

Mode II Delamination Analysis of Asymmetrical Four Point Bend Layered Beams with Considering Material Non-linearity

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An analytical study is performed of mode II delamination fracture in the asymmetrical Four Point Bend (FPB) beam configurations with considering non-linear material behavior. The beam mechanical behavior is described by using two non-linear constitutive models (ideally elastic-plastic model and model with power law stress-strain relation). The crack is located arbitrarily along the beam height. Fracture is analyzed by applying the J -integral approach. By using the classical beam theory, analytical solutions of the J -integral are obtained at characteristic levels of the external load. The solutions derived are compared with the strain energy release rate. The influence of material non-linearity and crack location along the beam height on the fracture behaviour is evaluated. The J -integral solutions derived are very useful for parametric investigations, since the simple formulae obtained capture the essentials of fracture in asymmetrical FPB beams that have non-linear material behaviour.

1 Introduction

The integrity and load bearing capacity of layered beam structures depends upon their delamination fracture properties. Appearance and propagation of delamination cracks between layers significantly deteriorates the structure performance. For instance, cracking considerably reduces the stiffness of beams and may lead to catastrophic failure.

Considerable efforts have been made in the analysis of delamination fracture in layered beam structures by many researchers (Guadette *et al.*, 2001; Guo *et al.*, 2006; Her and Su, 2015; Hsueh *et al.*, 2009; Jiao *et al.*, 1998; Sørensen and Jacobsen, 2003; Suo *et al.*, 1992; Szekrenyes, 2016; Szekrenyes, 2016; Tkacheva, 2008; Yeung *et al.*, 2000).

Influence of residual stress (hygroscopic, thermal and piezoelectric induced stresses) on the strain energy release rate in layered beam specimens has been analyzed by Guo *et al.*, 2006. Methods of linear-elastic fracture mechanics have been applied. Closed form analytical solutions for determination of the strain energy release rate have been derived by using the classical beam theory. The effects of geometrical and material parameters of the specimens on the delamination fracture behaviour have been evaluated.

Delamination fracture behaviour of laminated composite beams have been investigated by applying the J -integral approach by Sørensen and Jacobsen, 2003. Determination of cohesive laws for different beam specimens has been discussed. The effects of specimen shape on strength have been predicted by using the cohesive laws. Failure of adhesive joints has also been analyzed. Micromechanisms of splitting of laminated composites have been studied both experimentally and theoretically.

The effects of various bridging mechanisms on the delamination fracture resistance of laminated composites have been evaluated by Suo *et al.*, 1992. A family of steady-state mixed mode delamination beams has been considered. A complete solution has been obtained which has been used to construct R -curves for given model parameters. Double cantilever beams loaded by moments and by wedge forces have been compared in order to elucidate the significance of steady state fracture for understanding delamination R -curves. It has been recommended to use R -curves when studying localized damage response.

Dynamic delamination fracture in layered materials has been studied analytically by Tkacheva, 2008. Linear-elastic behaviour of the material has been assumed. Solutions have been obtained by using the classical beam theory. An equation for balance of energy has been used as a criterion for delamination crack growth.

The steady-state delamination fracture behaviour of multilayered beams has been analyzed assuming linear-elastic material behaviour by Hsueh *et al.*, 2009. The fracture has been studied in terms of the strain energy release rate by applying the classical beam theory. A closed form analytical solution has been derived for a delamination crack located arbitrary along the beam height. The solution can be used for multilayered beam configurations with any number of layers. Besides, each layer can have individual thickness and modulus of elasticity.

The delamination fracture toughness of a four-layer linear-elastic beam structure has been studied by Her and Su, 2015. It has been assumed that a delamination crack is located between the second and the third layers. A formula for the strain energy release rate has been derived by using the classical beam theory. The fracture has been studied in a function of modulus of elasticity and thickness of the layers.

