \sum_{n=1}^{\infty} \left(\varphi_n^l e^{inbr} + \varphi_{-n}^l e^{-inbr}\right), \qquad l = 1, 2. \tag{1}$$

where, φ_n^l and φ_{-n}^l are the coefficients of Fourier series expansion, $2\pi/b$ is wavelength of corrugation, n is the order of series expansion.

when the Fourier series expansion coefficients are given by

$$\varphi_{\pm n}^l = \begin{cases} \frac{\xi_1}{2} & \text{if } n = 1, l = 1 \\ \frac{\xi_2}{2} & \text{if } n = 1, l = 2 \\ 0 & \text{if } n \neq 1 \end{cases}$$

Eq. (1) leads to the expressions for corrugated common interfaces in the concerned problem in cosine term i.e. $\varphi_1(r) = \xi_1 \cos(br)$, $\varphi_2(r) = \xi_2 \cos(br)$; where ξ_1 and ξ_2 are the corresponding amplitudes of corrugation.

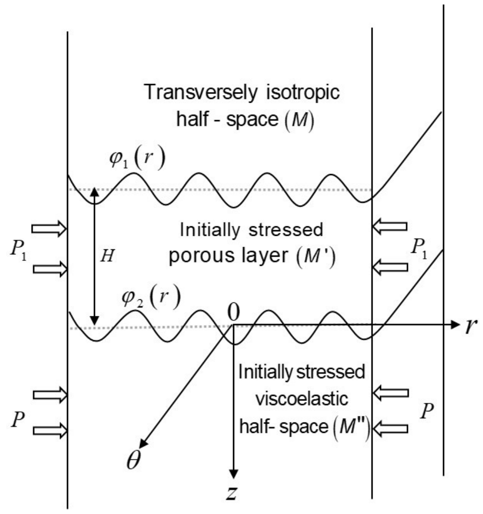

Figure 1: Schematic diagram of the problem

2.1 Formulation and Solution for Transversely Isotropic Half-Space *(M)*

The equation of motion for transversely isotropic half-space is taken from Vishwakarma *et al.* (2013) as

$$\begin{aligned}
&\frac{\partial \tau_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\tau_{rr}-\tau_{\theta\theta}}{r}={\rho'}_1\frac{\partial^2 u_1'}{\partial t^2},\\
&\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}={\rho'}_1\frac{\partial^2 v_1'}{\partial t^2},\\
&\frac{\partial \tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}+\frac{\tau_{rz}}{r}={\rho'}_1\frac{\partial^2 w_1'}{\partial t^2},
\end{aligned} \tag{2}$$

where, $\tau_{ij}(i,j=r,\theta,z)$ are components of stress; u_1', v_1', w_1' are displacement components in radial (r), azimuthal (θ), and axial (z) direction respectively; t is time and ρ_1' is density of uppermost transversely isotropic half-space.

For torsional wave propagation along r- direction and causing displacement in θ- direction only, we have

$$u_1'=0, w_1'=0, v_1'=v_1'(r,z,t),\ \frac{\partial}{\partial \theta}\equiv 0. \tag{3}$$

The constitutive stress-strain relations for isotropic medium are given by

$$\begin{aligned}
&\tau_{rr}=\lambda_1\Delta+2\mu_1 e_{rr}, \tau_{\theta\theta}=\lambda_1\Delta+2\mu_1 e_{\theta\theta},\ \tau_{zz}=\lambda_1\Delta+2\mu_1 e_{zz}, \tau_{r\theta}=2\mu_1 e_{r\theta},\\
&\tau_{rz}=2\mu_1 e_{rz},\ \tau_{\theta z}=2\mu_1 e_{\theta z},
\end{aligned} \tag{4}$$

where, λ_1, μ_1 are Lames constants and $\Delta=\frac{\partial u_1'}{\partial r}+\frac{1}{r}\frac{\partial v_1'}{\partial \theta}+\frac{u_1'}{r}+\frac{\partial w_1'}{\partial z}$ is dilatation.

Strain and displacement components are related as

$$\begin{aligned}
&e_{rr}=\frac{\partial u_1'}{\partial r},\ e_{\theta\theta}=\frac{1}{r}\frac{\partial v_1'}{\partial \theta}+\frac{u_1'}{r},\ e_{zz}=\frac{\partial w_1'}{\partial z},\ e_{r\theta}=\frac{1}{r}\frac{\partial u_1'}{\partial \theta}+\frac{\partial v_1'}{\partial r}-\frac{v_1'}{r},\ e_{\theta z}=\frac{\partial v_1'}{\partial z}+\frac{1}{r}\frac{\partial w_1'}{\partial \theta},\\
&e_{zr}=\frac{\partial w_1'}{\partial r}+\frac{\partial u_1'}{\partial z}.
\end{aligned} \tag{5}$$

In view of Eqs. (2 - 5), non-vanishing equation of motion and stress components are

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{\partial \tau_{z\theta}}{\partial z}+\frac{2}{r}\tau_{r\theta}=\rho_1'\frac{\partial^2 v_1'}{\partial t^2}. \tag{6}$$

$$\tau_{r\theta}=\mu_L\left(\frac{\partial {v'}_1}{\partial r}-\frac{{v'}_1}{r}\right),\ \tau_{z\theta}=\mu_T\frac{\partial {v'}_1}{\partial z}. \tag{7}$$

where, μ_L and μ_T are directional rigidities along radial (r) and axial (z) directions respectively.

