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On the Generalization of Uniaxial Thermoviscoplasticity with Damage to Finite Deformations Based on Enhanced Rheological Models

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The enhanced concept of rheological models, as proposed in Bröcker and Matzenmiller (2013), is generalized systematically to finite deformations. The basic bodies are defined individually for large deformations, and a rheological network of thermoviscoplasticity is assembled, representing nonlinear isotropic and kinematic hardening as well as an improved description of energy storage in metal plasticity. The constitutive equations are deduced in an analogous procedure as for the uniaxial model in Bröcker and Matzenmiller (2013). Furthermore, damage evolution is additionally accounted for.

1 Introduction

In a recent publication of Bröcker and Matzenmiller (2013), an enhanced concept of rheological networks has been proposed for the constitutive modeling of thermoviscoplasticity. New basic elements are introduced to account for nonlinear isotropic and kinematic hardening as well as to advance the description of energy storage and dissipation during plastic deformations by means of rheological models. Furthermore, damage evolution is represented as well - see Bröcker and Matzenmiller (2012). The yield condition and the viscoplastic flow rule are deduced from the equilibrium of stresses of the rheological network by means of simple algebraic calculations in the framework of thermomechanically consistent material modeling. Moreover, different approaches are given to improve the energy storage behavior by relating storage of inelastic work to ideal plasticity. The subsequent validation agrees well with the experimental test data.

However, the enhanced concept of rheological modeling, as presented by Bröcker and Matzenmiller (2013), is restricted to the uniaxial case only, but already requires a complex structure. Hence, the question consequently arises: How can the one-dimensional constitutive equations be generalized for the spatial application for both small and large deformations. Possible means for enhancing the rheological modeling to a three-dimensional formulation are given by Palmow (1984), Krawietz (1986), Altenbach and Altenbach (1994) - see also Haupt (1996). In particular Lion (1997), Lion and Sedlan (1998), Lion (2000) present a clear and elegant way for transferring the uniaxial equations of a rheological model of thermoviscoplasticity to large deformations. This method is also utilized by Shutov and Ihlemann (2011) and, moreover, enhanced for distortional hardening by Shutov et al. (2011). Based on a rheological network of endochronic viscoplasticity, another novel, attractive procedure is proposed by Kletschkowski et al. (2002, 2005) for the spatial generalization of the uniaxial constitutive model to finite deformations by means of the concept of material isomorphisms of Bertram (1999, 2003).

The paper at hand provides the representation of rheological networks for finite deformations as a straightforward generalization of the enhanced concept of rheological models proposed by Bröcker and Matzenmiller (2013). For this purpose, a modified tensor notation is used in comparison to the literature in order to distinguish easily between the different stress and strain measures, operating on the various configurations of the material body. The basic components of the uniaxial model - such as springs, dashpots, friction elements - are individually generalized as bodies with tensor-valued definition on either the initial or current configuration, which allows for assembling a rheological network of thermoviscoplasticity for finite strains by means of series and parallel connections of the elementary components and, hence, generates the free energy of the entire model. This arrangement of ideal bodies and the concept of dual variables of Haupt and Tsakmakis (1989) form the basis for the derivation of the constitutive material equations in finite deformation viscoplasticity, providing the yield function and the flow rule in an analogous procedure as in the uniaxial case, see Bröcker and Matzenmiller (2013). In this process, the thermomechanical consistency of the model is satisfied. Moreover, in the framework of the concept of effective stresses, introduced by Lemaitre and Chaboche (1990), Lemaitre and Desmorat (2005), the constitutive model is enhanced for damage evolution as well.

The content of this paper is arranged as following: In Section 2, the tensor notation is introduced and the basic rheological bodies are defined for finite deformations. Furthermore, the connection principles of elements in parallel and series arrangement are discussed with their main characteristic relations and a spring-network is considered as a simple example. In the next Section, a rheological network of finite thermoviscoplasticity is assembled, the associated constitutive equations are deduced in the framework of thermomechanically consistent material modeling and transferred to the initial configuration. In Section 5, the material model is enhanced for damage evolution.

2 Basic Rheological Bodies

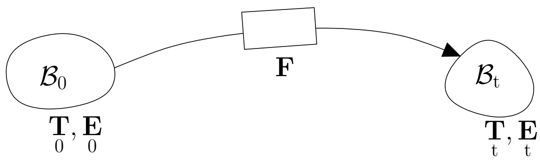

Figure 1: General representation of rheological element of finite deformations

In the first step, a general definition is given for the rheological elements of finite deformations. Considered individually, each basic component is placed between the initial $\mathcal{B}_0$ and the current configuration $\mathcal{B}_{\rm t}$ of the material body $\mathcal{B}$ - see Figure 1. Between both configurations, the deformation gradient $\mathbf{F}$ operates as a two-field tensor and maps material line elements $\mathrm{d}\vec{X}$ from $\mathcal{B}_0$ into spatial ones $\mathrm{d}\vec{x}_{\rm t}$ in the current configuration $\mathcal{B}_{\rm t}$ according to

$$\mathrm{d}\vec{x}_{\rm t} = \mathbf{F}\,\mathrm{d}\vec{X}\,. \tag{1}$$

2.1 Notation and Definitions

Compared to the literature - e.g. Haupt (2002), a modified tensor notation is applied for the stress and strain measures of the various (intermediate-) configurations for reasons of an increased clarity in the subsequent steps of mathematical rearrangements. The symbol $\underset{0}{\mathbf{E}}$ denotes the GREEN strain tensor of the initial configuration $\mathcal{B}_0$ according to $\underset{0}{\mathbf{E}} = \frac{1}{2}\big(\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{1}\big) = \frac{1}{2}\big(\underset{0}{\mathbf{C}} - \mathbf{1}\big)$ with the right CAUCHY-GREEN tensor $\underset{0}{\mathbf{C}} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$. The ALMANSI strain at the current configuration $\mathcal{B}_{\rm t}$ is represented as $\underset{\rm t}{\mathbf{E}} = \frac{1}{2}\big(\mathbf{1} - \mathbf{F}^{-\mathsf{T}}\mathbf{F}^{-1}\big)$ and the transformation property $\underset{0}{\mathbf{E}} = \mathbf{F}^{\mathsf{T}}\underset{\rm t}{\mathbf{E}}\mathbf{F}$ is valid. Moreover, the 2nd PIOLA-KIRCHHOFF stress tensor is represented as $\underset{0}{\mathbf{T}}$ and the KIRCHHOFF stress tensor is denoted as $\underset{\rm t}{\mathbf{T}}$ with the property[1] $\underset{0}{\mathbf{T}} = \mathbf{F}^{-1}\underset{\rm t}{\mathbf{T}}\mathbf{F}^{-\mathsf{T}}$. This means that the subscripted index indicates on which configuration the tensorial quantities operate[2]. Thus, the well-known push-forward and pull-backward operations for the stress and strain measures - see Marsden and Hughes (1994) - may be represented as

$$\underset{0}{\mathbf{E}} = \mathbf{F}^{\mathsf{T}}\underset{\rm t}{\mathbf{E}}\mathbf{F} \qquad , \qquad \underset{\rm t}{\mathbf{E}} = \mathbf{F}^{-\mathsf{T}}\underset{0}{\mathbf{E}}\mathbf{F}^{-1} \tag{2}$$

and

$$\underset{0}{\mathbf{T}} = \mathbf{F}^{-1}\underset{\rm t}{\mathbf{T}}\mathbf{F}^{-\mathsf{T}} \qquad , \qquad \underset{\rm t}{\mathbf{T}} = \mathbf{F}\underset{0}{\mathbf{T}}\mathbf{F}^{\mathsf{T}}\,. \tag{3}$$

The mechanical work w, which is applied to a basic element during the loading process, is either stored as free energy ψ in the rheological component or dissipated and, hence, typically converted into heat. However, the dissipated energy might in general be also consumed e.g. for microstructural changes of the material like endothermic

[1]The superscripts $(.)^{\mathsf{T}}$ and $(.)^{-1}$ indicate the transpose and the inverse of the tensor.