It can be concluded that delamination fracture in layered beam configurations, including the FPB, has been analyzed in terms of the strain energy release rate mainly assuming linear-elastic material behaviour. In reality, however, the beam structures may have non-linear behaviour of material. Therefore, the purpose of present article is to perform an analytical study of mode II delamination fracture in the asymmetrical FPB beam configuration under static loading with taking into account the material non-linearity. Fracture is analyzed by applying the J -integral approach. Two non-linear constitutive models (ideally elastic-plastic model and elastic-plastic model with power law stress-strain relation) are used to describe the mechanical response of beam considered. Closed form analytical solutions of the J -integral are obtained by applying the conventional beam theory. The influence of crack location and material non-linearity on the fracture behaviour is discussed.

2 Analysis of the Asymmetrical FPB Beam

The FPB beam configuration considered is illustrated in Figure 1. The beam is in four point bending. The beam cross-section is a rectangle of width, b , and height, $2h$. There is a delamination crack of length, a , located arbitrarily along the beam height. The lower and upper crack arm thicknesses are h_1 and h_2 , respectively (Figure 1). It is assumed that $h_1 \leq h_2$.

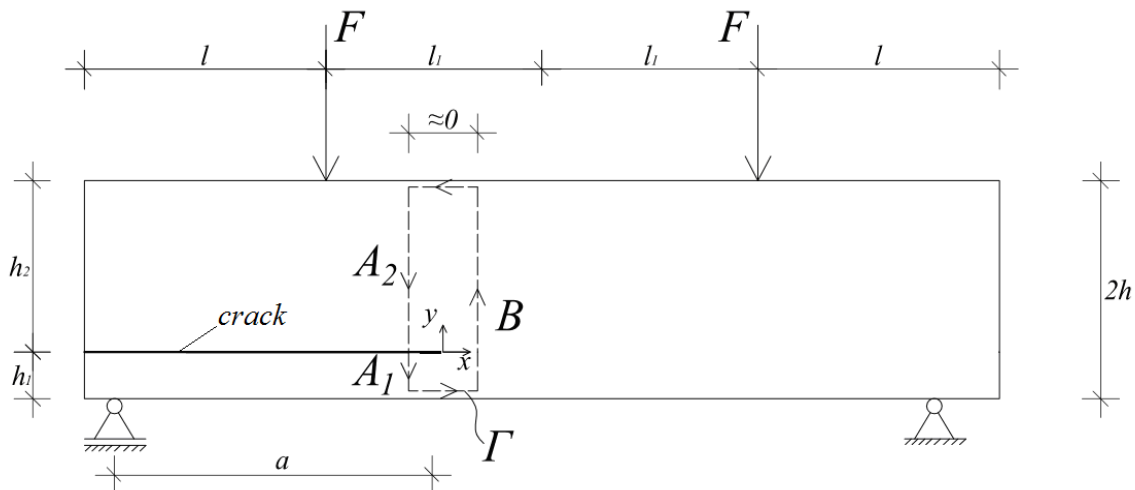


Figure 1. Geometry and loading of the asymmetrical FPB beam

The fracture behaviour is analyzed by using the J -integral approach (Rice, 1968; Rice *et al.*, 1973; Broek, 1986):

$$J = \int_{\Gamma} \left[u_0 \cos \alpha - \left(p_x \frac{\partial u}{\partial x} + p_y \frac{\partial v}{\partial x} \right) \right] ds, \quad (1)$$

where Γ is a contour of integration going from one crack face to the other in the counter clockwise direction, u_0 is the strain energy density, α is the angle between the outwards normal vector to the contour of integration and the crack direction, p_x and p_y are the components of the stress vector, u and v are the components of the displacement vector with respect to the crack tip coordinate system xy , and ds is a differential element along the contour Γ .

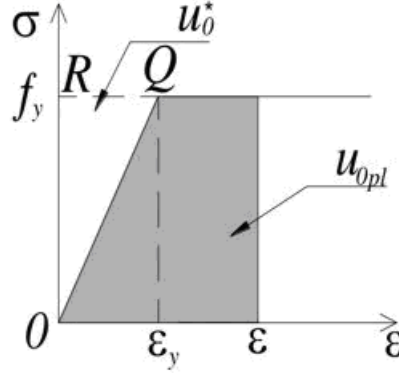


Figure 2. Stress-strain diagram of the ideally elastic-plastic model (u_{0pl} and u_0^* are the strain energy density and complimentary strain energy density, respectively)

It should be specified that the present analysis is based on the small strain assumption (this assumption has been frequently used in analyses of delamination fracture in layered beam configurations (Hsueh *et al.*, 2009)).

