

Bifurcations of Relative Equilibria Sets of a Massive Point on Rough Rotating Surfaces

A. Burov, E. Shalimova

Dynamics of a massive point on a rotating wire or surface under dry friction force action is considered. Existence, stability and bifurcations of non-isolated relative equilibria sets of the point located

- *on a sphere uniformly rotating about an inclined fixed axis;*
- *on a thin circular hoop rotating about an inclined fixed axis;*
- *on a paraboloidal bowl uniformly rotating about its axis*

are studied. The results are represented in the form of bifurcation diagrams.

1 Introduction

Problems similar to those considered in the present paper arise when we study dynamics of mechanical systems with rotating parts performing various operations such as mixing, grinding, drying, etc. of diverse substances (Joshi et al. (1995)), as well as, self-compensating systems (van de Wouw et al. (2005)).

Modern advances in computer simulation (Fleissner et al., 2010; Alkhaldi et al., 2008) allow to investigate dynamics of systems with a large number of moving parts numerically. However it is usually difficult to perform the qualitative analysis of the obtained results. That is why it is reasonable to consider some simple problems such as the motion of a material particle on some mobile surface or curve under the action of a friction force. It turns out that appropriate mechanical systems exhibit rather complicated behaviour even be considered in this simple formulation.

One of the main points relates to existence of non-isolated equilibria, in particular, of relative equilibria. Though existence of non-isolated equilibria for systems with dry friction has been known for a long time (cf. Kauderer (1958); Magnus (1976)), systematic investigation of their stability properties and dependence on parameters attracted attention of specialists much later (Pozharitsky, 1962; Matrosov and Finogenko, 1998; Leine and van de Wouw, 2008; Ivanov, 2009; Burov, 2010; Biemond et al., 2012; Burov and Yakushev, 2014; Ivanov, 2015; Burov and Shalimova, 2015, 2016; Várkonyi and Or, 2016)).

2 Formulation of problem and equations of motion in redundant coordinates

Consider a massive point P of mass m moving on a rotating surface A . Let O be an origin of a moving frame $Ox_1x_2x_3$ (MF). In this frame $\overrightarrow{OP} = \mathbf{x} = (x_1, x_2, x_3)^T$, and the surface A is given as

$$\varphi = \varphi(\mathbf{x}) = 0. \quad (1)$$

Furthermore we assume that the point P is moving under the action of potential active forces with a potential

$$U'(\mathbf{x}) = mU(\mathbf{x}). \quad (2)$$

A Coulomb friction force \mathbf{F} with a coefficient of friction μ appears between the point P and the surface A . This surface is assumed being in rotation with an angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$.

The motion of the point can be described by

$$\ddot{\mathbf{x}} = \mathbf{F}_C + \mathbf{F}_c + \mathbf{F}_N + \mathbf{N} + \mathbf{F}. \quad (3)$$

Here, $\mathbf{F}_C = 2\dot{\mathbf{x}} \times \boldsymbol{\omega}$, $\mathbf{F}_c = (\boldsymbol{\omega} \times \mathbf{x}) \times \boldsymbol{\omega}$, $\mathbf{F}_N = -U_{\mathbf{x}}$, $\mathbf{N} = \lambda \varphi_{\mathbf{x}} = N\mathbf{n}$ and \mathbf{F} are Coriolis force, centrifugal force, potential force, normal reaction force and friction force, respectively;

$$\mathbf{n} = \varphi_{\mathbf{x}} |\varphi_{\mathbf{x}}|^{-1}, \quad \left(\varphi_{\mathbf{x}} \stackrel{\text{def}}{=} \frac{\partial \varphi}{\partial \mathbf{x}} \right)$$

is a unit external normal to the surface. All these forces are divided by m . A magnitude of the normal reaction reads

$$N = (\mathbf{N}, \mathbf{n}) = \lambda (\varphi_{\mathbf{x}}, \varphi_{\mathbf{x}}) |\varphi_{\mathbf{x}}|^{-1} = \lambda |\varphi_{\mathbf{x}}|.$$

The multiplier λ can be determined via a standard way. It reads

$$\lambda = -(\varphi_{\mathbf{x}}, \varphi_{\mathbf{x}})^{-1} [(\varphi_{\mathbf{x}\mathbf{x}} \dot{\mathbf{x}}, \dot{\mathbf{x}}) + (\varphi_{\mathbf{x}}, \mathbf{F}_C + \mathbf{F}_c + \mathbf{F}_N)], \quad \left(\varphi_{\mathbf{x}\mathbf{x}} \stackrel{\text{def}}{=} \frac{\partial^2 \varphi}{\partial \mathbf{x}^2} \right). \quad (4)$$

The sign of λ determines the direction of the normal reaction.