[2]There is no need to indicate that the deformation gradient is operating between the initial $\mathcal{B}_0$ and the current configuration $\mathcal{B}_{\rm t}$. Hence, this two-field tensor is simply denoted with the symbol $\mathbf{F}$ without any subscripted index of associated configurations in order of clarity of the formula. Furthermore, the unity tensor $\mathbf{1}$ is written without any subscripted index of the configuration as well.

phase transformations. The mechanical work may be calculated as the time integral of the stress power in terms of the initial or the current configuration according to

$$w = \frac{1}{\rho_{\mathrm{R}}} \int_0^t \underset{0}{\mathbf{T}} \cdot \underset{0}{\dot{\mathbf{E}}} \, \mathrm{d}\tau = \frac{1}{\rho_{\mathrm{R}}} \int_0^t \underset{\mathrm{t}}{\mathbf{T}} \cdot \underset{\mathrm{t}}{\mathbf{D}} \, \mathrm{d}\tau \,. \tag{4}$$

In (4), ρ_{R} is the mass density of the initial configuration, $\underset{0}{\dot{\mathbf{E}}}$ gives the time rate of the GREEN strain tensor and $\underset{\mathrm{t}}{\mathbf{D}}$ is the strain rate tensor, which is equal to the symmetric part of the spatial velocity gradient $\underset{\mathrm{t}}{\mathbf{L}}$ and to the covariant OLDROYD rate of the ALMANSI strain tensor $\underset{\mathrm{t}}{\overset{\triangle}{\mathbf{E}}}$

$$\underset{\mathrm{t}}{\mathbf{D}} = \frac{1}{2}\big(\underset{\mathrm{t}}{\mathbf{L}} + \underset{\mathrm{t}}{\mathbf{L}}^{\mathsf{T}}\big) = \underset{\mathrm{t}}{\overset{\triangle}{\mathbf{E}}} = \mathbf{F}^{-\mathsf{T}} \underset{0}{\dot{\mathbf{E}}} \mathbf{F}^{-1} \qquad , \qquad \underset{\mathrm{t}}{\mathbf{L}} = \dot{\mathbf{F}} \mathbf{F}^{-1} \,. \tag{5}$$

An arbitrary tensor $\mathbf{A}$ can be decomposed into its spherical and deviatoric fractions

$$\mathbf{A} = \mathbf{A}^{\mathsf{S}} + \mathbf{A}^{\mathsf{D}} \qquad , \qquad \mathbf{A}^{\mathsf{S}} = \frac{1}{3}\,\mathrm{tr}(\mathbf{A})\,\mathbf{1} \qquad , \qquad \mathbf{A}^{\mathsf{D}} = \mathbf{A} - \frac{1}{3}\,\mathrm{tr}(\mathbf{A})\,\mathbf{1} \,, \tag{6}$$

where $\mathrm{tr}(.)$ denotes the trace operator $\mathrm{tr}(\mathbf{A}) = \mathbf{1} \cdot \mathbf{A}$ with $\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}$ representing the scalar product of both tensors $\mathbf{A}$ and $\mathbf{B}$. Furthermore, if the tensor $\mathbf{A}$ is invertible, it may be split multiplicatively into an unimodular (volume preserving) part $\widehat{\mathbf{A}}$ and a volumetric contribution $\mathbf{A}^{\mathsf{V}}$ as

$$\mathbf{A} = \widehat{\mathbf{A}}\,\mathbf{A}^{\mathsf{V}} \qquad , \qquad \widehat{\mathbf{A}} = (\det \mathbf{A})^{-\frac{1}{3}} \mathbf{A} \qquad , \qquad \mathbf{A}^{\mathsf{V}} = (\det \mathbf{A})^{\frac{1}{3}}\,\mathbf{1} \,, \tag{7}$$

which yields the properties of the determinants

$$\det\Big(\widehat{\mathbf{A}}\Big) = 1 \qquad , \qquad \det\Big(\mathbf{A}^{\mathsf{V}}\Big) = \det \mathbf{A} \,. \tag{8}$$

Moreover, the tensor $\mathbf{A}$ can be decomposed according to

$$\mathbf{A} = \|\mathbf{A}\|\,\mathsf{N}(\mathbf{A}) \tag{9}$$

with the definitions of the direction $\mathsf{N}(\mathbf{A})$ and the norm $\|\mathbf{A}\|$ of the tensor $\mathbf{A}$ as[3]

$$\mathsf{N}(\mathbf{A}) = \frac{1}{\|\mathbf{A}\|}\,\mathbf{A} \qquad , \qquad \|\mathbf{A}\| = (\mathbf{A} \cdot \mathbf{A})^{\frac{1}{2}} \,. \tag{10}$$

Note that the operator $\mathsf{N}(\mathbf{A})$ exhibits the properties

$$\|\mathsf{N}(\mathbf{A})\| = 1 \qquad , \qquad \mathsf{N}(\mathbf{A}) \cdot \mathsf{N}(\mathbf{A}) = 1 \tag{11}$$

for every non-zero tensor $\mathbf{A}$. The decomposition (9) and the relations (11) are of key importance for deducing the constitutive equations in the following sections.

Furthermore, the arclength of an arbitrary strain measure $\mathbf{A}$ is defined as

$$\bar{\mathrm{A}} := \sqrt{\frac{2}{3}} \int_0^t \|\dot{\mathbf{A}}\| \, \mathrm{d}\tau = \int_0^t \dot{\bar{\mathrm{A}}} \, \mathrm{d}\tau \qquad , \qquad \dot{\bar{\mathrm{A}}} = \sqrt{\frac{2}{3}}\,\|\dot{\mathbf{A}}\| \,. \tag{12}$$

This yields for the GREEN strain tensor $\underset{0}{\mathbf{E}}$ to the relation

$$\bar{\mathrm{E}} := \sqrt{\frac{2}{3}} \int_0^t \|\underset{0}{\dot{\mathbf{E}}}\| \, \mathrm{d}\tau = \int_0^t \dot{\bar{\mathrm{E}}} \, \mathrm{d}\tau \qquad , \qquad \dot{\bar{\mathrm{E}}} = \sqrt{\frac{2}{3}}\,\|\underset{0}{\dot{\mathbf{E}}}\| \tag{13}$$

and motivates a similar expression for the arclength of the strain rate tensor $\underset{\mathrm{t}}{\mathbf{D}}$ according to

$$\bar{\varepsilon} := \sqrt{\frac{2}{3}} \int_0^t \|\mathbf{F}^{-\mathsf{T}} \underset{0}{\dot{\mathbf{E}}} \mathbf{F}^{-1}\| \, \mathrm{d}\tau = \sqrt{\frac{2}{3}} \int_0^t \|\underset{\mathrm{t}}{\mathbf{D}}\| \, \mathrm{d}\tau = \int_0^t \dot{\bar{\varepsilon}} \, \mathrm{d}\tau \qquad , \qquad \dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3}}\,\|\underset{\mathrm{t}}{\mathbf{D}}\| \,. \tag{14}$$

In the case of rigid body motions $\underset{\mathrm{t}}{\mathbf{L}} = \underset{\mathrm{t}}{\mathbf{\Omega}} = -\underset{\mathrm{t}}{\mathbf{\Omega}}^{\mathsf{T}}$, the strain rate tensor $\underset{\mathrm{t}}{\mathbf{D}}$ vanishes identically, and hence, both arclengths $\bar{\varepsilon}$ and $\bar{\mathrm{E}}$ remain constant.

[3] It is defined that $\mathsf{N}(\mathbf{A}) = \mathbf{0}$ holds for the special case of $\mathbf{A} = \mathbf{0}$.