The J -integral solution is derived by using integration contour, Γ , consisting of the cross-sections ahead and behind the crack tip (Figure 1). Therefore, the J -integral value is obtained by summation:

$$J = J_{A_1} + J_{A_2} + J_B, \quad (2)$$

where J_{A_1} , J_{A_2} and J_B are the J -integral values in segments A_1 , A_2 and B , respectively.

2.1 Analysis by Using the Ideally Elastic-plastic Model

First, the mechanical response of the FPB beam is described by using the ideally elastic-plastic model (Figure 2). At low magnitudes of the external load, the beam deforms linear-elastically. The normal stresses, induced by bending are distributed linearly along the beam height. Thus, the components of the J -integral in the segment A_1 of the integration contour are written as

$$p_x = -\sigma = -\frac{M_1}{I_1} z_1, p_y = 0, ds = dz_1, \cos \alpha = -1, \quad (3)$$

where $I_1 = \frac{bh_1^3}{12}$ is the principal moment of inertia of the lower crack arm cross-section. The coordinate, z_1 , originates from the lower crack arm cross-section centre and is directed downwards (z_1 varies in the interval $\left[-\frac{h_1}{2}; +\frac{h_1}{2}\right]$). The cross-sectional bending moment in the lower crack arm, M_1 , that participates in (3), is obtained by using the following formula (Rzhanitsyn, 1986)

$$M_1 = M \frac{h_1^3}{h_1^3 + h_2^3}, \quad (4)$$

where

$$M = Fl \quad (5)$$

is the bending moment in the crack tip beam cross-section (Figure 1). The partial derivative, $\frac{\partial u}{\partial x}$, is written as

$$\frac{\partial u}{\partial x} = \varepsilon = \frac{\sigma}{E} = \frac{M_1}{EI_1} z_1. \quad (6)$$

where E is the modulus of elasticity. The strain energy density is obtained as

$$u_0 = \frac{\sigma^2}{2E} = \frac{1}{2E} \left(\frac{M_1}{I_1} z_1 \right)^2 = \frac{M_1^2}{2EI_1^2} z_1^2. \quad (7)$$

By substitution of (3), (6) and (7) in (1), we obtain the following solution of the J -integral in segment A_1

$$J_{A_1} = \frac{6M_1^2}{Eb^2h_1^3}. \quad (8)$$

The J -integral solution in the segment A_2 (Figure 1) is found also by (8). For this purpose, M_1 and h_1 are replaced with M_2 and h_2 , respectively

$$J_{A_2} = \frac{6M_2^2}{Eb^2h_2^3}, \quad (9)$$

where

$$M_2 = M \frac{h_2^3}{h_1^3 + h_2^3} \quad (10)$$

is the cross-sectional bending moment in the upper crack arm (Rzhanitsyn, 1986).

The J -integral components in the segment B of the integration contour are written as (Figure 1)

$$p_x = \sigma = \frac{M}{I_3} z_3, \quad p_y = 0, \quad ds = -dz_3, \quad \cos \alpha = 1, \quad (11)$$

$$\frac{\partial u}{\partial x} = \frac{M}{EI_3} z_3, \quad u_0 = \frac{M^2}{2EI_3^2} z_3^2, \quad (12)$$

where $I_3 = \frac{b(2h)^3}{12}$ is the principal moment of inertia of the un-cracked beam cross-section, M is obtained by (5). The coordinate, z_3 , originates from the beam cross-section centre and is directed downwards (z_3 varies in the interval $[-h; +h]$). After substitution of (11) and (12) in (1), we obtain

$$J_B = -\frac{3F^2 l^2}{4Eb^2 h^3}. \quad (13)$$

The final solution is found by substitution of (8), (9) and (13) in (2)

$$J = \frac{6M_1^2}{Eb^2 h_1^3} + \frac{6M_2^2}{Eb^2 h_2^3} - \frac{3F^2 l^2}{4Eb^2 h^3}. \quad (14)$$

It should be mentioned that at $h_1 = h_2 = h$ equation (14) transforms into

$$J = \frac{21F^2 l^2}{4Eb^2 h^3}. \quad (15)$$

Equation (15) coincides with the formula for the strain energy release rate when the crack is located in the FPB beam mid-plane (Hutchinson and Suo, 1992).