Eqs. (6) and (7) leads to

$$\mu_L\left(\frac{\partial^2 v_1'}{\partial r^2}-\frac{v_1'}{r^2}+\frac{1}{r}\frac{\partial v_1'}{\partial z}\right)+\frac{\partial}{\partial z}\left(\mu_T\frac{\partial v_1'}{\partial z}\right)=\rho_1'\frac{\partial^2 v_1'}{\partial t^2}. \tag{8}$$

We consider inhomogeneity in half-space by considering density (ρ_1') and directional rigidities (μ_L,μ_T) as function of z given by

$$\mu_L=\mu_{L0}e^{\frac{z}{\alpha}},\ \mu_T=\mu_{T0}e^{\frac{z}{\alpha}},\ \rho_1'=\rho_{10}'e^{\frac{z}{\alpha}}. \tag{9}$$

where, α is a real valued positive parameter, having dimension same that of length and is termed as inhomogeneity parameter.

For a harmonic wave propagation along r- direction, we may assume the solution of Eq. (8) as

$$v_1'=V_1'(z)J_1(k\,r)e^{i\omega t} \tag{10}$$

Where, J_1 is Bessel's function of first order and first kind, k is wave number, ω is frequency.

Using Eq. (10), in Eq. (8) gives

$$\frac{d^2V_1'}{dz^2}+\frac{1}{\mu_T}\frac{d\mu_T}{dz}\frac{dV_1'}{dz}-\frac{k^2\mu_L}{\mu_T}\left(1-\frac{c^2{\rho'}_1}{\mu_L}\right)V_1'=0. \tag{11}$$

On substituting $V_1'=\frac{V_1''}{\sqrt{\mu_T}}$ and using Eq. (9) in Eq. (11) leads to

$$\frac{d^2V_1''}{dz^2}-n_1^2V_1''=0, \tag{12}$$

Where, $n_1^2=k^2\left[\frac{1}{4\alpha^2k^2}+\frac{\mu_{L0}}{\mu_{T0}}\left(1-\frac{c^2}{c_0^2}\right)\right],\ \ c_0=\sqrt{\frac{\mu_{L0}}{\rho_{10}'}}.$

The solution of Eq. (12) is given by $V_1''\ (z)=B_1e^{n_1z}+B_1'e^{-n_1z}$, where B_1 and B_1' are arbitrary constants.

Using the fact that displacement component must vanish as $z\to-\infty$ which requires $\lim\limits_{z\to-\infty}{V_1}''(z)=0$, thus the appropriate solution of Eq. (12) is

$${V_1}''(z)=B_1e^{n_1z}. \tag{13}$$

Hence the component of displacement in the uppermost transversely isotropic half-space (M) is obtained as

$$v_1'=B_1e^{\left(n_1-\ \frac{1}{2\alpha}\right)z}\frac{J_1(kr)}{\sqrt{\mu_{T0}}}e^{i\omega t}. \tag{14}$$

2.2 Formulation and Solution for Initially Stressed Porous Layer *(M′)*

Dynamical equations of motion for the porous layer under the influence of initial stress (P_1) in absence of body forces and fluid viscosity are given by following Biot (1965) as

$$\begin{aligned}
&\frac{\partial s_{rr}}{\partial r}+\frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta}+\frac{\partial s_{rz}}{\partial z}+\frac{s_{rr}-s_{\theta\theta}}{r}-P_1\left(\frac{\partial \omega_z}{\partial \theta}-\frac{\partial \omega_\theta}{\partial z}\right)=\frac{\partial^2}{\partial t^2}\left(d_{rr}u_2'+d_{r\theta}U_1\right),\\
&\frac{\partial s_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta}+\frac{\partial s_{\theta z}}{\partial z}+\frac{2}{r}s_{r\theta}-P_1\frac{\partial \omega_z}{\partial r}=\frac{\partial^2}{\partial t^2}\left(d_{rr}v_2'+d_{r\theta}V_1\right),\\
&\frac{\partial s_{rz}}{\partial r}+\frac{1}{r}\frac{\partial s_{\theta z}}{\partial \theta}+\frac{\partial s_{zz}}{\partial z}+\frac{s_{rz}}{r}-P_1\frac{\partial \omega_\theta}{\partial r}=\frac{\partial^2}{\partial t^2}\left(d_{rr}w_2'+d_{r\theta}W_1\right),\\
&\frac{\partial s}{\partial r}=\frac{\partial^2}{\partial t^2}\left(d_{r\theta}u_2'+d_{\theta\theta}U_1\right),\ \frac{\partial s}{\partial \theta}=\frac{\partial^2}{\partial t^2}\left(d_{r\theta}v_2'+d_{\theta\theta}V_1\right),\ \frac{\partial s}{\partial z}=\frac{\partial^2}{\partial t^2}\left(d_{r\theta}w_2'+d_{\theta\theta}W_1\right),
\end{aligned} \tag{15}$$

Where, $s_{ij},\ (i,j=r,\theta,z)$ are components of stress acting on solid part and s is the stress acting on fluid part of porous material; (u_2',v_2',w_2') and (U_1,V_1,W_1) are the displacement components for solid and fluid part respectively; ω_j is rotational component; P_1 denote the initial stress acting on porous layer saturated with fluid along radial direction.