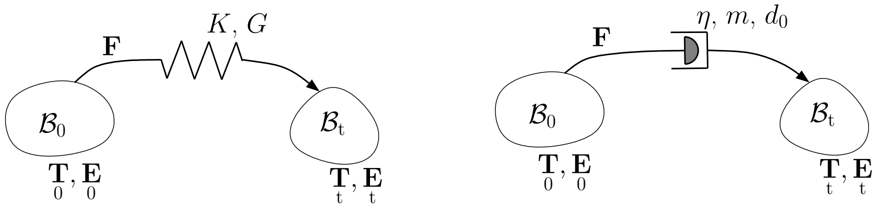

Figure 2: Elastic spring with parameter K, G (left) and nonlinear dashpot with parameter η, m, d_0 (right)

2.2 Basic Elements

In the next step, the rheological basic components, as utilized for the uniaxial case by Bröcker and Matzenmiller (2013), are generalized to finite deformations. For this purpose, each stress relation of the individual elements has to be redefined on either the initial or the current configuration[4]. The graphical representation of each rheological basic element of finite deformation is chosen identically as in the case of the uniaxial model of Bröcker and Matzenmiller (2013).

Elastic Spring

The elastic spring (Figure 2, left) is supposed to capture finite hyperelasticity. The free energy is simply assumed as consisting of a volumetric contribution $U(J)$ taken from Hartmann and Neff (2003) and an isochoric fraction $v(\underset{0}{\widehat{\mathbf{C}}})$ of NEO-HOOKE type as utilized by Shutov and Kreißig (2008) according to

$$\psi = U(J) + v(\underset{0}{\widehat{\mathbf{C}}}) = \psi(\underset{0}{\mathbf{C}}) = \psi(\underset{0}{\mathbf{E}}) \quad , \qquad U(J) = \frac{K}{2\rho_\mathrm{R}}(J-1)^2 \quad , \qquad v(\underset{0}{\widehat{\mathbf{C}}}) = \frac{G}{2\rho_\mathrm{R}}\left(\mathbf{1}\cdot\underset{0}{\widehat{\mathbf{C}}} - 3\right) \tag{15}$$

with the material parameters K and G as well as the abbreviation $J = \det\mathbf{F} = (\det\underset{0}{\mathbf{C}})^{\frac{1}{2}}$ and the unimodular tensor $\underset{0}{\widehat{\mathbf{C}}} = J^{-\frac{2}{3}}\underset{0}{\mathbf{C}}$. The associated stress relation is obtained for the initial configuration $\mathcal{B}_0$ as

$$\underset{0}{\mathbf{T}} = \rho_\mathrm{R}\frac{\partial\psi}{\partial\underset{0}{\mathbf{E}}} = 2\rho_\mathrm{R}\frac{\partial\psi}{\partial\underset{0}{\mathbf{C}}} = \left(K\,J(J-1)\,\mathbf{1} + G\,\underset{0}{\widehat{\mathbf{C}}}^{\mathsf{D}}\right)\underset{0}{\mathbf{C}}^{-1}\,, \tag{16}$$

in which $\underset{0}{\widehat{\mathbf{C}}}^{\mathsf{D}}$ denotes the deviator of the unimodular tensor $\underset{0}{\widehat{\mathbf{C}}}$.

Nonlinear Dashpot

The nonlinear dashpot (Figure 2, right) is generalized, operating on the current configuration with a stress definition in terms of the strain rate tensor according to

$$\underset{\mathrm{t}}{\mathbf{T}} = \sqrt{\frac{2}{3}}\left(\eta\,\|\underset{\mathrm{t}}{\mathbf{D}}\|\right)^{1/m} d_0\,\mathsf{N}\left(\underset{\mathrm{t}}{\mathbf{D}}\right) \tag{17}$$

with the operator $\mathsf{N}(.)$ as given in (10). The parameter η is a kind of viscosity, the exponent m yields nonlinear rate dependency and the parameter d_0 is only used to introduce stress units. All mechanical work, applied to the dashpot, is dissipated completely.

[4]At this point, the choice of the configuration for the definition of the basic elements may initially appear some kind of randomly. However, later on, these component expressions will provide well-known constitutive equations of viscoplasticity.

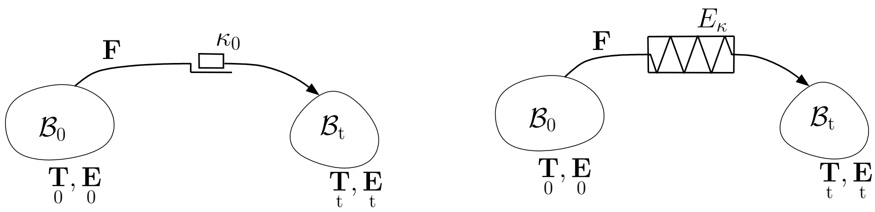

Figure 3: Friction body with initial yield limit κ_0 (left) and hardening element with stiffness E_κ (right)

Friction Element

The friction body (Figure 3, left) is assumed as an energy storing element with the stress relation - cf. Bröcker and Matzenmiller (2013):

$$\begin{aligned} \underset{\mathrm{t}}{\mathbf{T}} &= \sqrt{\tfrac{2}{3}}\,\kappa_0\,\mathsf{N}\Big(\underset{\mathrm{t}}{\mathbf{D}}\Big) && \text{for} \quad \underset{\mathrm{t}}{\mathbf{D}} \neq \mathbf{0}\,, \\ \|\underset{\mathrm{t}}{\mathbf{T}}\| &< \sqrt{\tfrac{2}{3}}\,\kappa_0 && \text{for} \quad \underset{\mathrm{t}}{\mathbf{D}} = \mathbf{0}\,, \end{aligned} \tag{18}$$

where κ_0 gives the initial yield limit. The free energy ψ is obtained by integrating $(4)_2$ and using (9) as well as $(11)_2$ according to

$$\psi = \frac{1}{\rho_\mathrm{R}} \int_0^t \sqrt{\frac{2}{3}}\,\kappa_0\,\mathsf{N}\Big(\underset{\mathrm{t}}{\mathbf{D}}\Big) \cdot \underset{\mathrm{t}}{\mathbf{D}}\,\mathrm{d}\tau = \frac{1}{\rho_\mathrm{R}}\,\kappa_0 \int_0^t \sqrt{\frac{2}{3}}\,\|\underset{\mathrm{t}}{\mathbf{D}}\|\,\mathrm{d}\tau = \frac{1}{\rho_\mathrm{R}}\,\kappa_0\,\bar{\varepsilon} \tag{19}$$

with $\bar{\varepsilon}$ representing the arclength of the strain rate tensor as given in (14).

Remark:
The friction body is typically regarded as a perfectly dissipative element. Hence, most state of art models of viscoplasticity account for energy storage due to elasticity as well as isotropic and kinematic hardening only, which yields an overestimation of dissipation for metals during inelastic deformation. For this reason, some authors - , e.g., Chaboche (1993a,b), Helm (1998, 2006) and others - already assign partial energy storage related to ideal plasticity in order to advance the prediction of energy transformation processes observed experimentally during viscoplastic loading. As demonstrated by Bröcker and Matzenmiller (2013), the non-dissipative friction body allows to model a process-dependent energy storage behavior by arranging such a modified friction component in series connection to the dissipative strain element, specified below. This modeling approach is validated by Bröcker and Matzenmiller (2013), showing a very good agreement with experimental energy transformation data. Note that the same characteristic of energy conversion could be realized alternatively by applying the classical, dissipative friction body in series arrangement with a novel conservative strain element, which could be defined analogous to the dissipative strain body, however, as its counterpart with energy storage.