When the external load magnitude increases, the yield stress limit f_y will be attained first in the lower crack arm, since $h_1 \leq h_2$.

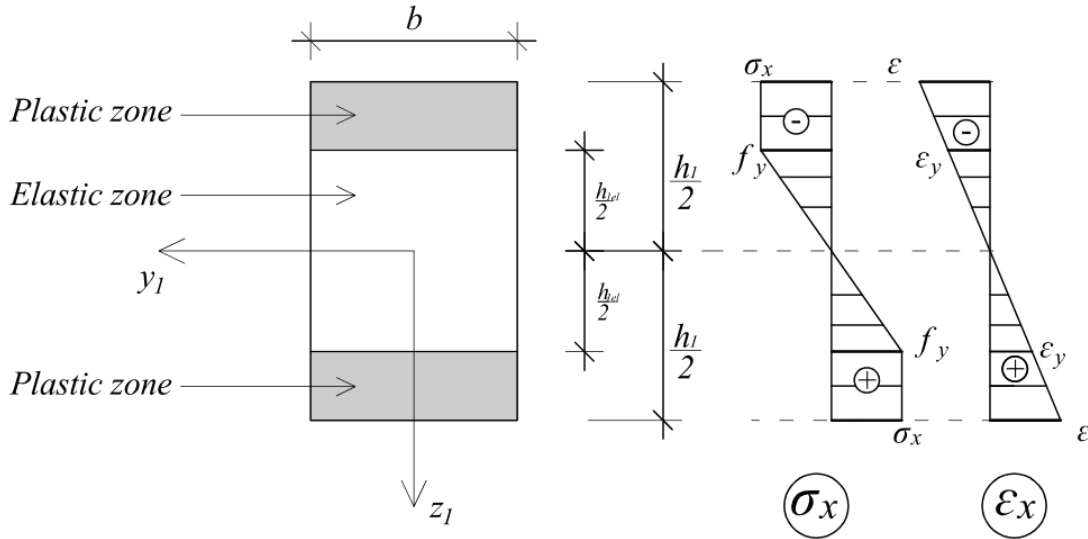


Figure 3. Elastic-plastic distribution of stresses and strains in cross-section A_1 of the lower crack arm

Two symmetric plastic zones will develop in the lower crack arm, while the rest of the beam will continue to deform linear-elastically. The normal stresses and the longitudinal strain distribution in the segment, A_1 , of the integration contour is shown in Figure 3. The J -integral solution in A_1 is obtained by integration along the elastic and plastic zones

$$J_{A_1} = J_{A_1 el} + J_{A_1 pl}. \quad (16)$$

By performing the necessary mathematical operations, we derive

$$J_{A_1} = \frac{6M_{1ep}(3M_{1el} - 2M_{1ep})}{Eb^2h_1^3} + \frac{18M_{1el}^2}{Eb^2h_1^3} \left(1 - \sqrt{3 - 2\frac{M_{1ep}}{M_{1el}}} \right), \quad (17)$$

where

$$M_{1ep} = Q_1 + Q_2 + \frac{1}{6}(3M_{1el} + 4Fl) \quad (18)$$

$$Q_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{Q_3}} \quad (19)$$

$$Q_2 = \sqrt[3]{-\frac{q}{2} - \sqrt{Q_3}} \quad (20)$$

$$Q_3 = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \quad (21)$$

$$p = -\frac{1}{12}(3M_{1el} + 4Fl)^2 + 3FlM_{1el} + F^2l^2 \quad (22)$$

$$q = -\frac{1}{108}(3M_{1el} + 4Fl)^3 + \frac{1}{6}(3M_{1el} + 4Fl)(3FlM_{1el} + F^2l^2) + \frac{1}{2}M_{1el}^3\frac{I_2^2}{I_1^2} - \frac{3}{2}F^2l^2M_{1el} \quad (23)$$