Determining the friction force, one must distinguish two cases (see, for example, Levi-Civita and Amaldi (1930)). In the case of slipping of the point P on the rotating surface the friction force \mathbf{F} is perpendicular to the vector of the normal reaction \mathbf{N} , i.e. $(\varphi_{\mathbf{x}}, \mathbf{F}) \equiv 0$, and the direction of this force is opposite to direction of slipping.

In the case of the resting point P with respect to the rotating surface the friction force is directed to the side, opposite to direction of possible slipping. Its magnitude does not exceed the magnitude of the normal reaction, multiplied by the coefficient of friction. It means that

$$\mathbf{F}_c + \mathbf{F}_N + \mathbf{N} + \mathbf{F} = 0, \quad \text{and} \quad |\mathbf{F}| \leq \mu |\mathbf{N}| \quad (5)$$

yields

$$|\mathbf{F}_c + \mathbf{F}_N + \mathbf{N}| \leq \mu |\mathbf{N}|, \quad (6)$$

which determines a set of relative equilibria.

Further assume that coefficient of static friction is equal to coefficient of sliding friction.

2.1 Equations of relative equilibria in redundant coordinates

Inequality (6) gives the condition of existence of relative equilibria and depends, in particular, on the vector $\boldsymbol{\omega}$. It allocates regions on the surface A , 'filled' by relative equilibria (RFbRE). By virtue of inequality (6) boundaries Σ of these regions are determined as

$$f = 0, \quad (7)$$

$$\begin{aligned} f &= (\mathbf{F}_c + \mathbf{F}_N + \mathbf{N}, \mathbf{F}_c + \mathbf{F}_N + \mathbf{N}) - \mu^2 (\mathbf{N}, \mathbf{N}) = \\ &= (\varphi_{\mathbf{x}} \times ((\boldsymbol{\omega} \times \mathbf{x}) \times \boldsymbol{\omega} - U_{\mathbf{x}}), \varphi_{\mathbf{x}} \times ((\boldsymbol{\omega} \times \mathbf{x}) \times \boldsymbol{\omega} - U_{\mathbf{x}})) - \mu^2 (\varphi_{\mathbf{x}}, (\boldsymbol{\omega} \times \mathbf{x}) \times \boldsymbol{\omega} - U_{\mathbf{x}})^2. \end{aligned}$$

The curve $\Gamma = \Sigma \cap A$ bounds a set of relative equilibria. Equilibria correspond to points $P : f \leq 0$ for all instants of time.

Remark. The similar approach can be used for describing the sliding of the point P along a curve, if this curve is given as an intersection of a couple of surfaces.

3 A point on a sphere

3.1 Description of the mechanical system

A heavy material point P moves on the 'standard' sphere S^2 under the action of dry friction force. The sphere is rotating with a constant angular velocity ω about a fixed axis, passing through the center of the sphere O . Let α be an angle of inclination of the axis. A dry friction force F with a friction coefficient μ acts between the point and the sphere (Figure 1).

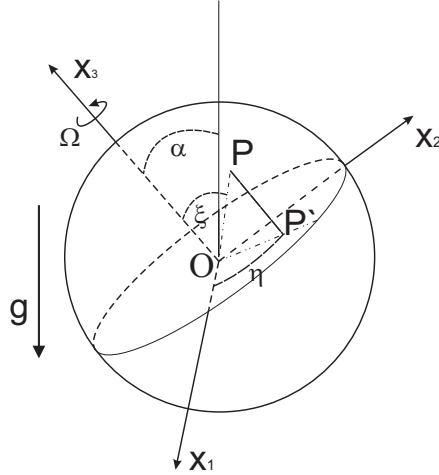


Figure 1. A point on a rotating sphere.

Let $Ox_1x_2x_3$ be a rectangular coordinate system fixed to the sphere with the axis Ox_3 coinciding with rotation axis. The position of the point in this system will be specified by two spherical angles ξ and η . Let ξ be an angle between the axis Ox_3 and \overrightarrow{OP} and η be an angle between the axis Ox_1 and $\overrightarrow{OP'}$, where P' is the projection of the point P onto the plane Ox_1x_2 .

