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# Classical Mechanics recast with Mach's Principle

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*Newton introduced the absolute space as a basis of his mechanics. Inertia causes the resistance of masses against an acceleration with respect to this space. Mach raised the objection that this space is unphysical since it is not tangible. He proposed to describe inertial forces in a similar manner like gravitational ones, namely as produced by all masses of the universe. The paper at hand gives the mathematical exploitation of Mach's concept. Only relative motions of masses enter the resulting equations. Their solutions, however, do not differ from those of Newton's approach.*

## 1 Introduction

According to Newton (1687), the absolute space is a basic concept of mechanics. Forces cause an acceleration with respect to this space. However, the acceleration is the same with respect to any other frame moving against the absolute space with constant velocity, *i.e.* any inertial frame of reference. Thus, the equations enjoy Galilean invariance. The question arises, which of the inertial frames coincides with the absolute space? It cannot be answered experimentally. The description of the effect of forces on a motion becomes more intricate with respect to an arbitrary frame. Mach (1883) proposed to replace the motion with respect to the absolute space by the motion with respect to all masses of the universe in order to remove these difficulties. Then only relative motions between masses occur, the equations are the same for any observer and hence enjoy Euclidean invariance, and the concept of an absolute space with distinguishable physical properties is eliminated. The solutions, however, are the same as those of Newtonian mechanics.

The first aim of the investigation at hand is to find a connection between the absolute space and the masses of the universe. Second, reinterpreting our results, we show how the inertial force can be computed from the position, the velocity and the acceleration of all masses of the universe in a similar manner as the gravitational force is computed from the position alone.

Mach's ideas have been considered by Einstein when formulating the general theory of relativity, but finally discarded. Since that time, papers dealing with Mach's principle generally try to include relativistic effects. A good survey can be found in Barbour and Pfister (1995). The investigation at hand disregards any relativistic effect and deals with classical mechanics, simply reformulating its laws. Physicists are no longer interested in reformulations of 17th century mechanics. Engineering mechanics, on the other hand, continues to be classical, and so a sound basis of this discipline is surely of interest to its teachers and students.

## 2 Classical Kinematics

On the basis of classical space-time, geometrical and physical objects such as the arrow from one material point to another or a force are described by an observer as a vector of a vector space $\mathcal{V}$, often called his frame of reference. So he denotes a force by $\mathbf{f} \in \mathcal{V}$ while another observer denotes it by $\bar{\mathbf{f}} \in \bar{\mathcal{V}}$. There exists an invertible linear mapping $\mathbf{Q}$ between the vector spaces $\mathcal{V}$ and $\bar{\mathcal{V}}$ of the two observers. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$ be fixed orthonormal right-handed bases of the two vector spaces, then we have

$$\bar{\mathbf{f}} = \sum_j \bar{f}_j \bar{\mathbf{e}}_j \,, \qquad \mathbf{f} = \sum_i f_i \mathbf{e}_i \,, \qquad \mathbf{Q} = \sum_k \sum_l Q_{kl} \bar{\mathbf{e}}_k \otimes \mathbf{e}_l \tag{1}$$

and

$$\bar{\mathbf{f}} = \mathbf{Q} \cdot \mathbf{f} \qquad \Longleftrightarrow \qquad \bar{f}_j = \sum_i Q_{ji} f_i \tag{2}$$

The dot $(\cdot)$ denotes a contraction, the cross $(\times)$ the vector product and the symbol $\otimes$ the tensor product. Now, the inner product is preserved under the mapping $\mathbf{Q}$. This implies

$$\bar{\mathbf{a}} \cdot \bar{\mathbf{b}} = (\mathbf{Q} \cdot \mathbf{a}) \cdot (\mathbf{Q} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{Q}^* \cdot \mathbf{Q} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} \tag{3}$$

and hence

$$\mathbf{Q}^* \cdot \mathbf{Q} = \sum_k \sum_l Q_{kl} \mathbf{e}_l \otimes \bar{\mathbf{e}}_k \cdot \sum_m \sum_n Q_{mn} \bar{\mathbf{e}}_m \otimes \mathbf{e}_n = \sum_l \sum_n \sum_k Q_{kl} Q_{kn} \mathbf{e}_l \otimes \mathbf{e}_n = \sum_l \sum_n \delta_{ln} \mathbf{e}_l \otimes \mathbf{e}_n = \mathbf{1} \tag{4}$$

where the asterisk denotes the adjoint mapping, and $\mathbf{1}$ is the identity on $\mathcal{V}$. We will also need $\mathbf{Q} \cdot \mathbf{Q}^* = \bar{\mathbf{1}}$ with $\bar{\mathbf{1}}$ the identity on $\bar{\mathcal{V}}$. The outer product is also preserved

$$\bar{\mathbf{a}} \times \bar{\mathbf{b}} = (\mathbf{Q} \cdot \mathbf{a}) \times (\mathbf{Q} \cdot \mathbf{b}) = \mathbf{Q} \cdot (\mathbf{a} \times \mathbf{b}) \tag{5}$$

So the $Q_{kl}$ are seen to be elements of a proper orthogonal matrix. The mapping $\mathbf{Q}$ is called an isometry, and its adjoint equals its inverse. Vectors that satify the transformation (2) are then called objective.

The motion of a material point P is described by its time-dependent position vector. The connection between the descriptions of two different observers is given by the transformation

$$\bar{\mathbf{r}}(t) = \bar{\mathbf{r}}_0(t) + \mathbf{Q}(t) \cdot \mathbf{r}(t) \tag{6}$$

where $\bar{\mathbf{r}}_0$ is the position vector of the origin of the observer without a bar as seen by the observer with a bar and $\mathbf{Q}$ an isometry, both time-dependent. The derivative with respect to time yields the transformation of the velocity

$$\dot{\bar{\mathbf{r}}}(t) = \dot{\bar{\mathbf{r}}}_0(t) + \dot{\mathbf{Q}}(t) \cdot \mathbf{r}(t) + \mathbf{Q}(t) \cdot \dot{\mathbf{r}}(t) = \dot{\bar{\mathbf{r}}}_0(t) - \mathbf{Q}(t) \cdot \Big(\boldsymbol{\omega}(t) \times \mathbf{r}(t)\Big) + \mathbf{Q}(t) \cdot \dot{\mathbf{r}}(t) \tag{7}$$

We expressed the skew tensor $\dot{\mathbf{Q}}^*(t) \cdot \mathbf{Q}(t)$ by a vector $\boldsymbol{\omega}(t)$ of angular velocity according to

$$\dot{\mathbf{Q}}^*(t) \cdot \mathbf{Q}(t) = -\mathbf{Q}^*(t) \cdot \dot{\mathbf{Q}}(t) = \boldsymbol{\omega}(t) \times \mathbf{1} = \mathbf{1} \times \boldsymbol{\omega}(t) \tag{8}$$

The second derivative is the acceleration

$$\ddot{\bar{\mathbf{r}}}(t) = \ddot{\bar{\mathbf{r}}}_0(t) - \mathbf{Q}(t) \cdot \Big(\dot{\boldsymbol{\omega}}(t) \times \mathbf{r}(t)\Big) + \mathbf{Q}(t) \cdot \Big(\boldsymbol{\omega}(t) \times \Big(\boldsymbol{\omega}(t) \times \mathbf{r}(t)\Big)\Big) - 2\,\mathbf{Q}(t) \cdot \Big(\boldsymbol{\omega}(t) \times \dot{\mathbf{r}}(t)\Big) + \mathbf{Q}(t) \cdot \ddot{\mathbf{r}}(t) \tag{9}$$

Note that neither the position nor the velocity or the acceleration are objective vectors.