The stress acting on fluid can be defined in terms of fluid pressure p through the relation as

$$-s=fp, \tag{16}$$

where, f is layer porosity.

The mass coefficients $d_{rr},\ d_{r\theta}$ and $d_{\theta\theta}$ are related to the mass densities ρ_2',d_s,d_l of layer, solid and fluid respectively by

$$d_{rr}+d_{r\theta}=(1-f)\,d_s,\ d_{r\theta}+d_{\theta\theta}=fd_l,\ \rho_2'=d_s+f\left(d_l-d_s\right). \tag{17}$$

Biot (1956(I), 1956(II)) has also shown that the said mass coefficients satisfy the following inequalities

$$d_{rr}>0,\ d_{\theta\theta}>0,\ d_{r\theta}<0,\ d_{rr}d_{\theta\theta}-d_{r\theta}^2>0\,. \tag{18}$$

The constitutive stress - strain relationship for porous layer are as

$$\begin{aligned}
&s_{rr}=\left(A_1+P_1\right)e_{rr}+\left(A_1-2N'+P_1\right)e_{\theta\theta}+\left(F_1+P_1\right)e_{zz}+Q_1\varepsilon,\\
&s_{\theta\theta}=\left(A_1-2N'\right)e_{rr}+A_1e_{\theta\theta}+F_1e_{zz}+Q_1\varepsilon,\\
&s_{zz}=F_1e_{rr}+F_1e_{\theta\theta}+C_1e_{zz}+Q_1\varepsilon,\\
&s_{r\theta}=2N'e_{r\theta}\,,\ s_{\theta z}=2L'e_{\theta z}\,,\ s_{rz}=2L'e_{zr}\,,\varepsilon=\frac{\partial U_1}{\partial r}+\frac{\partial V_1}{\partial \theta}+\frac{\partial W_1}{\partial z},
\end{aligned} \tag{19}$$

where A_1,F_1,C_1 are elastic constants of the medium; Q_1 is a positive quantity which represents the extent of coupling between the change of volume of the solid and fluid; N',L' are rigidities of porous medium along r and z directions respectively.

Rotational components are given by

$$\omega_r=\frac{1}{2r}\left(\frac{\partial w_2'}{\partial \theta}-r\frac{\partial v_2'}{dz}\right),\ \omega_\theta=\frac{1}{2}\left(\frac{\partial u_2'}{\partial z}-\frac{\partial w_2'}{dr}\right),\ \omega_z=\frac{1}{2r}\left(\frac{\partial\left(rv_2'\right)}{\partial r}-\frac{\partial u_2'}{d\theta}\right). \tag{20}$$

Strain and displacement components are related as

$$\begin{aligned}
&e_{rr}=\frac{1}{2}\frac{\partial u_2'}{\partial r},\ e_{zz}=\frac{1}{2}\frac{\partial w_2'}{\partial z},\ e_{\theta\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial v_2'}{\partial \theta}+\frac{u_2'}{r}\right),e_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_2'}{\partial \theta}+\frac{\partial v_2'}{\partial r}-\frac{v_2'}{r}\right),\\
&e_{\theta z}=\frac{1}{2}\left(\frac{\partial v_2'}{\partial z}+\frac{1}{r}\frac{\partial w_2'}{\partial \theta}\right)\ e_{rz}=\frac{1}{2}\left(\frac{\partial w_2'}{\partial r}+\frac{\partial u_2'}{\partial z}\right).
\end{aligned} \tag{21}$$

The displacement components for the considered wave whose direction of propagation is along r- axis is given by

$$u'_2 = 0,\ v'_2 = v'_2(r,z,t),\ w'_2 = 0,$$
$$U_1 = 0,\ V_1 = V_1(r,z,t),\ W_1 = 0. \tag{22}$$

Using Eq. (22) in Eqs. (20) and (21), the non-vanishing strain and rotational components are

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial v'_2}{\partial r} - \frac{v'_2}{r}\right),\ e_{\theta z} = \frac{1}{2}\left(\frac{\partial v'_2}{\partial z}\right) \tag{23}$$
$$\omega_r = -\frac{1}{2}\frac{\partial v'_2}{dz}, \omega_z = \frac{1}{2r}\frac{\partial}{\partial r}\left(r v'_2\right). \tag{24}$$

Using Eqs. (23) and (22) in Eq. (19), the non-vanishing stress components are

$$s_{r\theta} = N'\left(\frac{\partial v'_2}{\partial r} - \frac{v'_2}{r}\right),\ s_{\theta z} = L'\left(\frac{\partial v'_2}{\partial z}\right). \tag{25}$$