Now introduce dimensionless time $t \mapsto t\sqrt{\ell/g}$ and dimensionless parameter $\Omega^2 = \omega^2\ell/g$. The derivatives with respect to the new time will be denoted by a stroke. Then the system of motion in the relative coordinate system can be written as

$$\begin{aligned} \sin^2 \xi \eta'^2 + \xi'^2 &= (\sin \xi \sin \alpha \sin(\omega t + \eta) + \cos \xi \cos \alpha) - \tilde{N}_r - \Omega \sin^2 \xi (\Omega + 2\eta'), \\ \xi'' - \sin \xi \cos \xi \eta'^2 &= -(\cos \xi \sin \alpha \cos \eta \sin(\omega t + \eta) - \sin \xi \cos \alpha) - \tilde{F}_\xi + \Omega \sin \xi \cos \xi (\Omega + 2\eta'), \\ \sin \xi \eta'' + 2\xi' \eta' \cos \xi &= -\sin \alpha \cos(\omega t + \eta) - \tilde{F}_\eta - 2\Omega \cos \xi \xi', \end{aligned} \quad (8)$$

where $\tilde{F}_\xi = F_\xi/mg$, $\tilde{F}_\eta = F_\eta/mg$ are dimensionless projections of the friction force on the coordinate axes \vec{e}_ξ and \vec{e}_η , $\tilde{N}_r = N_r/mg$ is a dimensionless normal reaction.

3.2 Sets of equilibria

Introducing the angle $\gamma = \eta - \pi/2 + \omega t$, the equilibria can be found from these equations by supposing $\xi' = 0$, $\eta' = 0$, $\xi'' = 0$, $\eta'' = 0$. If the point is in a state of equilibrium then

$$F_\xi^2 + F_\eta^2 \leq \mu^2 N_r^2, \quad (9)$$

$$N_r = \sin \xi \sin \alpha \cos \gamma + \cos \xi \cos \alpha - \Omega^2 \sin^2 \xi,$$

$$F_\xi = \cos \xi \sin \alpha \cos \gamma + \sin \xi \cos \alpha + \Omega^2 \cos \xi \sin \xi, \quad F_\eta = \sin \alpha \sin \gamma.$$

Using these expressions and inequality (9) one obtains

$$\begin{aligned} & (-\cos \xi \sin \alpha \cos \gamma + \sin \xi \cos \alpha + \Omega^2 \cos \xi \sin \xi)^2 + \sin^2 \alpha \sin^2 \gamma \leq \\ & \leq \mu^2 (\sin \xi \sin \alpha \cos \gamma + \cos \xi \cos \alpha - \Omega^2 \sin^2 \xi)^2. \end{aligned} \quad (10)$$

Figure 2 represents the bifurcation diagrams for different values of the inclination angle α and $\mu = 0.7$. The equilibria sets can be obtained by a rotation of the angle 2π around an axis that coincides with the rotation axis. When $\alpha = 0$ the diagram represents a half of a ‘fat fork’, denoted with F , and an equilibrium set in a form of a ‘needle’, denoted with G , near the axis $\xi = 0$ that converges to zero when $\omega \rightarrow \infty$ (Figure 2a, see also Burov (2010)). With increasing of the angle α the area G and the middle ‘prong’ converge to zero and $\xi = \pi$ respectively (Figure 2b), then if $\alpha = \alpha_*$ there is only one point between the ‘prongs’ (Figure 2c), and when $\alpha > \alpha_*$ there is only the bigger jag left (Figure 2d) that straightens itself with the further increase of α (Figure 2e). When $\omega \rightarrow \infty$ the bifurcation diagram for every α is a strip with the boundaries $\xi = \pi/2 - \alpha_*$ and $\xi = \pi/2 + \alpha_*$.