Hardening Element

The stress of the hardening body (Figure 3, right) is defined as - cf. Bröcker and Matzenmiller (2013):

$$\begin{aligned} \underset{\mathrm{t}}{\mathbf{T}} &= \sqrt{\tfrac{2}{3}}\,E_\kappa\,\bar{\varepsilon}\,\mathsf{N}\Big(\underset{\mathrm{t}}{\mathbf{D}}\Big) && \text{for} \quad \underset{\mathrm{t}}{\mathbf{D}} \neq \mathbf{0}\,, \\ \|\underset{\mathrm{t}}{\mathbf{T}}\| &< \sqrt{\tfrac{2}{3}}\,E_\kappa\,\bar{\varepsilon} && \text{for} \quad \underset{\mathrm{t}}{\mathbf{D}} = \mathbf{0}\,. \end{aligned} \tag{20}$$

The related free energy results from $(4)_2$ and using (9), $(11)_2$ as well as $(14)_4$ to:

$$\psi = \frac{1}{\rho_\mathrm{R}} \int_0^t \sqrt{\frac{2}{3}}\,E_\kappa\,\bar{\varepsilon}\,\mathsf{N}\Big(\underset{\mathrm{t}}{\mathbf{D}}\Big) \cdot \underset{\mathrm{t}}{\mathbf{D}}\,\mathrm{d}\tau = \frac{1}{\rho_\mathrm{R}} E_\kappa \int_0^t \bar{\varepsilon}\,\dot{\bar{\varepsilon}}\,\mathrm{d}\tau = \frac{1}{\rho_\mathrm{R}} E_\kappa \int_0^{\bar{\varepsilon}} \bar{\epsilon}\,\mathrm{d}\bar{\epsilon} = \frac{1}{2\rho_\mathrm{R}} E_\kappa\,\bar{\varepsilon}^2\,. \tag{21}$$

Dissipative Strain Element

The dissipative strain element (Figure 4, left) serves for limiting the strain and the associated storage of energy in other rheological component - see Bröcker and Matzenmiller (2013). Hence, it is defined that all mechanical

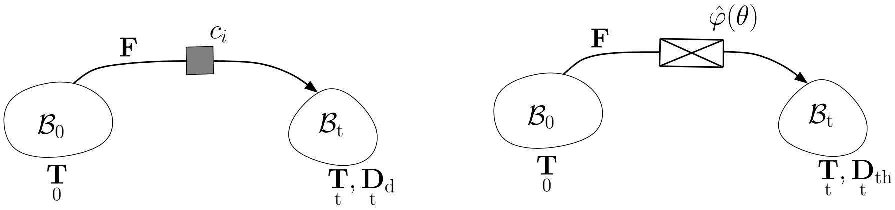

Figure 4: Dissipative strain element with internal deformation rate $\underset{\mathrm{t}}{\mathbf{D}}_{\mathrm{d}}$ and parameters c_i (left), thermal strain element with associated deformation rate $\underset{\mathrm{t}}{\mathbf{D}}_{\mathrm{th}}$ and general function $\hat{\varphi}(\theta)$ specifying thermal deformation (right)

work, applied to this ideal body, is dissipated completely. Moreover, the element is assumed to transfer the stress state $\underset{\mathrm{t}}{\mathbf{T}}$ due to external loads. However, the internal strains, evolving in this rheological component, are exclusively controlled by an evolution equation of the structure

$$\underset{\mathrm{t}}{\mathbf{D}}_{\mathrm{d}} = \underset{\mathrm{t}}{\boldsymbol{\zeta}}(\underset{\mathrm{t}}{\mathbf{T}}, a_j, ...)\,, \tag{22}$$

which may in general be a function of the stress state $\underset{\mathrm{t}}{\mathbf{T}}$ as well as other internal variables a_j of the material model with the associated parameters c_i. Appropriate approaches (22) will be constructed later on.

Thermal Strain Element

In the case of purely mechanical loading, the thermal strain element (Figure 4, right) only transfers the external load like a rigid rod. However, the rheological component responds to every change of the temperature θ with a strain rate tensor $\underset{\mathrm{t}}{\mathbf{D}}_{\mathrm{th}}$ according to

$$\underset{\mathrm{t}}{\mathbf{D}}_{\mathrm{th}} = \underset{\mathrm{t}}{\hat{\boldsymbol{\alpha}}}_{\mathrm{th}}\,\dot{\theta}\,. \tag{23}$$

It is reasonably assumed that a thermal vorticity $\underset{\mathrm{t}}{\mathbf{W}}_{\mathrm{th}} = -\underset{\mathrm{t}}{\mathbf{W}}_{\mathrm{th}}^{\mathsf{T}}$ does not occur due to temperature changes, i.e. the properties $\underset{\mathrm{t}}{\mathbf{W}}_{\mathrm{th}} = \mathbf{0}$ and

$$\underset{\mathrm{t}}{\mathbf{L}}_{\mathrm{th}} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \underset{\mathrm{t}}{\mathbf{D}}_{\mathrm{th}} \tag{24}$$

hold for the spacial velocity gradient $\underset{\mathrm{t}}{\mathbf{L}}_{\mathrm{th}}$ of this rheological component. Assuming an isotropic state of thermal deformation

$$\mathbf{F} = \hat{\varphi}(\theta)\,\mathbf{1}\,, \tag{25}$$

the tensor of thermal expansion[5] $\underset{\mathrm{t}}{\hat{\boldsymbol{\alpha}}}_{\mathrm{th}}$ in (23) results with (24) and (25) as

$$\underset{\mathrm{th}}{\hat{\boldsymbol{\alpha}}}_{\mathrm{th}} = \frac{\partial_\theta \hat{\varphi}(\theta)}{\hat{\varphi}(\theta)}\,\mathbf{1}\,. \tag{26}$$

The function of thermal deformation $\hat{\varphi}(\theta)$ may be characterized e.g. by means of the simple expression

$$\hat{\varphi}(\theta) = 1 + \hat{\alpha}(\theta)(\theta - \theta_0)\,. \tag{27}$$

The free energy, stored in the thermal strain element due to temperature changes, is introduced as a general function of the temperature according to

$$\psi = \hat{Q}(\theta)\,. \tag{28}$$

Remark:
Notice that both the dissipative and the thermal strain element may not be deformed arbitrarily - neither stress nor strain controlled, since the state of deformation in these components is completely governed by the evolution equations (23) and (22).

[5]The small hat ($\hat{}$) on top of the variables indicates their temperature dependency.

3 Network Assemblages

The basic elements of finite deformation may be also assembled in series or in parallel connection in order to construct networks for modeling complex material behavior.

3.1 Parallel Connection

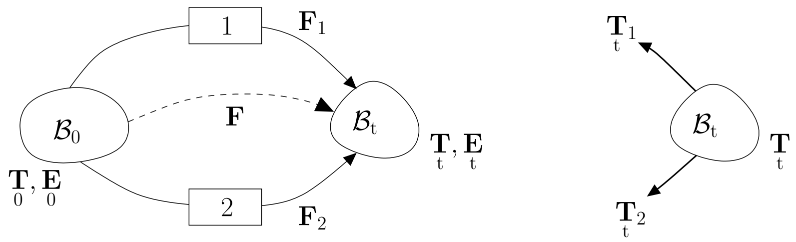

Figure 5: Parallel connection of basic elements 1 and 2 of finite deformations (left) with associated stress decomposition at current configuration $\mathcal{B}_\mathrm{t}$ (right)

Figure 5 shows a sketch of two basic elements arranged in parallel connection. The deformation gradient in both individual elements 1 and 2 is equal to the total one

$$\mathbf{F} = \mathbf{F}_1 = \mathbf{F}_2 \,, \tag{29}$$

which means that the strain in both rheological components is equal to the total strain in the network as well:

$$\underset{0}{\mathbf{E}} = \underset{0}{\mathbf{E}}_1 = \underset{0}{\mathbf{E}}_2 \qquad , \qquad \underset{\mathrm{t}}{\mathbf{E}} = \underset{\mathrm{t}}{\mathbf{E}}_1 = \underset{\mathrm{t}}{\mathbf{E}}_2 \,. \tag{30}$$

However, the total stress at the initial and the current configuration is decomposed additively into both contributions from the individual elements 1 and 2 according to

$$\underset{0}{\mathbf{T}} = \underset{0}{\mathbf{T}}_1 + \underset{0}{\mathbf{T}}_2 \qquad , \qquad \underset{\mathrm{t}}{\mathbf{T}} = \underset{\mathrm{t}}{\mathbf{T}}_1 + \underset{\mathrm{t}}{\mathbf{T}}_2 \,. \tag{31}$$

3.2 Series Connection

In the following, the concept of dual variables (Haupt and Tsakmakis, 1989; Haupt, 2002) is extensively used to obtain the kinematical, kinetic and energetic relations for the various configurations of the material body $\mathcal{B}$.