We consider the universe made up of celestial masses, which we call particles. A simple model of these particles are stars idealized as point masses. If this seems too coarse — since interstellar gases and the rotation of stars are not included —, one may choose a finer realization of particles, maybe down to atoms, without altering the conclusions of the following text.

The position and velocity vectors of the particle with number $j$ in both frames are connected by

$$\bar{\mathbf{r}}_j(t) = \bar{\mathbf{r}}_0(t) + \mathbf{Q}(t) \cdot \mathbf{r}_j(t) \tag{10}$$

$$\dot{\bar{\mathbf{r}}}_j(t) = \dot{\bar{\mathbf{r}}}_0(t) - \mathbf{Q}(t) \cdot \Big(\boldsymbol{\omega}(t) \times \mathbf{r}_j(t)\Big) + \mathbf{Q}(t) \cdot \dot{\mathbf{r}}_j(t) \tag{11}$$

We infer from eqs. (6) and (10) that the difference vector

$$\bar{\mathbf{r}}_j - \bar{\mathbf{r}} = \mathbf{Q} \cdot (\mathbf{r}_j - \mathbf{r}) \tag{12}$$

is objective, and obtain from eqs. (7) and (11)

$$\dot{\bar{\mathbf{r}}}_j = \mathbf{Q} \cdot \Big( - \mathbf{v} - \boldsymbol{\omega} \times (\mathbf{r}_j - \mathbf{r}) + \dot{\mathbf{r}}_j - \dot{\mathbf{r}}\Big) \qquad \text{with} \qquad \mathbf{v} \equiv -\,\mathbf{Q}^* \cdot \dot{\bar{\mathbf{r}}} \tag{13}$$

## 3 Newton's Dynamics

Let the gravitational force $m^\dagger \bar{\mathbf{b}}$ and, say, a spring force $\bar{\mathbf{f}}$ act on the material point P with the heavy mass $m^\dagger$ and the inert mass $m$ . Newton assumes a preferential frame of reference, called absolute space, in which we will mark all quantities by a bar.
**Assumption 1 (Newton)** The balance of momentum is

$$m\ddot{\bar{\mathbf{r}}} = m^\dagger \bar{\mathbf{b}} + \bar{\mathbf{f}} \tag{14}$$

Introduction of eq. (9) in eq. (14) yields the version of this balance for an arbitrary observer

$$\mathbf{Q}^* \cdot \left( \frac{m^\dagger}{m} \bar{\mathbf{b}} + \frac{1}{m} \bar{\mathbf{f}} \right) = \frac{m^\dagger}{m} \mathbf{b} + \frac{1}{m} \mathbf{f} = \mathbf{Q}^* \cdot \ddot{\bar{\mathbf{r}}} = \mathbf{Q}^* \cdot \ddot{\bar{\mathbf{r}}}_0 - \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}} + \ddot{\mathbf{r}} \tag{15}$$

**Assumption 2 (Newton)** The body force per unit heavy mass is produced by a long-distance effect of particles with the heavy mass $m_j^\dagger$, where $G$ denotes the gravitational constant.

$$\mathbf{b} = G \sum_j m_j^\dagger \frac{\mathbf{r}_j - \mathbf{r}}{|\mathbf{r}_j - \mathbf{r}|^3} \tag{16}$$

It is objective according to eq. (12).
**Assumption 3 (Newton)** The forces between any two point masses cancel each other out according to the law of action and reaction.

Eq. (15) may be reinterpreted in two ways. On the one hand, we may write

$$m\ddot{\mathbf{r}} = m^\dagger \mathbf{b} + \mathbf{f} + m\Big( - \mathbf{Q}^* \cdot \ddot{\bar{\mathbf{r}}}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}} \Big) \tag{17}$$

The given forces $m^\dagger \mathbf{b}$ and $\mathbf{f}$ are objective. However, the term with the bracket results from inertia, and its constituents are called *fictitious forces*. They are not objective since the left-hand side of the equation is not objective either. On the other hand, we may also write

$$\mathbf{0} = m^\dagger \mathbf{b} + \mathbf{f} + m\mathbf{i} \qquad \text{with} \qquad \mathbf{i} \equiv -\mathbf{Q}^* \cdot \ddot{\bar{\mathbf{r}}} \tag{18}$$

If this is interpreted as a statement of equilibrium of forces, then $\mathbf{i}$ represents the so-called *inertial force per unit of inert mass*. In contrast to the fictitious forces, the inertial force is objective. Therefore, we will take the representation (18) as the basis of our following investigation. The idea to formulate the equations of motion as equilibrium conditions dates back to d'Alembert.

Remark: We are not obliged to assume a fixed ratio of inert and heavy mass. This equivalence, based on the observation of equal acceleration during a free fall of all bodies, is a corner stone of general relativity, but is not a constituent of classical mechanics. Physicists point out that, taking the equivalence $m^\dagger = m$ for granted, the expression $m(\mathbf{b} + \mathbf{i})$ cannot be split into its gravitational and inertial part by means of an experimental device. Nevertheless, Newtonian mechanics provides the two distinct prescriptions (16) and (18) for the computation of the two components.

## 4 Characteristic Values of the Universe

**Assumption 4** The values of all following quantities are finite. This is at variance with the older idea of an infinite homogeneous universe but does not imply the existence of a finite radius of the universe provided that its mean mass density is decaying rapidly enough.

The total mass of the universe and the vector from the origin O to its center of mass as recorded by a general observer are

$$M \equiv \sum_j m_j\,, \qquad \mathbf{r}_C \equiv \frac{1}{M} \sum_j m_j\,\mathbf{r}_j \tag{19}$$

In classical mechanics, conservation of mass is postulated, and so $m_j$ for all $j$ and hence also $M$ are constant in time. The momentum and the moment of momentum with respect to the chosen origin O are defined as

$$\mathbf{j} \equiv \sum_j m_j\,\dot{\mathbf{r}}_j = M\dot{\mathbf{r}}_C\,, \qquad \mathbf{d}_O \equiv \sum_j m_j\,\mathbf{r}_j \times \dot{\mathbf{r}}_j \tag{20}$$

and

$$\mathbf{J}_O \equiv \sum_j m_j \Big( |\mathbf{r}_j|^2 \, \mathbf{1} - \mathbf{r}_j \otimes \mathbf{r}_j \Big) \tag{21}$$

is the inertia tensor of the universe with respect to O. Also useful are the same quantities referred to the locus P of the material point