$$M_{1el} = \frac{bh_1^2f_y}{6}. \quad (24)$$

The upper crack arm deforms linear-elastically. Therefore, the J -integral solution in the segment A_2 of the integration contour can be found by (9). For this purpose, M_2 has to be replaced with M_{2e}

$$J_{A_2} = \frac{6M_{2e}^2}{Eb^2h_2^3} \quad (25)$$

where

$$M_{2e} = Fl - M_{1ep}. \quad (26)$$

The solution in segment B of the integration contour is obtained by (13), since the un-cracked beam portion, $x \geq 0$, deforms linear-elastically (Figure 1).

The J -integral final solution is found by combining of (2), (13), (17) and (25)

$$J = \frac{6M_{1ep}(3M_{1el} - 2M_{1ep})}{Eb^2h_1^3} + \frac{18M_{1el}^2}{Eb^2h_1^3} \left(1 - \sqrt{3 - 2\frac{M_{1ep}}{M_{1el}}} \right) + \frac{6M_{2e}^2}{Eb^2h_2^3} - \frac{3F^2l^2}{4Eb^2h^3}. \quad (27)$$

Solution (27) is valid if $h_1 < h_{1,\text{lim}} = h_2\sqrt{2/3}$, because plastic collapse of the lower crack arm will occur before the onset of plastic deformation of the upper crack arm.

If $h_1 \geq h_{1,\text{lim}}$, plastic strains will develop also in the upper crack arm before the plastic collapse of lower crack arm. In this case, the J -integral solution is found as

$$J = \frac{6M_{1ep}(3M_{1el} - 2M_{1ep})}{Eb^2h_1^3} + \frac{18M_{1el}^2}{Eb^2h_1^3} \left(1 - \sqrt{3 - 2\frac{M_{1ep}}{M_{1el}}} \right) + \frac{6M_{2ep}(3M_{2el} - 2M_{2ep})}{Eb^2h_2^3} + \frac{18M_{2el}^2}{Eb^2h_2^3} \left(1 - \sqrt{3 - 2\frac{M_{2ep}}{M_{2el}}} \right) - \frac{3F^2l^2}{4Eb^2h^3}, \quad (28)$$

where

$$M_{2el} = f_y \frac{bh_2^2}{6} \quad (29)$$

$$M_{1ep} = \frac{2M_{1el}^3 I_2^2 Fl + 3M_{2el}^3 I_1^2 M_{1el} - 3M_{1el}^3 I_2^2 M_{2el}}{2(M_{1el}^3 I_2^2 + M_{2el}^3 I_1^2)} \quad (30)$$

$$M_{2ep} = Fl - \frac{2M_{1el}^3 I_2^2 Fl + 3M_{2el}^3 I_1^2 M_{1el} - 3M_{1el}^3 I_2^2 M_{2el}}{2(M_{1el}^3 I_2^2 + M_{2el}^3 I_1^2)}. \quad (31)$$

It should be specified that equation (28) is applicable when the external load magnitude F is in the boundaries between the elastic limit load and the plastic collapse load F_{pc} for the lower crack arm, i.e.

$$\frac{2(h_1^3 + h_2^3)}{3h_1(h_1^2 + h_2^2)} \leq \frac{F}{F_{pc}} \leq 1. \quad (32)$$

2.2 Analysis by Using the Model with Power Law Stress-strain Relation

The delamination fracture is analyzed also assuming that the mechanical behaviour of FPB beam can be described by using the constitutive model with power law stress-strain relation. The stress-strain equation is written as (Dowling, 2007)

$$\sigma = H_1 \varepsilon^{n_1}, \quad (33)$$

where σ and ε are the normal stresses and longitudinal strains, H_1 and n_1 are material constants.