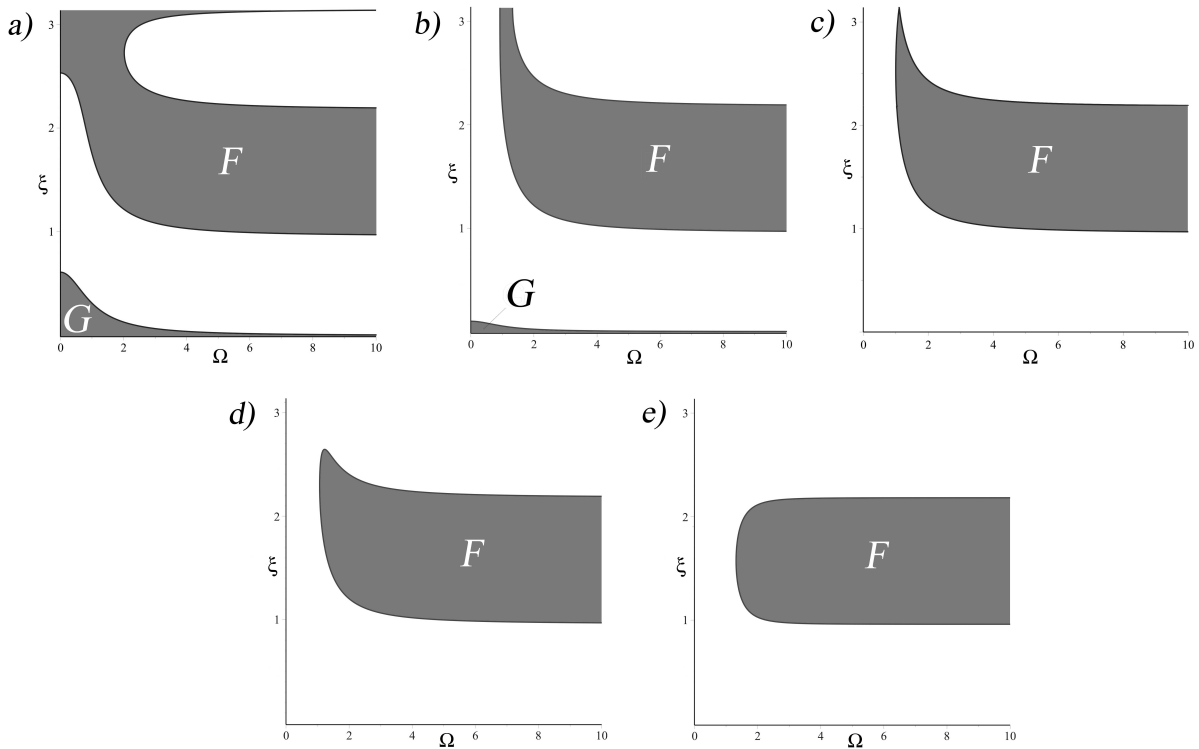


Figure 2. Bifurcation diagrams for $\alpha = 0$, $\alpha = \arctan(0.7) - 0.1$,
 $\alpha = \arctan(0.7)$, $\alpha = \arctan(0.7) + 0.1$, $\alpha = \pi/2$.

4 A bead on a circle

4.1 Description of the mechanical system

The motion of a heavy bead P of mass m on a circular hoop with radius ℓ with its center at the point O is considered. The hoop rotates with a constant angular velocity ω about an inclined axis lying in its plane and passing through its center. The angle of inclination of the axis α is constant. A dry friction force \mathbf{F} with a friction coefficient μ acts between the bead and the hoop (Figure 3).

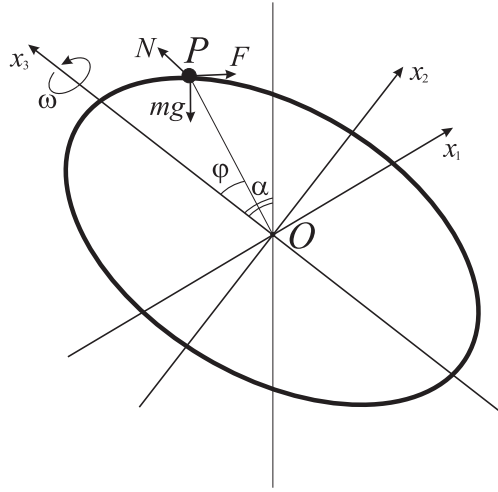


Figure 3. A bead on a circular hoop.

Suppose $Ox_1x_2x_3$ is a moving coordinate system (MCS) with origin at the center of the hoop, the x_3 axis of which is directed along its axis of rotation, the x_2 axis lies in the plane of the hoop and the x_1 axis is perpendicular to this plane.

In the MCS, the bead position is given by coordinates (x_1, x_2, x_3) . For the sake of convenience, represent the constraints restricting its motion as

$$f_1 = \frac{1}{2}(x_2^2 + x_3^2 - \ell^2) = 0, \quad f_2 = x_1 = 0. \quad (11)$$

Suppose $\mathbf{v}_r = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$ is the bead velocity in the MCS, $v_r = (\mathbf{v}_r, \mathbf{v}_r)^{1/2}$, and the transfer velocity $\mathbf{v}_e = (-\omega x_2, \omega x_1, 0)$. The kinetic energy of the system, free from constraints, and the potential energy in the MSC are given by the relations

$$\begin{aligned} T &= \frac{m}{2}((\dot{x}_1 - \omega x_2)^2 + (\dot{x}_2 + \omega x_1)^2 + \dot{x}_3^2), \\ U &= mg(x_1 \sin \omega t \sin \alpha + x_2 \cos \omega t \sin \alpha + x_3 \cos \alpha). \end{aligned} \quad (12)$$