$$\mathbf{p} \equiv \frac{1}{M} \sum_j m_j \,(\mathbf{r}_j - \mathbf{r}) = \mathbf{r}_C - \mathbf{r}\,, \qquad \mathbf{d}_P \equiv \sum_j m_j \,(\mathbf{r}_j - \mathbf{r}) \times \dot{\mathbf{r}}_j\,, \qquad \mathbf{J}_P \equiv \sum_j m_j \Big( |\mathbf{r}_j - \mathbf{r}|^2 \, \mathbf{1} - (\mathbf{r}_j - \mathbf{r}) \otimes (\mathbf{r}_j - \mathbf{r}) \Big) \tag{22}$$

and referred to the center of mass C

$$\mathbf{d}_C \equiv \sum_j m_j \,(\mathbf{r}_j - \mathbf{r}_C) \times \dot{\mathbf{r}}_j = \mathbf{d}_O - \mathbf{r}_C \times \mathbf{j} = \mathbf{d}_O - M \mathbf{r}_C \times \dot{\mathbf{r}}_C = \mathbf{d}_P - \mathbf{p} \times \mathbf{j} = \mathbf{d}_P - M \mathbf{p} \times \dot{\mathbf{r}}_C \tag{23}$$

$$\mathbf{J}_C \equiv \sum_j m_j \Big( |\mathbf{r}_j - \mathbf{r}_C|^2 \, \mathbf{1} - (\mathbf{r}_j - \mathbf{r}_C) \otimes (\mathbf{r}_j - \mathbf{r}_C) \Big) = \mathbf{J}_O - M \Big( |\mathbf{r}_C|^2 \, \mathbf{1} - \mathbf{r}_C \otimes \mathbf{r}_C \Big) = \mathbf{J}_P - M \Big( |\mathbf{p}|^2 \, \mathbf{1} - \mathbf{p} \otimes \mathbf{p} \Big) \tag{24}$$

The following rule of vector algebra will be needed in the sequel

$$\mathbf{a} \times (\boldsymbol{\omega} \times \mathbf{a}) \equiv \boldsymbol{\omega}\,(\mathbf{a} \cdot \mathbf{a}) - \mathbf{a}\,(\mathbf{a} \cdot \boldsymbol{\omega}) \equiv \Big( |\mathbf{a}|^2 \, \mathbf{1} - \mathbf{a} \otimes \mathbf{a} \Big) \cdot \boldsymbol{\omega} \tag{25}$$

## 5 The absolute Space inferred from the Masses of the Universe

**Assumption 5** The universe is closed so that no resulting force or torque is acting on it from the outside.

We note that the forces between any two point masses (gravitational forces, spring forces) are central forces.
The kinetic equation (14) for the particle with number $j$ under the influence of a resulting force $\bar{\mathbf{f}}_j$ can be given the form

$$m_j \ddot{\bar{\mathbf{r}}}_j = \bar{\mathbf{f}}_j \qquad \Longrightarrow \qquad m_j (\bar{\mathbf{r}}_j - \bar{\mathbf{r}}_C) \times \ddot{\bar{\mathbf{r}}}_j = (\bar{\mathbf{r}}_j - \bar{\mathbf{r}}_C) \times \bar{\mathbf{f}}_j \tag{26}$$

Summation of these equations over all particles gives

$$\sum_j m_j \ddot{\bar{\mathbf{r}}}_j \equiv \Big( \sum_j m_j \dot{\bar{\mathbf{r}}}_j \Big)^{\bullet} = \sum_j \bar{\mathbf{f}}_j = \mathbf{0} \tag{27}$$

and

$$\sum_j m_j (\bar{\mathbf{r}}_j - \bar{\mathbf{r}}_C) \times \ddot{\bar{\mathbf{r}}}_j \equiv \Big( \sum_j m_j (\bar{\mathbf{r}}_j - \bar{\mathbf{r}}_C) \times \dot{\bar{\mathbf{r}}}_j \Big)^{\bullet} = \sum_j (\bar{\mathbf{r}}_j - \bar{\mathbf{r}}_C) \times \bar{\mathbf{f}}_j = \mathbf{0} \tag{28}$$

where we made use of the fact that — because of eq. (20) —

$$\sum_j m_j (\bar{\mathbf{r}}_j - \bar{\mathbf{r}}_C)^{\bullet} \times \dot{\bar{\mathbf{r}}}_j = \sum_j m_j \dot{\bar{\mathbf{r}}}_j \times \dot{\bar{\mathbf{r}}}_j - M \dot{\bar{\mathbf{r}}}_C \times \dot{\bar{\mathbf{r}}}_C = \mathbf{0} \tag{29}$$

Integration with respect to time yields

$$\sum_j m_j \dot{\bar{\mathbf{r}}}_j = \bar{\mathbf{j}} = M \dot{\bar{\mathbf{r}}}_C = const. \tag{30}$$

$$\sum_j m_j (\bar{\mathbf{r}}_j - \bar{\mathbf{r}}_C) \times \dot{\bar{\mathbf{r}}}_j = \bar{\mathbf{d}}_C = const. \tag{31}$$

The vectors on the right-hand side represent the constant values of the momentum and the moment of momentum of the universe with respect to the absolute space. Insertion of eqs. (12) and (13) with (5) and (22) into eqs. (30) and (31) gives

$$\begin{aligned} \mathbf{Q}^* \cdot \bar{\mathbf{j}} &= \sum_j m_j \mathbf{Q}^* \cdot \dot{\bar{\mathbf{r}}}_j = \sum_j m_j \Big( -\mathbf{v} - \boldsymbol{\omega} \times (\mathbf{r}_j - \mathbf{r}) + \dot{\mathbf{r}}_j - \dot{\mathbf{r}} \Big) \\ &= M \mathbf{Q}^* \cdot \dot{\bar{\mathbf{r}}}_C = M \Big( -\mathbf{v} - \boldsymbol{\omega} \times \mathbf{p} + \dot{\mathbf{p}} \Big) \end{aligned} \tag{32}$$

and

$$\mathbf{Q}^* \cdot \Big(\bar{\mathbf{d}}_C + (\bar{\mathbf{r}}_C - \bar{\mathbf{r}}) \times \bar{\mathbf{j}}\Big) = \sum_j m_j \Big(\mathbf{Q}^* \cdot (\bar{\mathbf{r}}_j - \bar{\mathbf{r}})\Big) \times \Big(\mathbf{Q}^* \cdot \dot{\bar{\mathbf{r}}}_j\Big)$$

$$= \sum_j m_j (\mathbf{r}_j - \mathbf{r}) \times \Big( -\mathbf{v} - \boldsymbol{\omega} \times (\mathbf{r}_j - \mathbf{r}) + \dot{\mathbf{r}}_j - \dot{\mathbf{r}}\Big)$$

$$= -\sum_j m_j (\mathbf{r}_j - \mathbf{r}) \times (\mathbf{v} + \dot{\mathbf{r}}) - \sum_j m_j \Big( |\mathbf{r}_j - \mathbf{r}|^2\, \mathbf{1} - (\mathbf{r}_j - \mathbf{r}) \otimes (\mathbf{r}_j - \mathbf{r})\Big) \cdot \boldsymbol{\omega} + \sum_j m_j\, (\mathbf{r}_j - \mathbf{r}) \times \dot{\mathbf{r}}_j$$

$$= -M\, \mathbf{p} \times (\mathbf{v} + \dot{\mathbf{r}}) - \mathbf{J}_P \cdot \boldsymbol{\omega} + \mathbf{d}_P \tag{33}$$