Using Eqs. (22), (24) and (25) in Eq. (15), we get

$$\left(N' - \frac{P_1}{2}\right)\left(\frac{\partial^2 v'_2}{\partial r^2} + \frac{1}{r}\frac{\partial v'_2}{\partial r} - \frac{v'_2}{r^2}\right) + L'\frac{\partial^2 v'_2}{\partial z^2} = \frac{\partial^2}{\partial t^2}\left(d_{rr}v'_2 + d_{r\theta}V_1\right), \tag{26}$$
$$\frac{\partial^2}{\partial t^2}\left(d_{r\theta}v'_2 + d_{\theta\theta}V_1\right) = 0. \tag{27}$$

Eliminating V_1 from Eqs. (26) and (27), we obtain

$$\left(N' - \frac{P_1}{2}\right)\left(\frac{\partial^2 v'_2}{\partial r^2} + \frac{1}{r}\frac{\partial v'_2}{\partial r} - \frac{v'_2}{r^2}\right) + L'\frac{\partial^2 v'_2}{\partial z^2} = T_2\frac{\partial^2 v'_2}{\partial t^2}. \tag{28}$$

where, $T_2 = d_{rr} - \frac{d_{r\theta}^2}{d_{\theta\theta}}$.

We assume the solution of Eq. (28) as

$$v'_2 = V'_2(z)J_1(kr)e^{i\omega t}. \tag{29}$$

Using Eq. (29) in Eq. (28), we have

$$\frac{d^2 V'_2}{dz^2} + \lambda_2^2 V'_2 = 0, \tag{30}$$

where, $\lambda_2^2 = \frac{k^2 N'\left(1-\frac{P_1}{2N'}\right)}{L'}\left\{\frac{T}{\left(1-\frac{P_1}{2N'}\right)}\frac{c^2}{\beta_2^2} - 1\right\},\ T = \frac{T_2}{\rho'_2},\ \beta_2^2 = \frac{N'}{\rho'_2}$.

The solution of Eq. (30) is given by

$$V'_2 = \left(B_2 \sin\lambda_2 z + B'_2\cos\lambda_2 z\right), \tag{31}$$

where B_2 and B'_2 are arbitrary constants.

Thus the component of displacement in poroelastic layer is

$$v'_2 = \left(B_2\sin\lambda_2 z + B'_2\cos\lambda_2 z\right)J_1(kr)e^{i\omega t}. \tag{32}$$

2.3 Formulation and Solution for Pre-stressed Viscoelastic Half-Space *(M″)*

Let us consider (u_3', v_3', w_3') are components of displacement along r, θ, and z directions respectively for lower viscoelastic half-space influenced by initial stress. For the propagation of torsional surface wave we have

$$u_3' = w_3' = 0,\ v_3' = v_3'(r,z,t),\ \frac{\partial}{\partial\theta} \equiv 0. \tag{33}$$

In the pre-stressed viscoelastic half-space the only existing equation of motion is given by (Biot 1965)

$$\left(\mu_v + \mu_v'\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v_3'}{\partial r^2} + \frac{\partial^2 v_3'}{\partial z^2} + \frac{1}{r}\frac{\partial v_3'}{\partial r} - \frac{v_3'}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{P}{2}\frac{\partial v_3'}{\partial z}\right) = \rho_3'\frac{\partial^2 v_3'}{\partial t^2}. \tag{34}$$

where, μ_v is modulus of rigidity; μ_v' is parameter of internal friction as a result of viscoelasticity; ρ_3' is the density of viscoelastic half-space; P is initial stress on viscoelastic half-space acting along radial direction.

The stress-displacement equations for existing components of stress in the viscoelastic half space is given by

$$\sigma_{\theta z} = \left(\mu_v + \mu_v'\frac{\partial}{\partial t}\right)\frac{\partial v_3'}{\partial z},\ \ \sigma_{r\theta} = \left(\mu_v + \mu_v'\frac{\partial}{\partial t}\right)\left(\frac{\partial v_3'}{\partial r} - \frac{v_3'}{r}\right). \tag{35}$$

We consider the solution of Eq. (34) as

$$v_3' = V_3'(z)J_1(kr)e^{i\omega t}. \tag{36}$$

Using Eq. (36), Eq. (34) leads to

$$\frac{d^2V_3'}{dz^2} - \lambda_3^2\, V_3' = 0, \tag{37}$$

where, $\lambda_3^2 = k^2\left\{\frac{1-\frac{c^2}{\beta_3^2\left(1+\frac{i\omega\mu_v'}{\mu_v}\right)}}{1-\frac{P}{2\mu_v\left(1+\frac{i\omega\mu_v'}{\mu_v}\right)}}\right\}$, $\beta_3^2 = \frac{\mu_v}{\rho_3'}$.

On solving Eq. (37), we obtain

$$V_3'(z) = B_3e^{-\lambda_3 z} + B_3'e^{\lambda_3 z}, \tag{38}$$

where, B_3 and B_3' are arbitrary constants.

Using the fact that $\lim_{z\to\infty} V_3'(z) = 0$, the appropriate solution of Eq. (37) is $V_3'(z) = B_3e^{-\lambda_3 z}$.