Putting

$$\begin{aligned} x_1 &\mapsto x'_1 \ell, \quad x_2 \mapsto x'_2 \ell, \quad x_3 \mapsto x'_3 \ell, \quad t \mapsto t' \sqrt{\frac{\ell}{g}}, \quad \omega \mapsto \omega' \sqrt{\frac{g}{\ell}} \\ \lambda_1 &\mapsto \lambda'_1 m \frac{g}{\ell}, \quad \lambda_2 \mapsto \lambda'_2 mg, \quad L \mapsto L' mg \ell, \quad F_{x_2} \mapsto mg F'_{x_2}, \quad F_{x_3} \mapsto mg F'_{x_3}. \end{aligned} \quad (13)$$

where λ_1, λ_2 are Lagrange multipliers:

$$\lambda_1 = -(\dot{x}'_1{}^2 + \dot{x}'_2{}^2) - \omega'^2 x'_2{}^2 + x'_2 \cos \omega' t' \sin \alpha + x'_3 \cos \alpha, \quad \lambda_2 = -2\omega' x'_2 + \sin \omega' t' \sin \alpha, \quad (14)$$

and dropping the strokes we obtain the equations of motion in the form:

$$\ddot{x}_2 = \omega^2 x_2 - \cos \omega t \sin \alpha + \lambda_1 x_2 + F_{x_2}, \quad \ddot{x}_3 = -\cos \alpha + \lambda_1 x_3 + F_{x_3}, \quad (15)$$

According to Amontons-Coulomb law

$$F^2 = F_{x_2}^2 + F_{x_3}^2 \leq \mu^2 (\lambda_1^2 + \lambda_2^2). \quad (16)$$

4.2 Sets of equilibria

If the point is in equilibrium, then the friction force compensates the sum of the tangential components of the gravitational force and the centrifugal force, so

$$F = \omega^2 x_2 x_3 - x_3 \cos \omega t \sin \alpha + x_2 \cos \alpha.$$

Then condition (16) can be rewritten as

$$(\omega^2 x_2 x_3 - x_3 \cos \omega t \sin \alpha + x_2 \cos \alpha)^2 \leq \mu^2 (x_2 \cos \omega t \sin \alpha + x_3 \cos \alpha - \omega^2 x_2^2)^2 + \mu^2 \sin^2 \omega t \sin^2 \alpha. \quad (17)$$

For equilibria this inequality must be fulfilled identically for all time instances.

The sets of relative equilibria depend on three parameters (μ, α, ω) . Let us now introduce an angular coordinate φ , measured clockwise from the Oz axis of the MCS (Figure 3). Then $x_2 = \sin \varphi$, $x_3 = \cos \varphi$, and the condition (17) reads

$$\begin{aligned} & (\omega^2 \sin \varphi \cos \varphi - \cos \varphi \cos \omega t \sin \alpha + \sin \varphi \cos \alpha)^2 \leq \\ & \leq \mu^2 (\sin \varphi \sin \alpha \cos \omega t + \cos \varphi \cos \alpha - \omega^2 \sin^2 \varphi)^2 + \mu^2 \sin^2 \omega t \sin^2 \alpha. \end{aligned} \quad (18)$$

Putting in (18) $\cos \omega t = 1$ and $\cos \omega t = -1$, respectively, one obtains the regions Σ_+ and Σ_- in the (φ, ω) plane. These regions are bounded by the curves Γ_+ and Γ_- respectively. The intersection $\Sigma_+ \cap \Sigma_-$ is a set of relative equilibria.