So we obtain the velocity of the absolute space with respect to some material point P

$$\mathbf{v} \equiv -\mathbf{Q}^* \cdot \dot{\bar{\mathbf{r}}} = -\mathbf{Q}^* \cdot \dot{\bar{\mathbf{r}}}_C - \boldsymbol{\omega} \times \mathbf{p} + \dot{\mathbf{p}} = -\mathbf{Q}^* \cdot \dot{\bar{\mathbf{r}}}_C - \boldsymbol{\omega} \times (\mathbf{r}_C - \mathbf{r}) + \dot{\mathbf{r}}_C - \dot{\mathbf{r}} \tag{34}$$

and insert this into eq. (33) to find

$$\begin{aligned} \mathbf{Q}^* \cdot \bar{\mathbf{d}}_C &= M\, \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p} - \dot{\mathbf{r}}_C) - \mathbf{J}_P \cdot \boldsymbol{\omega} + \mathbf{d}_P \\ &= \Big( M(|\mathbf{p}|^2\, \mathbf{1} - \mathbf{p} \otimes \mathbf{p}) - \mathbf{J}_P \Big) \cdot \boldsymbol{\omega} + \Big( \mathbf{d}_P - M\mathbf{p} \times \dot{\mathbf{r}}_C \Big) = -\mathbf{J}_C \cdot \boldsymbol{\omega} + \mathbf{d}_C \end{aligned} \tag{35}$$

Therefore, the angular velocity of the absolute space with respect to our observer turns out to be

$$\boldsymbol{\omega} = \mathbf{J}_C^{-1} \cdot \Big( \mathbf{d}_C - \mathbf{Q}^* \cdot \bar{\mathbf{d}}_C \Big) \tag{36}$$

After all, we succeeded to characterize the absolute space by properties of all particles since the right-hand sides of the eqs. (34) and (36) are functions of the positions and velocities of all particles according to eqs. (19) to (24). They are, of course, time-dependent. In addition, they contain the two constants $\bar{\mathbf{d}}_C$ and $\dot{\bar{\mathbf{r}}}_C$, the role of which will be discussed later.

Our finding confirms what may be called the **weak form of Mach's principle**: *The absolute space is connected with the masses of the universe.*

It should, therefore, better be called a *privileged frame of reference* that allows a particular simple form of the equations of motion. This is a familiar phenomenon; the equations of motion of a rigid body are markedly simplified if referred to the center of mass and the principal axes of inertia.

## 6 The inertial Force induced by the Masses of the Universe

The time derivative of the definition of $\mathbf{v}$ according to eq. (13) is

$$\dot{\mathbf{v}} = -\mathbf{Q}^* \cdot \ddot{\bar{\mathbf{r}}} - \dot{\mathbf{Q}}^* \cdot \dot{\bar{\mathbf{r}}} = -\mathbf{Q}^* \cdot \ddot{\bar{\mathbf{r}}} + \boldsymbol{\omega} \times \mathbf{v} \tag{37}$$

which yields the inertial force according to eq. (18)

$$\mathbf{i} = -\mathbf{Q}^* \cdot \ddot{\bar{\mathbf{r}}} = \dot{\mathbf{v}} - \boldsymbol{\omega} \times \mathbf{v} \tag{38}$$

Introducing eq. (34) and taking notice of eq. (30) leads to the representations

$$\begin{aligned} \mathbf{i} &= (\dot{\mathbf{p}} - \boldsymbol{\omega} \times \mathbf{p})^{\bullet} - \boldsymbol{\omega} \times (\dot{\mathbf{p}} - \boldsymbol{\omega} \times \mathbf{p}) - \underline{\Big( \dot{\mathbf{Q}}^* \cdot \mathbf{Q} - \boldsymbol{\omega}(t) \times \mathbf{1} \Big)} \cdot \mathbf{Q}^* \cdot \dot{\bar{\mathbf{r}}}_C \\ &= \ddot{\mathbf{p}} - 2\,\boldsymbol{\omega} \times \dot{\mathbf{p}} - \dot{\boldsymbol{\omega}} \times \mathbf{p} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p}) \\ &= -\ddot{\mathbf{r}} + 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \ddot{\mathbf{r}}_C - 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}}_C - \dot{\boldsymbol{\omega}} \times \mathbf{r}_C + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_C) \end{aligned} \tag{39}$$

Note that the underlined bracket vanishes according to eq. (8). So the constant velocity $\dot{\bar{\mathbf{r}}}_C$ of the centre of mass and hence the momentum of the universe with respect to the privileged frame do not influence the inertial force. This reflects the well-known fact that all inertial frames are equivalent, and each of them could be presumed to be the "absolute space". On the other hand, the inertial force actually depends on the moment of momentum $\bar{\mathbf{d}}_C$ of the universe with respect to the privileged frame since this influences the angular velocity $\boldsymbol{\omega}$ according to eq. (36).

Eq. (39) may be interpreted as a **strong form of Mach's principle**: *The inertial force is induced by the masses of the universe by a long-range action in a similar way as the gravitational force.*
This statement is rendered precise by the following program. First, we choose any observer and compute the mass moments of degree 0, 1, and 2 and the moment of momentum, all referred to the origin O.

$$M = \sum_j m_j \,, \qquad \mathbf{s} = \sum_j m_j \mathbf{r}_j \,, \qquad \mathbf{J}_O = \sum_j m_j \left( |\mathbf{r}_j|^2 \, \mathbf{1} - \mathbf{r}_j \otimes \mathbf{r}_j \right) , \qquad \mathbf{d}_O = \sum_j m_j \, \mathbf{r}_j \times \dot{\mathbf{r}}_j \tag{40}$$

These four quantities are made up by all particles in an additive manner. Second, we use these quantities to get — note eqs. (23), (24) and $\mathbf{s} \equiv M \mathbf{r}_C$ —

$$\mathbf{J}_C = \mathbf{J}_O - \frac{1}{M} \left( |\mathbf{s}|^2 \, \mathbf{1} - \mathbf{s} \otimes \mathbf{s} \right) , \qquad \mathbf{d}_C = \mathbf{d}_O - \frac{1}{M} \mathbf{s} \times \dot{\mathbf{s}} \tag{41}$$

and hence from eq. (36)

$$\boldsymbol{\omega} \; = \mathbf{J}_C^{-1} \cdot \left( \mathbf{d}_C - \mathbf{Q}^* \cdot \bar{\mathbf{d}}_C \right) \tag{42}$$

Finally, we obtain the inertial force from eq. (39)

$$\mathbf{i} = (\dot{\mathbf{p}} - \boldsymbol{\omega} \times \mathbf{p})^{\bullet} - \boldsymbol{\omega} \times (\dot{\mathbf{p}} - \boldsymbol{\omega} \times \mathbf{p}) \qquad \text{with} \qquad \mathbf{p} = \frac{1}{M} \mathbf{s} - \mathbf{r} \tag{43}$$

One may regard equation (43) with (40), (41), (42) as a new law of nature, representing Machian instead of Newtonian mechanics since the inertial force is no longer explained by the existence of an absolute space. Nevertheless, the two formulations are equivalent.