Therefore, the components of displacement in the lowermost initially stressed viscoelastic half-space is given by

$$v_3' = B_3\ J_1(kr)\ e^{i\omega t-\lambda_3 z}. \tag{39}$$

3 Boundary Conditions and Dispersion Equation

Continuity of displacements and stresses gives the boundary conditions for the present problem

- At the common corrugated interface of layer (M') and upper half-space (M) i.e. at $z = \varphi_1(r) - H$

$$v_1' = v_2', \tag{40}$$
$$\tau_{z\theta} - \dot\varphi_1(r)\tau_{r\theta} = s_{z\theta} - \dot\varphi_1(r)s_{r\theta}. \tag{41}$$

- At the common corrugated interface of layer (M') and lower half-space (M'') i.e. at $z = \varphi_2(r)$

$$v_2' = v_3', \tag{42}$$
$$s_{z\theta} - \dot\varphi_2(r)s_{r\theta} = \sigma_{z\theta} - \dot\varphi_2(r)\sigma_{r\theta}. \tag{43}$$

Where, $\dot{\varphi}_i$ represents the derivative of φ_i with respect to r.

Using displacement components v_1', v_2', v_3' from Eqs. (14), (32), (39) and stress components from Eqs. (7), (25), (35) in the boundary conditions (40 - 43), we get

$$B_1 e^{\left(n_1-\frac{1}{2\alpha}\right)(\varphi_1-H)} - B_2\sqrt{\mu_{T0}}\ \sin\left(\lambda_2\left(\varphi_1-H\right)\right) - B_2'\sqrt{\mu_{T0}}\ \cos\left(\lambda_2\left(\varphi_1-H\right)\right) = 0, \tag{44}$$

$$\begin{aligned}&B_1 e^{\left(n_1+\frac{1}{2\alpha}\right)(\varphi_1-H)}\left\{\sqrt{\mu_{T0}}\left(n_1-\frac{1}{2\alpha}\right)-\dot{\varphi}_1\frac{\mu_{L0}k}{\sqrt{\mu_{T0}}}J\right\}\\ &-B_2\left\{L'\lambda_2\ \cos\left(\lambda_2\left(\varphi_1-H\right)\right)-\dot{\varphi}_1 N'k\,J\ \sin\left(\lambda_2\left(\varphi_1-H\right)\right)\right\}\\ &+B_2'\left\{L'\lambda_2\ \sin\left(\lambda_2\left(\varphi_1-H\right)\right)+\dot{\varphi}_1 N'k\,J\ \cos\left(\lambda_2\left(\varphi_1-H\right)\right)\right\}=0,\end{aligned} \tag{45}$$

$$B_2\ \sin\left(\lambda_2\varphi_2\right)+B_2'\cos\left(\lambda_2\varphi_2\right)-B_3e^{-\lambda_3\varphi_2}=0, \tag{46}$$

$$\begin{aligned}&B_2\left\{L'\lambda_2\ \cos\left(\lambda_2\varphi_2\right)-\dot{\varphi}_2N'k\,J\ \sin\left(\lambda_2\varphi_2\right)\right\}-B'_2\left\{L'\lambda_2\ \sin\left(\lambda_2\varphi_2\right)+\dot{\varphi}_2N'k\,J\ \cos\left(\lambda_2\varphi_2\right)\right\}\\ &+B_3e^{-\lambda_3\varphi_2}\left(\mu_v+i\mu_v'\omega\right)\left(\lambda_3+\dot{\varphi}_2k\,J\right)=0.\end{aligned} \tag{47}$$

where, $J=\frac{\dot{J}_1(kr)}{J_1(kr)}-\frac{1}{kr}$, $\dot{J}_1$ represents derivative of J_1 with respect to r.

Eliminating arbitrary constants B_1, B_2, B_3 and B_2' from Eqs. (44 - 47), we find the complex velocity equation as

$$\begin{aligned}&2L'\lambda_2e^{\left(n_1-\frac{1}{2\alpha}\right)(\varphi_1-H)}\alpha+\left(Q_1+2e^{\left(n_1-\frac{1}{2\alpha}\right)(\varphi_1-H)}\alpha kJN'\dot{\varphi}_1\right)\tan\left[\lambda_2\left(H-\varphi_1+\varphi_2\right)\right]=\\ &\frac{Q_2-i\,Q_3}{Q_2^2+Q_3^2}\left\{Q_4\tan\left[\lambda_2\left(H-\varphi_1+\varphi_2\right)\right]-L'\lambda_2\left(Q_1+2e^{\left(n_1-\frac{1}{2\alpha}\right)(\varphi_1-H)}\alpha kJN'\left(\dot{\varphi}_1-\dot{\varphi}_2\right)\right)\right\}.\end{aligned} \tag{48}$$