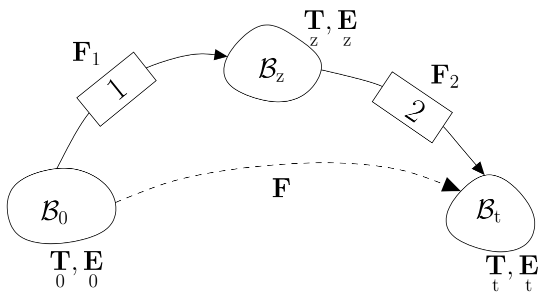

Figure 6: Series connection of two basic elements 1 and 2 of finite deformations

For the series connection of two basic components 1 and 2 (Figure 6), a multiplicative decomposition of the total deformation gradient $\mathbf{F}$ is assumed with the contributions $\mathbf{F}_1$ and $\mathbf{F}_2$ related to both individual elements - cf. Haupt (2002), Lion (2000), Landgraf and Ihlemann (2012):

$$\mathbf{F} = \mathbf{F}_2\mathbf{F}_1 \,, \tag{32}$$

which implies an intermediate configuration $\mathcal{B}_\mathrm{z}$ between the initial configuration $\mathcal{B}_0$ and the current one $\mathcal{B}_\mathrm{t}$. Hence, the transformation (1) turns into $\mathrm{d}\vec{x}_\mathrm{t} = \mathbf{F}_2\mathbf{F}_1\,\mathrm{d}\vec{X} = \mathbf{F}_2\,\mathrm{d}\vec{x}_\mathrm{z}$ with $\mathrm{d}\vec{x}_\mathrm{z} = \mathbf{F}_1\,\mathrm{d}\vec{X}$ representing the mapping of the material line element $\mathrm{d}\vec{X}$ into the intermediate configuration $\mathcal{B}_\mathrm{z}$.

The total strain tensor $\underset{\mathrm{z}}{\mathbf{E}}$ of the intermediate configuration $\mathcal{B}_\mathrm{z}$ results by means of the push-forward or pull-backward operations (2) as

$$\underset{\mathrm{z}}{\mathbf{E}} = \mathbf{F}_2^\mathsf{T} \underset{\mathrm{t}}{\mathbf{E}} \mathbf{F}_2 = \mathbf{F}_1^{-\mathsf{T}} \underset{0}{\mathbf{E}} \mathbf{F}_1^{-1} \tag{33}$$

and the multiplicative decomposition (32) directly yields an additive split of the strain state $\underset{\mathrm{z}}{\mathbf{E}}$ according to

$$\underset{\mathrm{z}}{\mathbf{E}} = \frac{1}{2}\big(\mathbf{F}_1^{-\mathsf{T}}\mathbf{F}_1^{-1} + \mathbf{F}_2^\mathsf{T}\mathbf{F}_2\big) = \underset{\mathrm{z}}{\mathbf{E}}_1 + \underset{\mathrm{z}}{\mathbf{E}}_2 \tag{34}$$

with contributions of $\underset{\mathrm{z}}{\mathbf{E}}_1$ as ALMANSI and $\underset{\mathrm{z}}{\mathbf{E}}_2$ as GREEN strain tensor related to the individual basic components 1 and 2

$$\underset{\mathrm{z}}{\mathbf{E}}_1 = \frac{1}{2}\big(\mathbf{1} - \mathbf{F}_1^{-\mathsf{T}}\mathbf{F}_1^{-1}\big) \qquad , \qquad \underset{\mathrm{z}}{\mathbf{E}}_2 = \frac{1}{2}\big(\mathbf{F}_2^\mathsf{T}\mathbf{F}_2 - \mathbf{1}\big)\,. \tag{35}$$

Similarly, the total GREEN and the ALMANSI strain tensors are decomposed additively on $\mathcal{B}_0$ and $\mathcal{B}_\mathrm{t}$ into

$$\underset{0}{\mathbf{E}} = \underset{0}{\mathbf{E}}_1 + \underset{0}{\mathbf{E}}_2 \qquad , \qquad \underset{\mathrm{t}}{\mathbf{E}} = \underset{\mathrm{t}}{\mathbf{E}}_1 + \underset{\mathrm{t}}{\mathbf{E}}_2 \tag{36}$$

with the contributions related to the basic elements 1 and 2 on $\mathcal{B}_0$

$$\underset{0}{\mathbf{E}}_1 = \mathbf{F}_1^\mathsf{T}\underset{\mathrm{z}}{\mathbf{E}}_1\mathbf{F}_1 = \frac{1}{2}\big(\mathbf{F}_1^\mathsf{T}\mathbf{F}_1 - \mathbf{1}\big) \qquad , \qquad \underset{0}{\mathbf{E}}_2 = \mathbf{F}_1^\mathsf{T}\underset{\mathrm{z}}{\mathbf{E}}_2\mathbf{F}_1 = \frac{1}{2}\big(\mathbf{F}^\mathsf{T}\mathbf{F} - \mathbf{F}_1^\mathsf{T}\mathbf{F}_1\big) \tag{37}$$

and on $\mathcal{B}_\mathrm{t}$

$$\underset{\mathrm{t}}{\mathbf{E}}_1 = \mathbf{F}_2^{-\mathsf{T}}\underset{\mathrm{z}}{\mathbf{E}}_1\mathbf{F}_2^{-1} = \frac{1}{2}\big(\mathbf{F}_2^{-\mathsf{T}}\mathbf{F}_2^{-1} - \mathbf{F}^{-\mathsf{T}}\mathbf{F}^{-1}\big) \qquad , \qquad \underset{\mathrm{t}}{\mathbf{E}}_2 = \mathbf{F}_2^{-\mathsf{T}}\underset{\mathrm{z}}{\mathbf{E}}_2\mathbf{F}_2^{-1} = \frac{1}{2}\big(\mathbf{1} - \mathbf{F}_2^{-\mathsf{T}}\mathbf{F}_2^{-1}\big)\,. \tag{38}$$

However, the total stress state is equal to the stresses in the basic elements 1 and 2

$$\underset{0}{\mathbf{T}} = \underset{0}{\mathbf{T}}_1 = \underset{0}{\mathbf{T}}_2 \qquad , \qquad \underset{\mathrm{z}}{\mathbf{T}} = \underset{\mathrm{z}}{\mathbf{T}}_1 = \underset{\mathrm{z}}{\mathbf{T}}_2 \qquad , \qquad \underset{\mathrm{t}}{\mathbf{T}} = \underset{\mathrm{t}}{\mathbf{T}}_1 = \underset{\mathrm{t}}{\mathbf{T}}_2 \tag{39}$$

and the stress tensor $\underset{\mathrm{z}}{\mathbf{T}}$ of the intermediate configuration $\mathcal{B}_\mathrm{z}$ is obtained by means of the push-forward or pull-back operations (3) according to

$$\underset{\mathrm{z}}{\mathbf{T}} = \mathbf{F}_2^{-1}\underset{\mathrm{t}}{\mathbf{T}}\mathbf{F}_2^{-\mathsf{T}} = \mathbf{F}_1\underset{0}{\mathbf{T}}\mathbf{F}_1^\mathsf{T}\,. \tag{40}$$

The multiplicative decomposition (32), moreover, provides an additive split of the spatial velocity gradient $\underset{\mathrm{t}}{\mathbf{L}}$ into - see Marin et al. (2006) -

$$\underset{\mathrm{t}}{\mathbf{L}} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{F}}_2\mathbf{F}_2^{-1} + \mathbf{F}_2\dot{\mathbf{F}}_1\mathbf{F}_1^{-1}\mathbf{F}_2^{-1} = \underset{\mathrm{t}}{\mathbf{L}}_2 + \underset{\mathrm{t}}{\mathbf{L}}_1 \tag{41}$$

with the contribution of both individual elements 1 and 2:

$$\underset{\mathrm{t}}{\mathbf{L}}_2 = \dot{\mathbf{F}}_2\mathbf{F}_2^{-1} \qquad , \qquad \underset{\mathrm{t}}{\mathbf{L}}_1 = \mathbf{F}_2\underset{\mathrm{z}}{\mathbf{L}}_1\mathbf{F}_2^{-1} \qquad , \qquad \underset{\mathrm{z}}{\mathbf{L}}_1 = \dot{\mathbf{F}}_1\mathbf{F}_1^{-1}\,. \tag{42}$$

Note that the second relation of (42) represents the deformation rate $\underset{\mathrm{z}}{\mathbf{L}}_1$ of the element 1 from the intermediate configuration $\mathcal{B}_\mathrm{z}$, which is transferred to the current configuration $\mathcal{B}_\mathrm{t}$ and, thus, denoted as $\underset{\mathrm{t}}{\mathbf{L}}_1$. Motivated by the various expressions $\underset{\mathrm{i}}{\mathbf{L}}_j$ of the configurations $\mathcal{B}_\mathrm{j}$, the operator

$$\underset{\mathrm{i}}{\mathbf{D}}_j = \mathsf{sym}\Big(\underset{\mathrm{i}}{\mathbf{L}}_j\Big) = \frac{1}{2}\big(\underset{\mathrm{i}}{\mathbf{L}}_j + \underset{\mathrm{i}}{\mathbf{L}}_j^\mathsf{T}\big) \tag{43}$$

is defined as the symmetric part of each measure $\underset{\mathrm{i}}{\mathbf{L}}_j$.