## 7 Simplification in Case of an isotropic Universe

Under the special assumption of an isotropic but not necessarily homogeneous universe, the inertia tensor $\mathbf{J}_C$ must be spherical.

$$\mathbf{J}_C = \frac{1}{3} \left( \text{trace } \mathbf{J}_C \right) \mathbf{1} \qquad \text{with} \qquad \text{trace } \mathbf{J}_C = 2 \sum_j m_j |\mathbf{r}_j - \mathbf{r}_C|^2 = 2M \, R^2 \qquad \Longrightarrow \qquad \mathbf{J}_C = \frac{2}{3} M \, R^2 \, \mathbf{1} \tag{44}$$

where $R$ denotes the polar radius of gyration, depending on time if the universe is expanding. If we consider $M$ and the actual value of $R$ to be known — similar to the gravitational constant — then we only need the following two quantities

$$\mathbf{s} = \sum_j m_j \mathbf{r}_j \,, \qquad \mathbf{d}_O = \sum_j m_j \, \mathbf{r}_j \times \dot{\mathbf{r}}_j \tag{45}$$

They determine the angular velocity

$$\boldsymbol{\omega} = \frac{3}{2R^2} \left( \frac{1}{M} \mathbf{d}_O - \frac{1}{M^2} \mathbf{s} \times \dot{\mathbf{s}} - \frac{1}{M} \mathbf{Q}^* \cdot \bar{\mathbf{d}}_C \right) \tag{46}$$

and the inertial force follows again from eq. (43).

While the original quantities $\mathbf{s}$ and $\mathbf{d}_O$ are made up in an additive manner by the single particles as was the case with the gravitational force of eq. (16), the product $\mathbf{s} \times \dot{\mathbf{s}}$ and hence also the vector $\boldsymbol{\omega}$ give rise to a coupling between all particles. This is even more true for the inertial force of eq. (43), which contains products of $\boldsymbol{\omega}$ and $\mathbf{s}$ and, therefore, is by no means the sum of single forces each created by only one particle. So, in contrast to the gravitational forces, no principle of superposition holds for the inertial forces. Nevertheless, they are real interactive forces between a material point and the remainder of the universe, obeying the law of action and reaction.

## 8 Simplification if the Stars are fixed in the absolute Space

Foucault's pendulum allows to measure the angular velocity of the privileged frame with respect to the earth and confirms, with high accuracy, the fact that it is identical to the angular velocity of distant celestial objects. This agrees with the naive view that identifies the privileged frame with that particular frame in which the stars are fixed. We want to exploit this experimental fact. However, since we know that all masses of the universe are actually moving, we replace the system of fixed stars by an observer to whom the sum of the squares of the velocities of all particles— weighted with their inert mass — is minimal (in the sense of Gauss' method of smallest error squares). So we postulate

$$\sum_j m_j \dot{\bar{\mathbf{r}}}_j^2 = \min \tag{47}$$

where $\dot{\bar{\mathbf{r}}}_j$ has to be evaluated according to eq. (13) and the velocity $\mathbf{v}$ and the angular velocity $\boldsymbol{\omega}$ of the privileged frame are to be found. Variation of eq. (13) yields

$$\delta\dot{\bar{\mathbf{r}}}_j = -\mathbf{Q}\cdot\delta\mathbf{v} - \mathbf{Q}\cdot\Big(\delta\boldsymbol{\omega}\times(\mathbf{r}_j-\mathbf{r})\Big) = -\mathbf{Q}\cdot\delta\mathbf{v} - (\mathbf{Q}\cdot\delta\boldsymbol{\omega})\times(\bar{\mathbf{r}}_j-\bar{\mathbf{r}}) \tag{48}$$

Inserting this into the postulate (47), we find

$$0 = \sum_j m_j \dot{\bar{\mathbf{r}}}_j\cdot\delta\dot{\bar{\mathbf{r}}}_j = -\sum_j m_j\dot{\bar{\mathbf{r}}}_j\cdot\mathbf{Q}\cdot\delta\mathbf{v} - \sum_j m_j\Big((\bar{\mathbf{r}}_j-\bar{\mathbf{r}})\times\dot{\bar{\mathbf{r}}}_j\Big)\cdot\mathbf{Q}\cdot\delta\boldsymbol{\omega} \tag{49}$$

Since the vectors $\mathbf{Q}\cdot\delta\mathbf{v}$ and $\mathbf{Q}\cdot\delta\boldsymbol{\omega}$ are arbitrary, we obtain the two statements — note (30), (31) —

$$\sum_j m_j\dot{\bar{\mathbf{r}}}_j \equiv M\dot{\bar{\mathbf{r}}}_C = \mathbf{0} \tag{50}$$

$$\sum_j m_j(\bar{\mathbf{r}}_j-\bar{\mathbf{r}})\times\dot{\bar{\mathbf{r}}}_j \equiv \bar{\mathbf{d}}_C + (\mathbf{Q}\cdot\mathbf{p})\times M\dot{\bar{\mathbf{r}}}_C = \mathbf{0} \tag{51}$$

Thus, the values of the momentum and of the moment of momentum of the universe are zero in the frame under discussion. The moment of momentum is the same with respect to all points since the momentum is zero. Within the more general framework of the preceding chapters, the moment of momentum $\bar{\mathbf{d}}_C$ plays the role of a vector-valued natural constant. The knowledge of its value is necessary in order to evaluate equation (36). Now Foucault's experiment shows that the fixed stars determine the privileged frame rather well so that we may attribute to this constant the trivial value zero and reduce eq.(36) to

$$\boldsymbol{\omega} = \mathbf{J}_C^{-1}\cdot\mathbf{d}_C \tag{52}$$

This special result was already obtained by Lynden-Bell (1995) on the basis of an energy minimization but without giving a motivation for this procedure.

## 9 Newton's Bucket

Newton (1687) discussed the behavior of a rotating bucket. The experiment shows that the surface of the rotating water within the bucket is curved to a paraboloid. Newton took this effect as a proof of the existence of absolute space. Now, what is the result in Mach's interpretation?

We go back to the general anisotropic case and choose a frame of reference that rotates with the bucket. The velocities of the celestial particles with respect to this frame are huge compared with those observed within the privileged frame, so that the latter contributions may be neglected. If the origin is situated on the axis of rotation then the velocity of a star is

$$\dot{\mathbf{r}}_j = \tilde{\boldsymbol{\omega}}\times\mathbf{r}_j \tag{53}$$

while the particles of the bucket have the velocity $\dot{\mathbf{r}}_j = \mathbf{0}$. This implies

$$\dot{\mathbf{r}}_C = \frac{1}{M}\sum_j m_j\dot{\mathbf{r}}_j = \frac{1}{M}\Bigg(\sum_j^{(S)} m_j\tilde{\boldsymbol{\omega}}\times\mathbf{r}_j + \underline{\tilde{\boldsymbol{\omega}}\times\sum_j^{(B)} m_j\mathbf{r}_j}\Bigg) = \tilde{\boldsymbol{\omega}}\times\frac{1}{M}\sum_j m_j\mathbf{r}_j = \tilde{\boldsymbol{\omega}}\times\mathbf{r}_C \tag{54}$$