where,

$$\begin{aligned}&Q_1=2e^{\left(n_1+\frac{1}{2\alpha}\right)(\varphi_1-H)}\left(2n_1\alpha\mu_{T0}-\mu_{T0}-2\alpha kJ\mu_{L0}\dot{\varphi}_1\right),\ \ Q_2=A_1+\mu_vJk\dot{\varphi}_2,\\ &Q_3=A_2+\mu_v'Jk\omega\dot{\varphi}_2,\ \ Q_4=Q_1kJN'\dot{\varphi}_2+2e^{\left(n_1-\frac{1}{2\alpha}\right)(\varphi_1-H)}\alpha\left(L'^2\lambda_2^2+k^2J^2N'^2\dot{\varphi}_1\dot{\varphi}_2\right),\\ &A_1=\frac{k}{\sqrt{2}}\left(\sqrt{\gamma_1^2+\gamma_2^2}+\gamma_1\right)^{1/2},\quad A_2=\frac{k}{\sqrt{2}}\left(\sqrt{\gamma_1^2+\gamma_2^2}-\gamma_1\right)^{1/2}\\ &\gamma_1=D_1D_3-D_2D_4,\ \ \gamma_2=D_2D_3+D_1D_4,\ \ D_1={\mu_v}^2-\omega^2{\mu_v'}^2,\ \ D_2=2\omega\mu_v\mu_v',\\ &D_3=\frac{\left(2\mu_v\beta_3^2-2\mu c^2\right)\left(2\mu_v\beta_3^2-P\beta_3^2\right)+\left(2\omega\mu_v'\beta_3^2\right)^2}{\left(2\mu_v\beta_3^2-P\beta_3^2\right)^2+\left(2\omega\mu_v'\beta_3^2\right)^2},\ \ D_4=\frac{\left(2\mu_vc^2-P\beta_3^2\right)\left(2\omega\mu_v'\beta_3^2\right)^2}{\left(2\mu_v\beta_3^2-P\beta_3^2\right)^2+\left(2\omega\mu_v'\beta_3^2\right)^2}.\end{aligned}$$

The real part of Eq. (48) gives the dispersion equation and the imaginary part gives damping equation associated with torsional surface wave propagation. In the present study we are concern about only the dispersive nature of considered surface wave and hence we consider only the dispersion equation for characterization of considered wave.

Equating the real parts of Eq. (48) we get the following relation

$$\begin{aligned}&2L'\lambda_2e^{\left(n_1-\frac{1}{2\alpha}\right)(\varphi_1-H)}\alpha+\left(Q_1+2e^{\left(n_1-\frac{1}{2\alpha}\right)(\varphi_1-H)}\alpha kJN'\dot{\varphi}_1\right)\tan\left[\lambda_2\left(H-\varphi_1+\varphi_2\right)\right]=\\ &\frac{Q_2}{Q_2^2+Q_3^2}\left\{Q_4\tan\left[\lambda_2\left(H-\varphi_1+\varphi_2\right)\right]-L'\lambda_2\left(Q_1+2e^{\left(n_1-\frac{1}{2\alpha}\right)(\varphi_1-H)}\alpha kJN'\left(\dot{\varphi}_1-\dot{\varphi}_2\right)\right)\right\}.\end{aligned} \tag{49}$$

Eq. (49) is the dispersion equation for torsional surface waves propagating in pre-stressed poroelastic layer sandwiched between transversely isotropic and pre-stressed viscoelastic half-spaces with non-planer boundaries.

4 Particular Cases

4.1 Case 1

If the uppermost half-space is absent; only corrugated common interface is $\varphi_2(r)$ i.e. $\varphi_1(r) = 0$, then dispersion equation (49) reduces to

$$\frac{Q_2}{Q_2^2 + Q_3^2}\left\{L'\lambda_2 \tan\left[\lambda_2\left(H + \varphi_2\right)\right] + \left(kJN'\dot{\varphi}_2\right)\right\} = 1. \tag{50}$$

which is the dispersion equation for torsional surface wave propagation in pre-stressed poroelastic layer overlying pre-stressed viscoelastic half-spaces with corrugated interface.

4.2 Case 2

If the uppermost half-space is absent; intermediate layer is made free from porosity, initial stress and corrugated interface i.e. $T \to 1, P_1 = 0, \varphi_1(r) = \varphi_1(r) = 0$ and also $N' = L' = \mu'$, then dispersion equation (49) reduces to

$$\tan\left[kH\sqrt{\frac{c^2}{\beta_{02}^2} - 1}\right] = \frac{A_1^2 + A_2^2}{A_1}\frac{1}{\mu' k\sqrt{\frac{c^2}{\beta_{02}^2} - 1}}. \tag{51}$$

where, $\beta_{02}^2 = \frac{\mu'}{\rho_2'}$.

which is the dispersion equation for torsional surface wave propagation in an isotropic layer overlying pre-stressed viscoelastic half-spaces with corrugated interface. The obtained dispersion equation (51) is in well agreement with the result obtained by Singh *et. al.* (2015).

4.3 Case 3

If the uppermost half-space is absent; intermediate layer is made free from porosity (T), initial stress (P_1) and corrugated boundaries $(\varphi_1(r), \varphi_2(r))$ and also $N' = L' = \mu'$. Lower most half-space is made free from initial stress and viscoelasticity i.e. $P = 0$, $\mu_v' = 0$; then the considered problem becomes problem of homogeneous isotropic layer overlying a homogeneous isotropic half space and Eq. (49) reduces to

$$\tan kH\sqrt{\frac{c^2}{\beta_{02}^2} - 1} = \frac{\mu_v\sqrt{1 - \frac{c^2}{\beta_3^2}}}{\mu'\sqrt{\frac{c^2}{\beta_{02}^2} - 1}}. \tag{52}$$

Which represents the classical Love wave equation (Ewing *et. al.*,1957), which validates the obtained result.