The fracture is analyzed again by applying the J -integral approach (1). The J -integral solution is

$$J = \frac{H_1 n_1 \kappa_1^{n_1+1}}{(n_1+1)(n_1+2)} \left[\left(\frac{h_1}{2} \right)^{n_1+2} - \left(-\frac{h_1}{2} \right)^{n_1+2} \right] + \frac{H_1 n_1 \kappa_1^{n_1+1}}{(n_1+1)(n_1+2)} \left[\left(\frac{h_2}{2} \right)^{n_1+2} - \left(-\frac{h_2}{2} \right)^{n_1+2} \right] - \frac{H_1 n_1 \kappa_3^{n_1+1}}{(n_1+1)(n_1+2)} \left[(h)^{n_1+2} - (-h)^{n_1+2} \right], \quad (34)$$

where

$$\kappa_1 = \left[\frac{2^{n_1+1} (n_1+2) Fl}{bH_1 (h_1^{n_1+2} + h_2^{n_1+2})} \right]^{\frac{1}{n_1}} \quad (35)$$

$$\kappa_3 = \left[\frac{Fl(n_1 + 2)}{2bh^{n_1+2}H_1} \right]^{\frac{1}{n_1}}. \quad (36)$$

The J -integral solutions are compared with the strain energy release rate G determined with taking into account the material non-linearity by using the following equation

$$G = \frac{U_a^* - U_b^*}{b\Delta a} \quad (37)$$

where U_b^* and U_a^* are the complimentary strain energies before and after the increase of crack, respectively, Δa is a small increase of the crack length. The fact that the J -integral solutions coincide with the strain energy release rate is an indication for consistency of the non-linear delamination fracture analyses performed in the present paper.

3 Parametric Investigations

A parametric investigation is performed in order to evaluate the effect of non-linear behaviour of material on the mode II delamination fracture in the asymmetrical FPB layered beam configuration. For this purpose, calculations of the J -integral are carried-out by using formula (28).

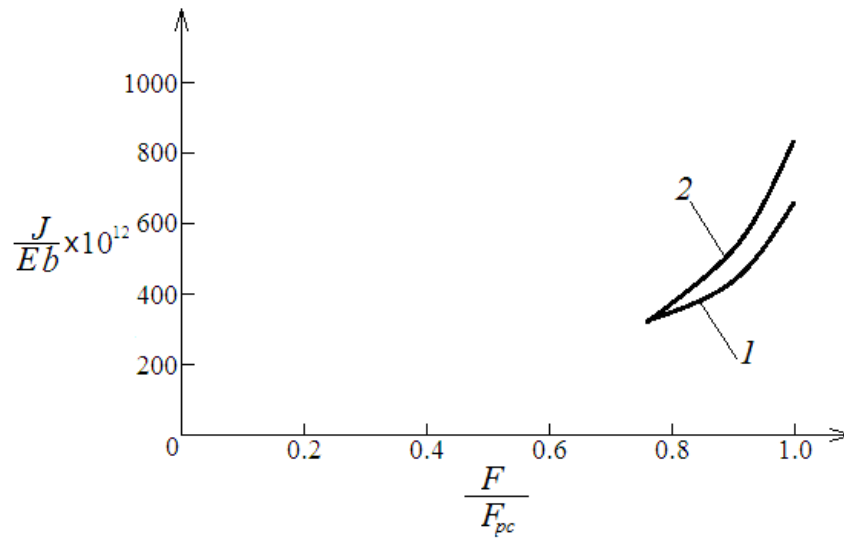


Figure 4. The J -integral value in non-dimensional form plotted against F/F_{pc} ratio (curve 1 – linear-elastic material behaviour, curve 2 – elastic-plastic material behaviour)

The J -integral value obtained is presented in non-dimensional form by using the formula $J_N = J/(Eb)$. It is assumed that $b=0.02$ m, $h=0.01$ m, $h_1 = 0.009$ m and $l=0.5$ m. The J -integral value in non-dimensional form is plotted against F/F_{pc} ratio in Fig. 4 (F/F_{pc} ratio varies in the boundaries determined by (32)). The lower boundary of F/F_{pc} ratio calculated by (32) at $h_1 = 0.009$ and $h_2 = 0.011$ is 0.755. In order to evaluate the effect of material non-linearity on the delamination fracture, the J -integral value obtained assuming linear-elastic material behaviour by formula (14) is also plotted in non-dimensional form against F/F_{pc} ratio in Figure 4 for comparison with the curve generated by non-linear solution. The curves in Figure 4 indicate that the material non-linearity leads to increase of the J -integral value. Therefore, the non-linear behaviour of material has to be taken into account in fracture mechanics based safety design of structural members composed by layered materials.