The superscripts $(S)$ and $(B)$ characterize summation over the stars and the bucket, respectively, and the underlined term vanishes and hence could be added without damage since the center of mass of the bucket is situated on the axis of rotation. We need

$$\mathbf{d}_O = \sum_j m_j\,\mathbf{r}_j\times\dot{\mathbf{r}}_j = \sum_j^{(S)} m_j\,\mathbf{r}_j\times(\tilde{\boldsymbol{\omega}}\times\mathbf{r}_j) = \sum_j^{(S)} m_j\left(|\mathbf{r}_j|^2\,\mathbf{1}-\mathbf{r}_j\otimes\mathbf{r}_j\right)\cdot\tilde{\boldsymbol{\omega}} = \mathbf{J}_O^{(S)}\cdot\tilde{\boldsymbol{\omega}} = \Big(\mathbf{J}_O-\mathbf{J}_O^{(B)}\Big)\cdot\tilde{\boldsymbol{\omega}} \tag{55}$$

$$\mathbf{d}_C = \mathbf{d}_O - M\mathbf{r}_C\times\dot{\mathbf{r}}_C = \mathbf{d}_O - M\mathbf{r}_C\times(\tilde{\boldsymbol{\omega}}\times\mathbf{r}_C) = \Big(\mathbf{J}_O-\mathbf{J}_O^{(B)}-M\Big(|\mathbf{r}_C|^2\,\mathbf{1}-\mathbf{r}_C\otimes\mathbf{r}_C\Big)\Big)\cdot\tilde{\boldsymbol{\omega}} = \Big(\mathbf{J}_C-\mathbf{J}_O^{(B)}\Big)\cdot\tilde{\boldsymbol{\omega}} \tag{56}$$

Inserting this into eq. (42) yields

$$\boldsymbol{\omega} = \left(\mathbf{1} - \mathbf{J}_C^{-1} \cdot \mathbf{J}_O^{(B)}\right) \cdot \tilde{\boldsymbol{\omega}} - \mathbf{J}_C^{-1} \cdot \mathbf{Q}^* \cdot \bar{\mathbf{d}}_C \tag{57}$$

Let us restrict ourselves to the case $\bar{\mathbf{d}}_C = \mathbf{0}$ and notice that the tensor of inertia $\mathbf{J}_O^{(B)}$ of a typical bucket is negligible with respect to that of the universe $\mathbf{J}_C$. Then we obtain the plausible result

$$\boldsymbol{\omega} = \tilde{\boldsymbol{\omega}} \tag{58}$$

which states that the rotation of the privileged frame with respect to the bucket is the same as that of the stars. Hence follows from eq. (54)

$$\dot{\mathbf{r}}_C - \boldsymbol{\omega} \times \mathbf{r}_C = \mathbf{0} \qquad \Longrightarrow \qquad \ddot{\mathbf{r}}_C - \dot{\boldsymbol{\omega}} \times \mathbf{r}_C - \boldsymbol{\omega} \times \dot{\mathbf{r}}_C = \mathbf{0} \tag{59}$$

and the expression of the inertial force (39) does no longer depend on the position $\mathbf{r}_C$ of the center of mass C of the universe, but reduces to

$$\mathbf{i} = -\ddot{\mathbf{r}} + 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{60}$$

The centrifugal force $-\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ and hence the curvature of the liquid surface was interpreted by Newton as a consequence of the rotation of the bucket with respect to an absolute space. We see, however, that we may also consider the centrifugal forces to be induced by the masses that rotate around the bucket at rest.

It is enlightening to read Mach's original text: *„Der Versuch Newtons mit dem rotierenden Wassergefäß lehrt nur, daß die Relativdrehung des Wassers gegen die Gefäßwände keine merklichen Zentrifugalkräfte weckt, daß dieselben aber durch die Relativdrehung gegen die Masse der Erde und die übrigen Himmelskörper geweckt werden. Niemand kann sagen, wie der Versuch quantitativ und qualitativ verlaufen würde, wenn die Gefäßwände immer dicker und massiger, zuletzt mehrere Meilen dick würden.* " (Newton's experiment with the rotating vessel of water simply informs us, that the relative rotation of the water with respect to the sides of the vessel produces no noticeable centrifugal forces, but that such forces are reproduced by its relative rotation with respect to the mass of the earth and other celestial bodies. No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick. — Translation quoted from Barbour and Pfister (1995). )

Now, having taken Newtonian mechanics for granted, we learned that the privileged frame is determined by the masses of the universe. The bucket, too, is part of this universe. Usually, its influence can be ignored with respect to that of the stars. But if the mass of the bucket increases, its influence may become observable, at least in principle. If the axis of rotation is a principal axis of inertia of both the bucket and the universe, and we assume again $\bar{\mathbf{d}}_C = \mathbf{0}$, then eq. (57) becomes

$$\boldsymbol{\omega} = \left(\mathbf{1} - \mathbf{J}_C^{-1} \cdot \mathbf{J}_O^{(B)}\right) \cdot \tilde{\boldsymbol{\omega}} \qquad \Longrightarrow \qquad \omega = \left(1 - \frac{M_B r_B^2}{M r^2}\right) \tilde{\omega} \tag{61}$$

where $M$, $M_B$, $r$, $r_B$ are the masses and the radii of gyration about the corresponding axes of the universe and the bucket, respectively. The value of the fraction is about $10^{-90}$ in case of a steel bucket with a wall thickness of one kilometer. So the bucket rotates in one direction with respect to the inertial frames and the remainder of the universe rotates in the mean in the opposite direction since the value of the resulting moment of momentum of the universe must retain the value zero. The value of $\omega$ and hence the centrifugal force acting on the water is reduced. Physicists term such an influence of a rotating body on the inertial behavior gravitomagnetic.

Remark: When accelerating the gross bucket from the state of rest, the opposite moment of momentum is surely only applied to the earth — modifying its angular velocity — and not to any fixed star. In this sense, $\tilde{\boldsymbol{\omega}}$ is a mean value over the remainder of the universe.

Remark: Only the inertia tensor of the bucket but not its shape enters the formula. This has an interesting consequence. If the bucket has the special shape of a homogeneous hollow sphere, then it is well-known that it does not induce any gravitational forces on point masses in its interior, even if rotating. On the contrary, the rotating bucket of any shape influences the angular velocity $\boldsymbol{\omega}$ and hence the inertial forces everywhere in the universe. This is not surprising since gravity and inertia are totally separate phenomena in our setting.