The part of the (φ, ω) plane enclosed between the lines $\varphi = 0$ and $\varphi = \pi$ is shown in Figure 4. The relation between the equilibrium positions and the angular velocity of rotation of the sphere ω is depicted in this figure for different values of the inclination angle of the rotation axis, assuming the friction coefficient $\mu = 0.7$. The Σ_+ regions are denoted by the lightest shading, the Σ_- regions are denoted by the darker shading, and their intersection is distinguished by the darkest shading. It has been shown by Burov (2010) that, when $\alpha = 0$, the set of relative equilibria is the half region F of the ‘bold-face fork’ that is symmetrical about the axis $\varphi = \pi$, and also the region of equilibria G in the form of a ‘needle’ stretched along the line $\varphi = 0$ that converges to zero when $\omega \rightarrow \infty$ (Figure 4a). When α is increased, the regions Σ_+ and Σ_- , which coincide when $\alpha = 0$, diverge (in Figure 4b the case of $\alpha = \arctan \mu - 0.04$ was chosen for purposes of convenience), and, when $\alpha = \arctan \mu$, the region G vanishes and only one point of the middle prong of the ‘fork’ remains (Figure 4c). When $\arctan \mu < \alpha < \pi/2 - \arctan \mu$ these prongs split completely and diverge (Figure 4d). When $\alpha > \pi/2 - \arctan \mu$ a new region I appears for small ω (Figure 4e) that becomes larger as α increases (Figure 4f) and it joins the region F when $\alpha = \pi/2$. In this case the bifurcation diagram is a strip.

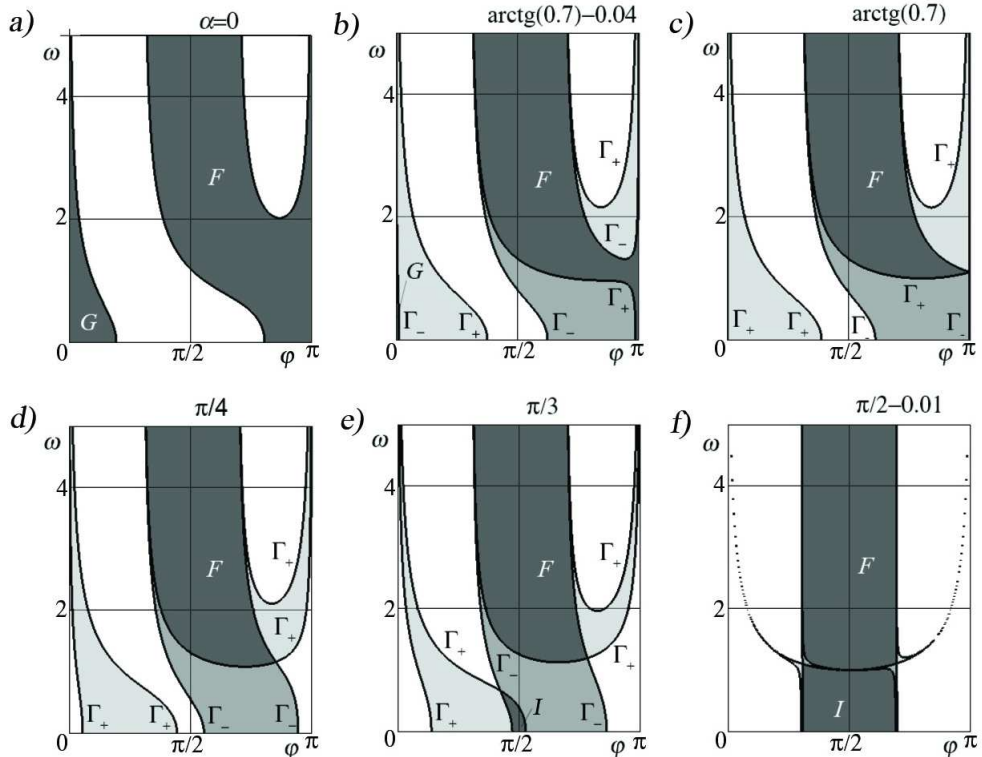


Figure 4. Bifurcation diagrams for different angles of inclination α , $\mu = 0.7$.

5 A point on a paraboloidal cup

5.1 Description of the mechanical system

A heavy material point P moves on the surface of a paraboloidal cup under the action of a dry friction force. The cup is rotating with a constant angular velocity ω about its symmetry axis (Figure 5). The coefficient of dry friction is μ .

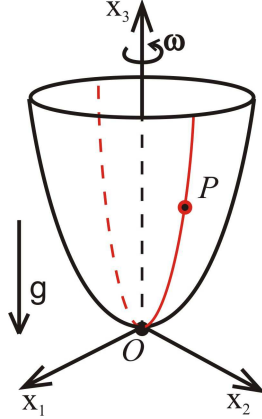


Figure 5. A point in a paraboloidal cup.