The effect of crack location along the beam height on the elastic-plastic fracture behaviour is analyzed too. For this purpose, the non-linear solution (34) is used. The J -integral in non-dimensional form, $J/(H_1b)$, is plotted against $h_1/2h$ ratio at $n_1 = 0.8$ and $F/F_{pc} = 0.95$ in Figure 5.

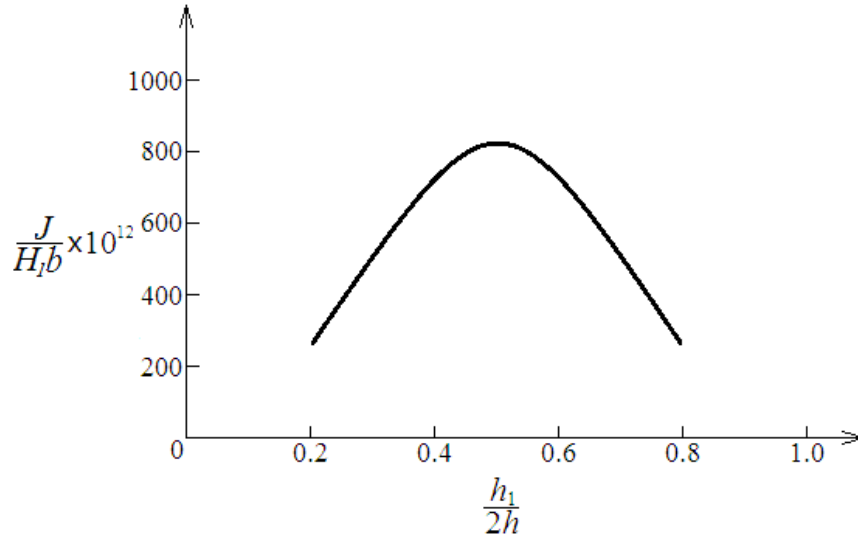


Figure 5. The J -integral value in non-dimensional form plotted against $h_1/2h$ ratio for the model with power law stress-strain relation

It can be observed that the J -integral value is maximum at $h_1/2h = 0.5$, i.e. when the crack is located in the beam mid-plane (Figure 5).

4 Conclusions

An analytical study is conducted of mode II delamination fracture in the asymmetrical FPB layered beams. The basic purpose is to analyze the fracture behaviour with considering the material non-linearity when the crack is located arbitrarily along the beam height. Fracture is investigated by applying the J -integral approach. The FPB beam mechanical behaviour is described by using two non-linear material models (ideally elastic-plastic model and model with power law stress-strain relation). Closed form analytical solutions of the J -integral are derived with the help of classical beam theory. The solutions are compared with the strain energy release rate derived with taking into account the material non-linearity. The effects of material non-linearity and crack location along the beam height on the fracture behaviour are analyzed. It is found that the J -integral value is maximal when the crack is in the beam mid-plane. The analytical solutions derived are very convenient for parametric investigations, since the simple formulae capture the essentials of delamination fracture with considering the non-linear material behaviour. The present study contributes for the understanding of delamination fracture behaviour of layered beams that exhibit material non-linearity. Also, the analysis developed in the present paper can be applied for studying the mode II strain energy release rate in the FPB adhesion test in which adherends are made of metallic materials with non-linear behaviour (for instance, the mechanical behaviour of mild steel and aluminium can be reasonably approximated by the ideally elastic-plastic model, the model with power law stress-strain relation can be applied for describing the mechanical behaviour of annealed copper (Denton, 1966)).

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