Applying $(41)_3$, the strain rate tensor $\underset{\mathrm{t}}{\mathbf{D}}$ decomposes additively

$$\underset{\mathrm{t}}{\mathbf{D}} = \frac{1}{2}\big(\underset{\mathrm{t}}{\mathbf{L}} + \underset{\mathrm{t}}{\mathbf{L}}^\mathsf{T}\big) = \underset{\mathrm{t}}{\mathbf{D}}_2 + \underset{\mathrm{t}}{\mathbf{D}}_1 \tag{44}$$

into its summands

$$\underset{\mathrm{t}}{\mathbf{D}}_2 = \frac{1}{2}\big(\underset{\mathrm{t}}{\mathbf{L}}_2 + \underset{\mathrm{t}}{\mathbf{L}}_2^{\mathsf{T}}\big) \qquad , \qquad \underset{\mathrm{t}}{\mathbf{D}}_1 = \frac{1}{2}\big(\underset{\mathrm{t}}{\mathbf{L}}_1 + \underset{\mathrm{t}}{\mathbf{L}}_1^{\mathsf{T}}\big)\,. \tag{45}$$

Pulling back the strain rate tensor (44) onto the intermediate configuration $\mathcal{B}_{\mathrm{z}}$ yields the covariant OLDROYD rate of the strain tensor $\underset{\mathrm{z}}{\mathbf{E}}$ according to

$$\overset{\triangle}{\underset{\mathrm{z}}{\mathbf{E}}} = \mathbf{F}_1^{-\mathsf{T}}\dot{\underset{0}{\mathbf{E}}}\mathbf{F}_1^{-1} = \mathbf{F}_2^{\mathsf{T}}\underset{\mathrm{t}}{\mathbf{D}}\mathbf{F}_2 = \mathbf{F}_2^{\mathsf{T}}\underset{\mathrm{t}}{\mathbf{D}}_2\mathbf{F}_2 + \mathbf{F}_2^{\mathsf{T}}\underset{\mathrm{t}}{\mathbf{D}}_1\mathbf{F}_2 \tag{46}$$

with the relations

$$\mathbf{F}_2^{\mathsf{T}}\underset{\mathrm{t}}{\mathbf{D}}_2\mathbf{F}_2 = \dot{\underset{\mathrm{z}}{\mathbf{E}}}_2 \qquad , \qquad \mathbf{F}_2^{\mathsf{T}}\underset{\mathrm{t}}{\mathbf{D}}_1\mathbf{F}_2 = \frac{1}{2}\big(\underset{\mathrm{z}}{\mathbf{C}}_2\underset{\mathrm{z}}{\mathbf{L}}_1 + \underset{\mathrm{z}}{\mathbf{L}}_1^{\mathsf{T}}\underset{\mathrm{z}}{\mathbf{C}}_2\big) \tag{47}$$

and the CAUCHY-GREEN type tensor $\underset{\mathrm{z}}{\mathbf{C}}_2 = \mathbf{F}_2^{\mathsf{T}}\mathbf{F}_2$.

Moreover, the special case is considered, when the velocity gradient is a deviator $\underset{\mathrm{t}}{\mathbf{L}} = \underset{\mathrm{t}}{\mathbf{L}}^{\mathsf{D}}$. Inserting $(41)_3$ into the definition of the deviator $(6)_3$ yields the property that $\underset{\mathrm{t}}{\mathbf{L}}^{\mathsf{D}}$ decomposes additively into two deviatoric contributions

$$\underset{\mathrm{t}}{\mathbf{L}}^{\mathsf{D}} = (\underset{\mathrm{t}}{\mathbf{L}}_2 + \underset{\mathrm{t}}{\mathbf{L}}_1) - \frac{1}{3}\Big(\mathbf{1}\cdot(\underset{\mathrm{t}}{\mathbf{L}}_2 + \underset{\mathrm{t}}{\mathbf{L}}_1)\Big)\,\mathbf{1} = \underset{\mathrm{t}}{\mathbf{L}}_2^{\mathsf{D}} + \underset{\mathrm{t}}{\mathbf{L}}_1^{\mathsf{D}} \tag{48}$$

associated with both individual elements 1 and 2. Similarly, the symmetric part $\underset{\mathrm{t}}{\mathbf{D}}^{\mathsf{D}}$ of the deviator $\underset{\mathrm{t}}{\mathbf{L}}^{\mathsf{D}}$ splits into

$$\underset{\mathrm{t}}{\mathbf{D}}^{\mathsf{D}} = \frac{1}{2}\big(\underset{\mathrm{t}}{\mathbf{L}}^{\mathsf{D}} + \underset{\mathrm{t}}{\mathbf{L}}^{\mathsf{D}\mathsf{T}}\big) = \underset{\mathrm{t}}{\mathbf{D}}_2^{\mathsf{D}} + \underset{\mathrm{t}}{\mathbf{D}}_1^{\mathsf{D}} \tag{49}$$

with two deviatoric, symmetric contributions of both elements 1 and 2

$$\underset{\mathrm{t}}{\mathbf{D}}_2^{\mathsf{D}} = \frac{1}{2}\big(\underset{\mathrm{t}}{\mathbf{L}}_2^{\mathsf{D}} + \underset{\mathrm{t}}{\mathbf{L}}_2^{\mathsf{D}\mathsf{T}}\big) \qquad , \qquad \underset{\mathrm{t}}{\mathbf{D}}_1^{\mathsf{D}} = \frac{1}{2}\big(\underset{\mathrm{t}}{\mathbf{L}}_1^{\mathsf{D}} + \underset{\mathrm{t}}{\mathbf{L}}_1^{\mathsf{D}\mathsf{T}}\big) \tag{50}$$

The deviator $\underset{\mathrm{t}}{\mathbf{L}}_1^{\mathsf{D}}$ may also be rewritten in terms of the transformation $(42)_2$ according to

$$\underset{\mathrm{t}}{\mathbf{L}}_1^{\mathsf{D}} = \mathbf{F}_2\underset{\mathrm{z}}{\mathbf{L}}_1\mathbf{F}_2^{-1} - \frac{1}{3}\,(\mathbf{1}\cdot\mathbf{F}_2\underset{\mathrm{z}}{\mathbf{L}}_1\mathbf{F}_2^{-1})\,\underset{\mathrm{t}}{\mathbf{1}} = \mathbf{F}_2\underset{\mathrm{z}}{\mathbf{L}}_1^{\mathsf{D}}\mathbf{F}_2^{-1}\,, \tag{51}$$

which states that the deviator $\underset{\mathrm{t}}{\mathbf{L}}_1^{\mathsf{D}}$ of the current configuration $\mathcal{B}_{\mathrm{t}}$ entails the deviator $\underset{\mathrm{z}}{\mathbf{L}}_1^{\mathsf{D}}$ on the intermediate configuration $\mathcal{B}_{\mathrm{z}}$ and vice versa.