## 10 Rates of Distances

Eqs. (42) and (43) show that the inertial force depends on the vectors $\mathbf{p}$ and $\boldsymbol{\omega}$ and their time derivatives, while $\boldsymbol{\omega}$ itself is depending on the tensor $\mathbf{J}_C$ and the vector $\mathbf{d}_C$. According to eq. (22), the vector

$$\mathbf{p} = \frac{1}{M}\sum_j m_j(\mathbf{r}_j - \mathbf{r}) \tag{62}$$

only contains the distance vectors from the position of the material point P to the positions of all particles. Next, we demonstrate that $\mathbf{J}_C$ and $\mathbf{d}_C$ and hence also $\boldsymbol{\omega}$ only depend on the distance vectors of the particles and their first derivative. According to (40), (41), we have

$$\begin{aligned}
\mathbf{J}_C &= \mathbf{J}_O - \frac{1}{M}\Big(|\mathbf{s}|^2\,\mathbf{1} - \mathbf{s}\otimes\mathbf{s}\Big)\\
&= \sum_j m_j\Big(|\mathbf{r}_j|^2\mathbf{1} - \mathbf{r}_j\otimes\mathbf{r}_j\Big) - \frac{1}{M}\Big(\sum_j m_j\mathbf{r}_j\cdot\sum_k m_k\mathbf{r}_k\;\mathbf{1} - \sum_j m_j\mathbf{r}_j\otimes\sum_k m_k\mathbf{r}_k\Big)\\
&= \frac{1}{2M}\sum_j\sum_k m_j m_k\Big(\Big(|\mathbf{r}_j|^2 + |\mathbf{r}_k|^2 - 2\mathbf{r}_j\cdot\mathbf{r}_k\Big)\mathbf{1} - \mathbf{r}_j\otimes\mathbf{r}_j - \mathbf{r}_k\otimes\mathbf{r}_k + \mathbf{r}_j\otimes\mathbf{r}_k + \mathbf{r}_k\otimes\mathbf{r}_j\Big)\\
&= \frac{1}{2M}\sum_j\sum_k m_j m_k\Big(|\mathbf{r}_j - \mathbf{r}_k|^2\;\mathbf{1} - (\mathbf{r}_j - \mathbf{r}_k)\otimes(\mathbf{r}_j - \mathbf{r}_k)\Big)
\end{aligned} \tag{63}$$

$$\begin{aligned}
\mathbf{d}_C &= \mathbf{d}_O - \frac{1}{M}\mathbf{s}\times\dot{\mathbf{s}} = \sum_j m_j\mathbf{r}_j\times\dot{\mathbf{r}}_j - \frac{1}{M}\;\sum_j m_j\mathbf{r}_j\times\sum_k m_k\dot{\mathbf{r}}_k\\
&= \frac{1}{2M}\sum_j\sum_k m_j m_k\Big(\mathbf{r}_j\times\dot{\mathbf{r}}_j + \mathbf{r}_k\times\dot{\mathbf{r}}_k - \mathbf{r}_j\times\dot{\mathbf{r}}_k - \mathbf{r}_k\times\dot{\mathbf{r}}_j\Big)\\
&= \frac{1}{2M}\sum_j\sum_k m_j m_k(\mathbf{r}_j - \mathbf{r}_k)\times(\mathbf{r}_j - \mathbf{r}_k)^\bullet
\end{aligned} \tag{64}$$

The kinetic energy of the universe with respect to the privileged frame with $M\dot{\bar{\mathbf{r}}}_C = \sum_j m_j\dot{\bar{\mathbf{r}}}_j = \mathbf{0}$ and $\bar{\mathbf{d}}_C = \mathbf{0}$ is

$$\begin{aligned}
E_{\text{kin}} &= \frac{1}{2}\sum_j m_j|\dot{\bar{\mathbf{r}}}_j|^2 = \frac{1}{2M}\left(\sum_k m_k\sum_j m_j|\dot{\bar{\mathbf{r}}}_j|^2 - \underbrace{\sum_k m_k\dot{\bar{\mathbf{r}}}_k}\cdot\underbrace{\sum_j m_j\dot{\bar{\mathbf{r}}}_j}\right)\\
&= \frac{1}{4M}\sum_j\sum_k m_j m_k\Big(|\dot{\bar{\mathbf{r}}}_j|^2 + |\dot{\bar{\mathbf{r}}}_k|^2 - 2\dot{\bar{\mathbf{r}}}_j\cdot\dot{\bar{\mathbf{r}}}_k\Big) = \frac{1}{4M}\sum_j\sum_k m_j m_k|(\bar{\mathbf{r}}_j - \bar{\mathbf{r}}_k)^\bullet|^2\\
&= \frac{1}{4M}\sum_j\sum_k m_j m_k\Big|(\mathbf{r}_j - \mathbf{r}_k)^\bullet - \boldsymbol{\omega}\times(\mathbf{r}_j - \mathbf{r}_k)\Big|^2
\end{aligned} \tag{65}$$

The underbraced sums vanish and the last expression is obtained from eq. (13).

The moment of momentum and the kinetic energy depend on the derivatives $(\mathbf{r}_j - \mathbf{r}_k)^\bullet$ of the distance vectors. These contain information on the rate of both the length and the direction of these vectors. We want to prove that the kinetic energy, but not the moment of momentum, actually depends only on the rate of length $|\mathbf{r}_j - \mathbf{r}_k|^\bullet$ of all distance vectors. For that purpose, we number the particles in such a way that any four consecutive particles do not lie in a plane. We want to demonstrate that $\dot{\mathbf{r}}_{n+4}\,(n \geq 0)$ can be computed from the knowledge of the three foregoing values $\dot{\mathbf{r}}_{n+j}\,(j = 1, 2, 3)$ and the time derivative of the three distances $|\mathbf{r}_{n+4} - \mathbf{r}_{n+j}|\,(j = 1, 2, 3)$. We introduce a vector basis and its dual basis by

$$\mathbf{a}_j \equiv \mathbf{r}_{n+4} - \mathbf{r}_{n+j}\,(j = 1, 2, 3), \quad \mathbf{a}^k \equiv \frac{1}{D}\mathbf{a}_{k+1}\times\mathbf{a}_{k+2}\,(k = 1, 2, 3,\ \text{cyclic numbering})\,, \quad D = (\mathbf{a}_1\times\mathbf{a}_2)\cdot\mathbf{a}_3 \tag{66}$$

We note

$$(\mathbf{r}_{n+4} - \mathbf{r}_{n+j})\cdot(\mathbf{r}_{n+4} - \mathbf{r}_{n+j})^\bullet = |\mathbf{r}_{n+4} - \mathbf{r}_{n+j}|\,|\mathbf{r}_{n+4} - \mathbf{r}_{n+j}|^\bullet \tag{67}$$

and hence, indeed, get the recursion formula

$$\dot{\mathbf{r}}_{n+4} \equiv \sum_{m=1}^{3} u_m \mathbf{a}^m \quad \Longrightarrow \quad u_j \equiv \dot{\mathbf{r}}_{n+4} \cdot \mathbf{a}_j = \dot{\mathbf{r}}_{n+j} \cdot \mathbf{a}_j + |\mathbf{a}_j|\,|\mathbf{r}_{n+4} - \mathbf{r}_{n+j}|^\bullet \tag{68}$$