Let $Ox_1x_2x_3$ be a moving frame, uniformly rotating about an axis Ox_3 directed along the upward vertical. In this system the position of the point is given by the coordinates (x_1, x_2, x_3) , and the constraint restricting its motion is defined by the relation

$$f = x_3 - \frac{1}{2} \left(\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} \right) = 0. \quad (19)$$

5.2 Relative equilibria sets

Introduce dimensionless variables and parameters

$$x_1 = \sqrt{a_1 a_2} x'_1, \quad x_2 = \sqrt{a_1 a_2} x'_2, \quad x_3 = \sqrt{a_1 a_2} x'_3, \\ p' = \sqrt{a_1 a_2} p, \quad p = \omega^2 / g, \quad b_1 = \sqrt{a_1 / a_2}, \quad b_2 = \sqrt{a_2 / a_1}, \quad (0 < b_1 \leq 1 \leq b_2).$$

Dropping the strokes over the symbols and using (6) with $\mathbf{F}_N = mg$ for determination of relative equilibria, we obtain the boundary of the equilibria sets as follows:

$$\left(p^2 (x_1^2 + x_2^2) + 1 \right) \left(\frac{x_1^2}{b_1^2} + \frac{x_2^2}{b_2^2} + 1 \right) - (1 + \mu^2) \left(p \left(\frac{x_1^2}{b_1} + \frac{x_2^2}{b_2} \right) + 1 \right)^2 = 0. \quad (20)$$

At first we consider the symmetrical case, i.e. $b_1 = b_2 = b$. If $x > 0$: $x^2 = x_1^2 + x_2^2$ is a new coordinate, then the boundary of the equilibria sets can be rewritten as

$$P(x, p, b, \mu) = x^2 \left(1 - \mu^2 \frac{x^2}{b^2} \right) p^2 - 2(1 + \mu^2) \frac{x^2}{b} p + \frac{x^2}{b^2} - \mu^2 = 0. \quad (21)$$

Consider this expression as an equation on $y = x^2$. It is possible to show that this equation has only one root if

$$p = p_{\pm} = \frac{\left(\sqrt{1 + \mu^2} \pm \mu \right)^2}{b} = \frac{1 \pm \sin \alpha_{\star}}{b(1 \mp \sin \alpha_{\star})}.$$

The relative equilibria sets on the (p, x) -plane are depicted on Figure 6. If $p = 0$ the set of equilibria is a disk that contains the point $(x_1, x_2) = (0, 0)$. If $0 < p < p_-$ the set of equilibria is a plane except for an annulus, the center

of which is the point $(x_1, x_2) = (0, 0)$. The radius of the outer circle of the annulus increases indefinitely when $p \mapsto 0$. If $p_- < p < p_+$, then the whole cup becomes the set of equilibria. When $p > p_+$ the set of equilibria is again a whole cup except for an annulus. When $p \mapsto \infty$ the radius of the inner circle of the annulus converges to zero and the radius of an outer circle approaches $\sqrt{b/\mu}$.

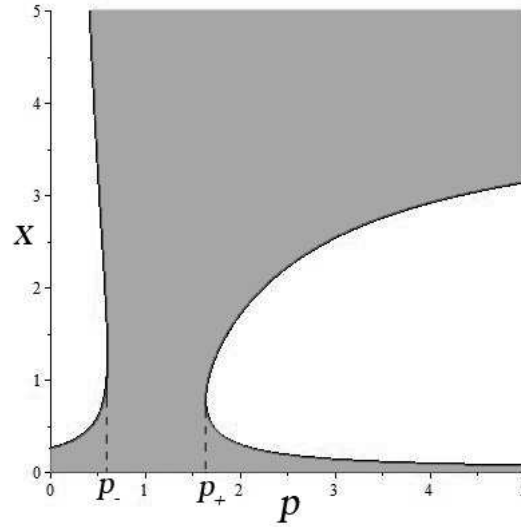


Figure 6. Relative equilibria sets in the symmetrical case.

Now, consider the asymmetrical case. Since Ox_1 and Ox_2 are symmetry axes of the boundary of equilibria sets, it is reasonable to study topological restructuring of the equilibria sets on these axes.

Let

$$p_{\pm k} = \frac{\left(\sqrt{1 + \mu^2} \pm \mu\right)^2}{b_k} = \frac{1 \pm \sin \alpha_*}{b_k(1 \mp \sin \alpha_*)}, \quad k = 1, 2$$

be parameters, similar to p_{\pm} used above.

Consider a section of sets of equilibria by a plane $x_2 = 0$. This section is a curve

$$P(x_1, p, b_1, \mu) = 0. \tag{22}$$

This curve bifurcates for $p = p_{\pm 1}$.

Similarly, the section of sets of equilibria by a plane $x_1 = 0$ is

$$P(x_2, p, b_2, \mu) = 0. \tag{23}$$

This curve bifurcates for $p = p_{\pm 2}$.