5 Numerical Example and Discussions

For graphical illustrations, we have considered dispersion equation (49) together with $\varphi_1(r) = \xi_1\cos(br)$, $\varphi_2(r) = \xi_2\cos(br)$ and the following data

- Transversely isotropic half-space (M): $\mu_{L0} = 8.665 \times 10^{10}\ N/m^2, \mu_{T0} = 6.53 \times 10^{10}\ N/m^2, \rho_{10}' = 2700\ kg/m^3$ (Acharya *et al.* (2009)).
- Initially stressed poroelastic layer (M'): $N' = 0.49 \times 10^9\ N/m^2, L' = 0.52 \times 10^9\ N/m^2, \rho_2' = 2435\ kg/m^3$ (Batugin and Nirenburg (1972)).
- Initially stressed viscoelastic half-space (M''): $\mu_v = 203.2 \times 10^9\ N/m^2, \rho_3' = 4744\ kg/m^3$ (Gubbins (1990)).

The variation of dimensionless wavenumber (kH) against dimensionless phase velocity (c/β_2) under the influence of various affecting parameters has been revealed by Figs. 2-6.

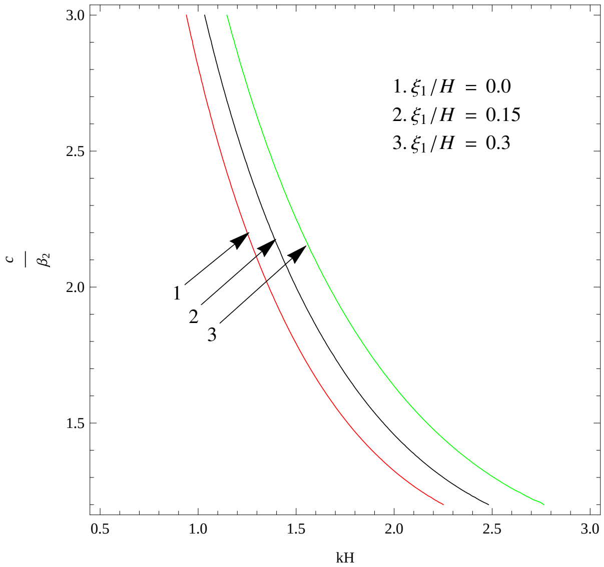

Figure 2: Variation of phase velocity (c/β_2) versus wave number (kH) for different values of flatness parameter (ξ_1/H) associated with common corrugated interface of medium M and M' .

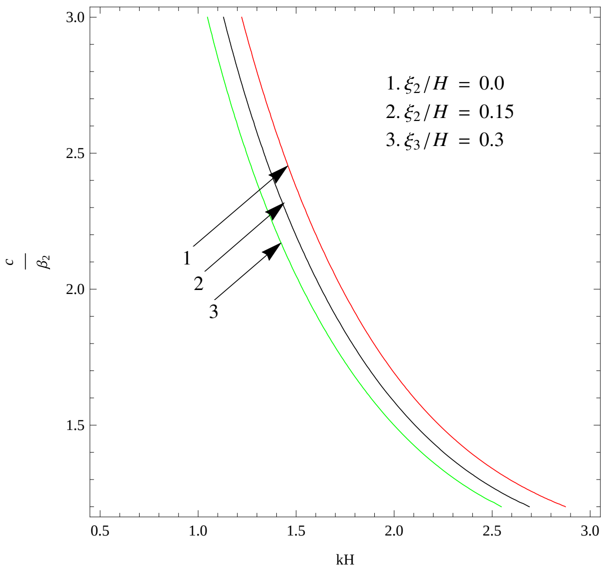

Figure 3: Variation of phase velocity (c/β_2) versus wave number (kH) for different values of flatness parameter (ξ_2/H) associated with common corrugated interface of medium M' and M'' .

It is clear from Fig. 2 to Fig. 6 that phase velocity (c/β_2) decreases with increase in dimensionless wave number (kH) Fig. 2 represents the effect of flatness parameter on the propagation of torsional surface wave associated with common corrugated interface of medium M and M', whereas Fig. 3 represents the influence of flatness parameter associated with corrugated common interface of medium M' and M''. The comparative study of Fig. 2 and Fig. 3 portrays that phase velocity (c/β_2) increases as flatness parameter (ξ_1/H) associated with corrugated common interface of medium M and M' increases; whereas flatness parameter (ξ_2/H) associated with corrugated common interface of medium M' and M'' has an adverse effect on phase velocity (c/β_2) of torsional wave. Fig. 4 shows the effect of inhomogeneity parameter (αk) associated with transversely isotropic half space M on phase velocity of torsional wave. It has been examined from Fig. 4 that phase velocity of torsional wave increases with the increase of inhomogeneity parameter. Fig. 5 reflects the effect of porosity parameter (T) associated with the sandwiched poroelastic layer M'. From Fig. 5, it has been observed that as the porosity parameter increases the phase velocity of torsional wave decreases. Fig. 6 depicts the variation of phase velocity against wave number for