By means of the kinematical and kinetic relations obtained above, the internal stress power l may be considered next for the case of the series connection of two basic elements:

$$l = \frac{1}{\rho_{\mathrm{R}}}\underset{0}{\mathbf{T}}\cdot\dot{\underset{0}{\mathbf{E}}} = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{t}}{\mathbf{T}}\cdot\underset{\mathrm{t}}{\mathbf{D}} = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{t}}{\mathbf{T}}\cdot\underset{\mathrm{t}}{\mathbf{L}} = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{t}}{\mathbf{T}}\cdot\underset{\mathrm{t}}{\mathbf{L}}_2 + \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{t}}{\mathbf{T}}\cdot\underset{\mathrm{t}}{\mathbf{L}}_1 = l_2 + l_1\,, \tag{52}$$

i.e. the total internal stress power l also decomposes into the fractions l_1 and l_2 of both individual basic components. Note, due to the symmetry property $\underset{\mathrm{t}}{\mathbf{T}} = \underset{\mathrm{t}}{\mathbf{T}}^{\mathsf{T}}$ of the KIRCHHOFF stress tensor, only the symmetric fraction of $\underset{\mathrm{t}}{\mathbf{L}}$ gives a contribution to the scalar product $\underset{\mathrm{t}}{\mathbf{T}}\cdot\underset{\mathrm{t}}{\mathbf{L}}$. Thus, in eq. (52), $\underset{\mathrm{t}}{\mathbf{D}}$ may be replaced by $\underset{\mathrm{t}}{\mathbf{L}}$ and vice versa. The first summand of $(52)_4$ can be transferred to the intermediate configuration $\mathcal{B}_{\mathrm{z}}$ with (40) and (47) as

$$l_2 = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{t}}{\mathbf{T}}\cdot\underset{\mathrm{t}}{\mathbf{L}}_2 = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{t}}{\mathbf{T}}\cdot\underset{\mathrm{t}}{\mathbf{D}}_2 = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{z}}{\mathbf{T}}\cdot\dot{\underset{\mathrm{z}}{\mathbf{E}}}_2\,. \tag{53}$$

Moreover, the second summand of $(52)_4$ may be rearranged using (40) and $(42)_2$ as

$$l_1 = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{t}}{\mathbf{T}}\cdot\underset{\mathrm{t}}{\mathbf{L}}_1 = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{t}}{\mathbf{T}}\cdot\underset{\mathrm{t}}{\mathbf{D}}_1 \tag{54}$$

$$= \frac{1}{\rho_{\mathrm{R}}}\mathbf{F}_2\underset{\mathrm{z}}{\mathbf{T}}\mathbf{F}_2^{\mathsf{T}}\cdot\mathbf{F}_2\underset{\mathrm{z}}{\mathbf{L}}_1\mathbf{F}_2^{-1} = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{z}}{\mathbf{C}}_2\underset{\mathrm{z}}{\mathbf{T}}\cdot\underset{\mathrm{z}}{\mathbf{L}}_1 = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{z}}{\mathbf{M}}\cdot\underset{\mathrm{z}}{\mathbf{L}}_1 \tag{55}$$

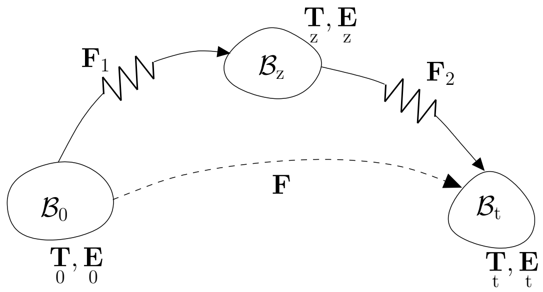

Figure 7: Series connection of two elastic springs

with the definition of the MANDEL stress tensor

$$\underset{z}{\mathbf{M}} = \underset{z}{\mathbf{C}}_2 \underset{z}{\mathbf{T}} \,. \tag{56}$$

In the special cases of $\underset{z}{\mathbf{M}} = \underset{z}{\mathbf{M}}^{\mathsf{T}}$, i.e. if $\underset{z}{\mathbf{T}}$ and $\underset{z}{\mathbf{C}}_2$ are coaxial to each other, the velocity gradient $\underset{z}{\mathbf{L}}_1$ may be replaced in (55) by its symmetric part $\underset{z}{\mathbf{D}}_1 = \overset{\triangle}{\underset{z}{\mathbf{E}}}_1 = \mathbf{F}_1^{-\mathsf{T}} \, \underset{0}{\dot{\mathbf{E}}}_1 \, \mathbf{F}_1^{-1}$ - cf. (5) - and, hence, the stress power l_1 can be further converted according to

$$l_1 = \frac{1}{\rho_R} \underset{z}{\mathbf{M}} \cdot \underset{z}{\mathbf{L}}_1 = \frac{1}{\rho_R} \underset{z}{\mathbf{M}} \cdot \underset{z}{\mathbf{D}}_1 = \frac{1}{\rho_R} \mathbf{F}_1^{-1} \, \underset{z}{\mathbf{M}} \, \mathbf{F}_1^{-\mathsf{T}} \cdot \underset{0}{\dot{\mathbf{E}}}_1 = \frac{1}{\rho_R} \underset{0}{\mathbf{M}} \cdot \underset{0}{\dot{\mathbf{E}}}_1 \tag{57}$$

with the transformation - cf. (3) - $\underset{0}{\mathbf{M}} = \mathbf{F}_1^{-1} \, \underset{z}{\mathbf{M}} \, \mathbf{F}_1^{-\mathsf{T}} = \underset{0}{\mathbf{C}}_1^{-1} \underset{0}{\mathbf{C}} \underset{0}{\mathbf{T}}$ of the MANDEL stress tensor $\underset{z}{\mathbf{M}}$ to the initial configuration $\mathcal{B}_0$ and the CAUCHY-GREEN type tensors $\underset{0}{\mathbf{C}}_1 = \mathbf{F}_1^{\mathsf{T}} \mathbf{F}_1$ as well as $\underset{0}{\mathbf{C}} = \mathbf{F}^{\mathsf{T}} \mathbf{F}$.

Landgraf and Ihlemann (2012) have presented a similar approach for the connection of finite rheological elements based on the multiplicative decomposition of the deformation gradient (32). They have postulated the balance of the stress power for the initial configuration in order to derive the equilibrium of stresses. However, in the paper at hand, the balance of stresses (39) is assumed, and hence, the decomposition of the stress power (52) is the outcome.

3.3 Example: Series Connection of Two Elastic Springs

Two elastic springs with different stiffness properties are connected in series - see Figure 7.

According to the general definition of a hyperelastic spring, its free energy is assumed as a function of the GREEN strain tensor $\underset{0}{\mathbf{E}} = \frac{1}{2}\big(\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{1}\big)$. That means, the free energy ψ of the entire network of springs in Figure 7 is obtained as a function of both GREEN strain type tensors

$$\underset{0}{\mathbf{E}}_1 = \frac{1}{2}\big(\, \mathbf{F}_1^{\mathsf{T}} \mathbf{F}_1 - \mathbf{1} \big) \quad , \qquad \underset{z}{\mathbf{E}}_2 = \frac{1}{2}\big(\, \mathbf{F}_2^{\mathsf{T}} \mathbf{F}_2 - \mathbf{1} \big) \tag{58}$$

according to

$$\psi = \psi_1\left(\underset{0}{\mathbf{E}}_1\right) + \psi_2\left(\underset{z}{\mathbf{E}}_2\right) = \psi\left(\underset{0}{\mathbf{E}}_1, \underset{z}{\mathbf{E}}_2\right) \,. \tag{59}$$

The temporal rate of the free energy is

$$\dot{\psi} = \frac{\partial \psi}{\partial \underset{0}{\mathbf{E}}_1} \cdot \underset{0}{\dot{\mathbf{E}}}_1 + \frac{\partial \psi}{\partial \underset{z}{\mathbf{E}}_2} \cdot \underset{z}{\dot{\mathbf{E}}}_2 \,. \tag{60}$$

In the case of isothermal conditions and by neglecting heat conduction, the internal dissipation δ results as - see Haupt (2002), p. 511 -

$$\delta = -\dot{\psi} + \frac{1}{\rho_R} \underset{0}{\mathbf{T}} \cdot \underset{0}{\dot{\mathbf{E}}} \geq 0 \tag{61}$$

and has to be non-negative for any admissible process.