Topologically, there are four types of “regular” sets of equilibria and three types of “singular” sets in this problem. The regular, topologically coarse, sets are

- D : a disk;
- $D \cup 4T$: a disk and four tongues;
- $S_i \cup 2T_j$: a strip and two tongues;
- C : a cross.

These sets are structurally stable.

The singular sets are

- $S_2S_i \cup 2T_j$: a strip with 2 straps;

- $C2S_i$: a cross with 2 straps;
- $C4S$: a cross with 4 straps,

respectively. Quarters of sets of these types are depicted in Figure 7.

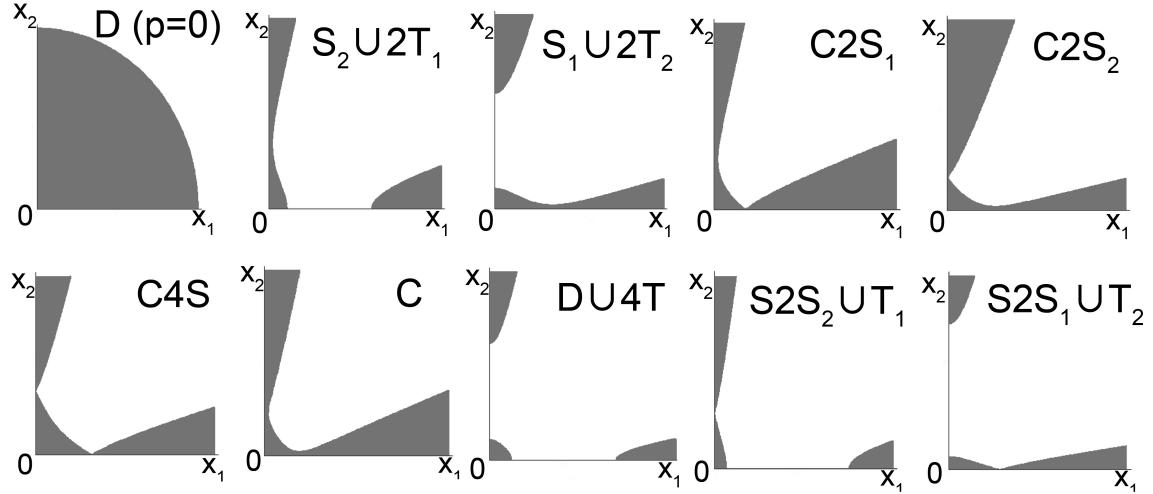


Figure 7. Possible types of equilibria sets.

It is supposed that $b_1 < b_2$, so $p_{+1} > p_{+2}$ and $p_{-1} > p_{-2}$. Thus there are three possible cases of mutual arrangement of p_{-1} and p_{+2} , and so there are three possible ways of the evolution of equilibria sets:

$$1) p_{+1} > p_{-1} > p_{+2} > p_{-2},$$

$$D \rightarrow D \cup 4T \rightarrow S2S2 \cup 2T1 \rightarrow S2 \cup 2T1 \rightarrow S2S2 \cup 2T1 \rightarrow D \cup 4T \rightarrow S2S1 \cup 2T2 \rightarrow S1 \cup 2T2 \rightarrow S2S1 \cup 2T2 \rightarrow D \cup 4T$$

$$2) p_{+1} > p_{-1} = p_{+2} > p_{-2}$$

$$D \rightarrow D \cup 4T \rightarrow S2S2 \cup 2T1 \rightarrow S2 \cup 2T1 \rightarrow C4S \rightarrow S1 \cup 2T2 \rightarrow S2S1 \cup 2T2 \rightarrow D \cup 4T$$

$$3) p_{+1} > p_{+2} > p_{-1} > p_{-2}$$

$$D \rightarrow D \cup 4T \rightarrow S2S2 \cup 2T1 \rightarrow S2 \cup 2T1 \rightarrow C2S1 \rightarrow C \rightarrow C2S2 \rightarrow S1 \cup 2T2 \rightarrow S2S1 \cup 2T2 \rightarrow D \cup 4T$$

6 Conclusions

In this paper three problems of motion of a heavy point on rotating surfaces under the action of dry friction force were considered. For these problems the equations of motion were obtained and the condition of equilibrium and its dependence on parameters of the system was studied. The evolution of relative equilibria sets was represented graphically.

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Address: 1, Michurinckiy prospect, Moscow, 119192, Russia
email: ekateryna-shalimova@yandex.ru, jtm@narod.ru