One possible set of starting values of the recursion is

$$\dot{\mathbf{r}}_1 = \mathbf{0}\,, \quad \dot{\mathbf{r}}_2 = \frac{|\mathbf{r}_2 - \mathbf{r}_1|^\bullet}{|\mathbf{r}_2 - \mathbf{r}_1|}\,(\mathbf{r}_2 - \mathbf{r}_1)\,, \tag{69}$$

$$\dot{\mathbf{r}}_3 = \frac{1}{\Delta}\Big((\mathbf{r}_3 - \mathbf{r}_1)\times(\mathbf{r}_3 - \mathbf{r}_2)\Big)\times\Big((\dot{\mathbf{r}}_2\cdot(\mathbf{r}_3 - \mathbf{r}_2) + |\mathbf{r}_3 - \mathbf{r}_2||\mathbf{r}_3 - \mathbf{r}_2|^\bullet)\,(\mathbf{r}_3 - \mathbf{r}_1) - |\mathbf{r}_3 - \mathbf{r}_1||\mathbf{r}_3 - \mathbf{r}_1|^\bullet\,(\mathbf{r}_3 - \mathbf{r}_2)\Big) \tag{70}$$

with $\Delta = |(\mathbf{r}_3 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_2)|^2$ since it satisfies the constraints

$$(\mathbf{r}_k - \mathbf{r}_l) \cdot (\mathbf{r}_k - \mathbf{r}_l)^\bullet = |\mathbf{r}_k - \mathbf{r}_l|\,|\mathbf{r}_k - \mathbf{r}_l|^\bullet\,, \qquad k, l = 1, 2, 3 \tag{71}$$

So we arrive at a set of velocities $\{\dot{\mathbf{r}}_j\}$ of all particles. However, the modified set with

$$\dot{\mathbf{r}}^*_j = \dot{\mathbf{r}}_j + \mathbf{c}^* + \boldsymbol{\beta}^* \times \mathbf{r}_j \tag{72}$$

is also admissible since it does not alter the rates of the distances.

$$(\mathbf{r}_j - \mathbf{r}_k) \cdot (\dot{\mathbf{r}}^*_j - \dot{\mathbf{r}}^*_k) = (\mathbf{r}_j - \mathbf{r}_k) \cdot (\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_k) = |\mathbf{r}_j - \mathbf{r}_k||\mathbf{r}_j - \mathbf{r}_k|^\bullet \tag{73}$$

So the knowledge of the rates of all distances only determines the set of velocities up to the two vectors $\mathbf{c}^*$ and $\boldsymbol{\beta}^*$, *i.e.* a rigid body motion. We infer from eqs. (63), (64)

$$\mathbf{d}^*_C = \frac{1}{2M}\sum_j\sum_k m_j m_k (\mathbf{r}_j - \mathbf{r}_k) \times (\dot{\mathbf{r}}^*_j - \dot{\mathbf{r}}^*_k) = \mathbf{d}_C + \mathbf{J}_C \cdot \boldsymbol{\beta}^* \quad \Longrightarrow \quad \boldsymbol{\omega}^* = \mathbf{J}_C^{-1} \cdot \mathbf{d}^*_C = \boldsymbol{\omega} + \boldsymbol{\beta}^* \tag{74}$$

and then from (65)

$$\begin{aligned} E^*_{\mathrm{kin}} &= \frac{1}{4M}\sum_j\sum_k m_j m_k \Big|\dot{\mathbf{r}}^*_j - \dot{\mathbf{r}}^*_k - \boldsymbol{\omega}^* \times (\mathbf{r}_j - \mathbf{r}_k)\Big|^2 \\ &= \frac{1}{4M}\sum_j\sum_k m_j m_k \Big|\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_k + \boldsymbol{\beta}^* \times (\mathbf{r}_j - \mathbf{r}_k) - (\boldsymbol{\omega} + \boldsymbol{\beta}^*) \times (\mathbf{r}_j - \mathbf{r}_k)\Big|^2 \\ &= \frac{1}{4M}\sum_j\sum_k m_j m_k \Big|\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_k - \boldsymbol{\omega} \times (\mathbf{r}_j - \mathbf{r}_k)\Big|^2 = E_{\mathrm{kin}} \end{aligned} \tag{75}$$

This completes the proof that the kinetic energy of Newtonian mechanics can, in principle, be computed from the knowledge of the rates of the distances between all particles. However, a simple formula of this dependence cannot be given. In the case of an isotropic universe, we obtain $\boldsymbol{\omega}$ from (46) and (64) and arrive at

$$E_{\mathrm{kin}} = \frac{1}{4M}\sum_j\sum_k m_j m_k \Big|(\mathbf{r}_j - \mathbf{r}_k)^\bullet - \frac{3}{4M^2R^2}\Big(\sum_l\sum_n m_l m_n (\mathbf{r}_l - \mathbf{r}_n) \times (\mathbf{r}_l - \mathbf{r}_n)^\bullet\Big) \times (\mathbf{r}_j - \mathbf{r}_k)\Big|^2 \tag{76}$$

We may compare this with the potential energy of the universe due to gravitation

$$E_{\mathrm{pot}} = -\frac{1}{2}\,G\sum_{j\neq k}\sum_k \frac{m^\dagger_j\, m^\dagger_k}{|\mathbf{r}_j - \mathbf{r}_k|} \tag{77}$$

While the potential energy is a polynomial of degree two of the heavy masses, the kinetic energy turns out to be a polynomial of degree six of the inert masses. If the assumption of an isotropic universe is released, then rational functions of the masses occur.

## 11 Historical Remarks

The idea of an absolute space is older than Newtonian mechanics. The dispute whether the earth or the sun is at rest, *i.e.* whether Ptolemaios or Copernicus is right, does not make sense without such a concept. Our investigation reveals that the two ways of thinking are not only kinematically but also kinetically equivalent. This was already pointed out by Mach himself. The serious consequence is that both Galileo's dialogo and his condemnation and publication ban lose their basis.
Mach himself did not present a satisfactory quantitative elaboration of what Einstein later called "Mach's principle". Three authors (Hofmann (1904), Hans Reissner (1914), Schrödinger (1925)) tried to base classical mechanics on a Lagrangean $L \equiv E_{\text{kin}} - E_{\text{pot}}$ with a kinetic energy of the form

$$E_{kin} \propto \sum_j \sum_k m_j m_k f(|\mathbf{r}_j - \mathbf{r}_k|)\Big(|\mathbf{r}_j - \mathbf{r}_k|^{\bullet}\Big)^2 \tag{78}$$

which is a polynomial of degree two of the inert masses. This is the simplest way of introducing the rates of the distances and to ensure Euclidean invariance of $L$. However, it creates a new framework of mechanics, not coincident with Newton's and not in accord with experiments. On the other hand, the Lagrangean of Newtonian mechanics with $E_{\text{kin}}$ according to eqs. (75) or (76) depends on the rates of the distances in a much more intricate manner, which cannot simply be guessed but must be inferred from a reformulation of Newton's equations. This was successfully done by Lynden-Bell (1995) who arrived at the representations (64) and (65).

## 12 Conclusion

It is shown that the inertial force on a material point may be calculated in any frame of reference from the relative motion of this point with respect to all other masses of the universe. Hence we have a quantification of Mach's idea, and Newton's nebulous concept of an absolute space is no longer needed. However, our result is of a purely epistemological character. The engineer will furthermore compute the inertial force from the acceleration against the absolute space — supposed to be known — (or some other inertial frame) and will surely not carry out a summation over billions of stars.
Some physicists argue that a simple reformulation of Newtonian mechanics must not be called Machian since the predictions of Machian mechanics should necessarily differ from those of Newtonian mechanics. This view is not supported by our preceding analysis, which presents a revised representation of classical mechanics that is no longer open to Mach's objections. So it seems justified to call the presented formulation Machian.

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