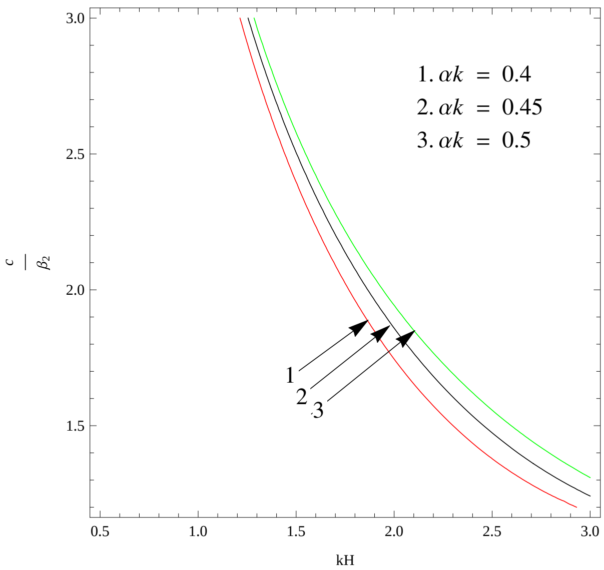

Figure 4: Variation of phase velocity (c/β_2) versus wave number (kH) for different values of inhomogeneity parameter (αk) associated with transversely isotropic half space (M).

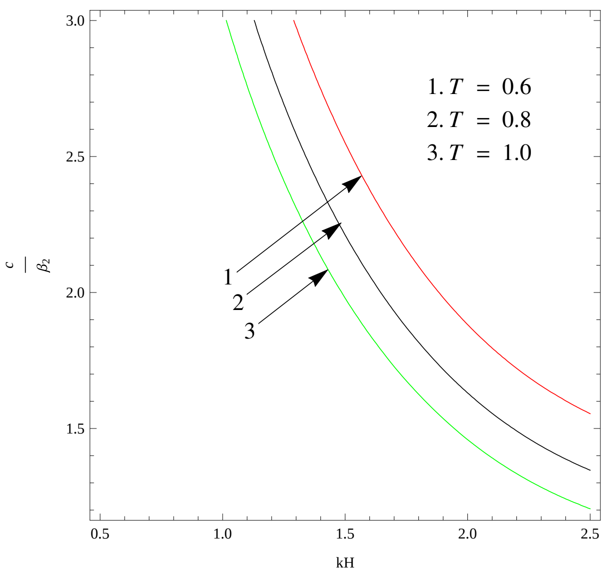

Figure 5: Variation of phase velocity (c/β_2) versus dimensionless wave number (kH) for different values of porosity parameter (T) associated with poroelastic sandwiched layer (M').

different values of initial stress in fluid saturated porous layer M'. It can be examined from Fig. 6 that the phase velocity of torsional waves decreases with increasing values of initial stress of the layer.

6 Conclusions

The present study investigates the possibility of torsional wave propagation in pre-stressed, corrugated poroelastic layer, sandwiched between a transversely isotropic and pre-stressed viscoelastic half-spaces. Frequency equation has been obtained analytically in closed form. Substantial effect of various affecting parameters (viz. flatness, inhomogeneity, porosity and initial stress) on the phase velocity of torsional surface wave has been highlighted. The compact form of obtained dispersion relation has been matched with classical Love wave equation, as a special case of present study. Finally, we may conclude with the following salient outcomes of the present investigation

- Torsional wave velocity is affected significantly by different parameters viz. corrugated boundaries, initial stress, porosity and inhomogeneity.

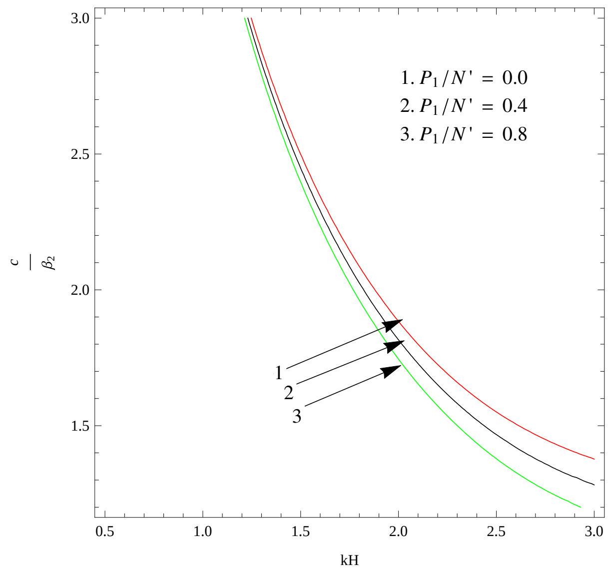

Figure 6: Variation of phase velocity (c/β_2) versus wave number (kH) for different values of initial stress P_1/N' acting on fluid saturated porous layer (M') along radial direction.

- Phase velocity of torsional wave decreases with wave number.
- Initial stress of poroelastic layer has an adverse effect on the phase velocity of torsional wave.
- As the magnitude of porosity parameter increases the phase velocity decreases.
- Phase velocity increases with increase in heterogeneity parameter of transversely isotropic half-space.

In view of the fact that earth is an initially stressed layered elastic medium, outcomes of the present analysis may contribute towards interpretation of wave profile during earthquakes. Due to specific model of the study, results are expected to be utilized for the purpose of seismic exploration.

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