Assuming that both tensors $\underset{\mathrm{z}}{\mathbf{C}}_2$ and $\underset{\mathrm{z}}{\mathbf{T}}$ are coaxial[6], the total stress power l of the system in Fig. 7 may be decomposed according to $(52)_5$, (53) and (57) into

$$l = \frac{1}{\rho_{\mathrm{R}}} \underset{0}{\mathbf{T}} \cdot \underset{0}{\dot{\mathbf{E}}} = \frac{1}{\rho_{\mathrm{R}}} \left(\underset{\mathrm{z}}{\mathbf{T}} \cdot \underset{\mathrm{z}}{\dot{\mathbf{E}}}_2 + \underset{0}{\mathbf{M}} \cdot \underset{0}{\dot{\mathbf{E}}}_1 \right) . \tag{62}$$

With (60) and (62), the internal dissipation (61) is evaluated as

$$\delta = \left(\frac{1}{\rho_{\mathrm{R}}} \underset{\mathrm{z}}{\mathbf{T}} - \frac{\partial \psi}{\partial \underset{\mathrm{z}}{\mathbf{E}}_2} \right) \cdot \underset{\mathrm{z}}{\dot{\mathbf{E}}}_2 + \left(\frac{1}{\rho_{\mathrm{R}}} \underset{0}{\mathbf{M}} - \frac{\partial \psi}{\partial \underset{0}{\mathbf{E}}_1} \right) \cdot \underset{0}{\dot{\mathbf{E}}}_1 \geq 0 \tag{63}$$

and yields for arbitrary rates of $\underset{0}{\dot{\mathbf{E}}}_1$ and $\underset{\mathrm{z}}{\dot{\mathbf{E}}}_2$ the potential properties for the stress on the intermediate configuration $\mathcal{B}_{\mathrm{z}}$ and the KIRCHHOFF stress on the current one $\mathcal{B}_{\mathrm{t}}$

$$\underset{\mathrm{z}}{\mathbf{T}} = \rho_{\mathrm{R}} \frac{\partial \psi}{\partial \underset{\mathrm{z}}{\mathbf{E}}_2} \qquad \text{or} \qquad \underset{\mathrm{t}}{\mathbf{T}} = \rho_{\mathrm{R}}\, \mathbf{F}_2 \frac{\partial \psi}{\partial \underset{\mathrm{z}}{\mathbf{E}}_2} \mathbf{F}_2^{\mathsf{T}} \tag{64}$$

as well as to the MANDEL type stress tensors on the initial $\mathcal{B}_0$ and the intermediate configuration $\mathcal{B}_{\mathrm{z}}$

$$\underset{0}{\mathbf{M}} = \rho_{\mathrm{R}} \frac{\partial \psi}{\partial \underset{0}{\mathbf{E}}_1} \qquad \text{or} \qquad \underset{\mathrm{z}}{\mathbf{M}} = \rho_{\mathrm{R}}\, \mathbf{F}_1 \frac{\partial \psi}{\partial \underset{0}{\mathbf{E}}_1} \mathbf{F}_1^{\mathsf{T}} . \tag{65}$$

If the free energy $\psi(\underset{0}{\mathbf{E}}_1)$ is an isotropic tensor function, e.g. as in (15), the relation (64) leads to the property that $\underset{\mathrm{z}}{\mathbf{C}}_2$ and $\underset{\mathrm{z}}{\mathbf{T}}$ are coaxial to each other, i.e. $\underset{\mathrm{z}}{\mathbf{M}} = \underset{\mathrm{z}}{\mathbf{C}}_2\, \underset{\mathrm{z}}{\mathbf{T}} = \underset{\mathrm{z}}{\mathbf{M}}^{\mathsf{T}}$ and $\underset{0}{\mathbf{M}} = \mathbf{F}_1^{-1}\, \underset{\mathrm{z}}{\mathbf{M}}\, \mathbf{F}_1^{-\mathsf{T}} = \underset{0}{\mathbf{M}}^{\mathsf{T}}$ hold. It remains $\delta = 0$ from the internal dissipation (63), i.e. a reversible material behavior is present in this example.

The interesting outcome from this quite simple network of elastic springs is the following:

- the second spring is naturally connected to the KIRCHHOFF stress $\underset{\mathrm{t}}{\mathbf{T}}$ at the current configuration $\mathcal{B}_{\mathrm{t}}$ as well as to the stress tensor $\underset{\mathrm{z}}{\mathbf{T}} = \mathbf{F}_2^{-1}\, \underset{\mathrm{t}}{\mathbf{T}}\, \mathbf{F}_2^{-\mathsf{T}}$ of the intermediate configuration $\mathcal{B}_{\mathrm{z}}$,
- however, the first spring is naturally associated to the MANDEL stress tensor $\underset{\mathrm{z}}{\mathbf{M}} = \underset{\mathrm{z}}{\mathbf{C}}_2\, \underset{\mathrm{z}}{\mathbf{T}}$ of the intermediate configuration $\mathcal{B}_{\mathrm{z}}$ as well as to the stress tensor $\underset{0}{\mathbf{M}} = \mathbf{F}_1^{-1}\, \underset{\mathrm{z}}{\mathbf{M}}\, \mathbf{F}_1^{-\mathsf{T}} = \underset{0}{\mathbf{C}}_1^{-1} \underset{0}{\mathbf{C}}\, \underset{0}{\mathbf{T}}$ at the initial configuration $\mathcal{B}_0$.

These conclusions may be generalized: In a series connection of two stress inducing basic elements, the second rheological body is associated to the KIRCHHOFF stress $\underset{\mathrm{t}}{\mathbf{T}}$ at the current configuration $\mathcal{B}_{\mathrm{t}}$, whereas the first element is naturally associated to the MANDEL stress tensor $\underset{\mathrm{z}}{\mathbf{M}} = \underset{\mathrm{z}}{\mathbf{C}}_2\, \underset{\mathrm{z}}{\mathbf{T}}$ of the intermediate configuration $\mathcal{B}_{\mathrm{z}}$.

The stress relations of the configurations $\mathcal{B}_0$, $\mathcal{B}_{\mathrm{z}}$ and $\mathcal{B}_{\mathrm{t}}$ may be summarized as

$$\underset{0}{\mathbf{T}} = \rho_{\mathrm{R}}\, \underset{0}{\mathbf{C}}^{-1} \underset{0}{\mathbf{C}}_1\, \frac{\partial \psi}{\partial \underset{0}{\mathbf{E}}_1} = \rho_{\mathrm{R}}\, \mathbf{F}_1^{-1} \frac{\partial \psi}{\partial \underset{\mathrm{z}}{\mathbf{E}}_2} \mathbf{F}_1^{-\mathsf{T}} , \tag{66}$$

$$\underset{\mathrm{z}}{\mathbf{T}} = \underset{\mathrm{z}}{\mathbf{C}}_2^{-1}\, \underset{\mathrm{z}}{\mathbf{M}} = \rho_{\mathrm{R}}\, \underset{\mathrm{z}}{\mathbf{C}}_2^{-1}\, \mathbf{F}_1 \frac{\partial \psi}{\partial \underset{0}{\mathbf{E}}_1} \mathbf{F}_1^{\mathsf{T}} = \rho_{\mathrm{R}} \frac{\partial \psi}{\partial \underset{\mathrm{z}}{\mathbf{E}}_2} \tag{67}$$

and

$$\underset{\mathrm{t}}{\mathbf{T}} = \rho_{\mathrm{R}}\, \underset{\mathrm{t}}{\mathbf{B}}_2^{-1}\, \mathbf{F} \frac{\partial \psi}{\partial \underset{0}{\mathbf{E}}_1} \mathbf{F}^{\mathsf{T}} = \rho_{\mathrm{R}}\, \mathbf{F}_2 \frac{\partial \psi}{\partial \underset{\mathrm{z}}{\mathbf{E}}_2} \mathbf{F}_2^{\mathsf{T}} . \tag{68}$$

4 Rheological Model of Thermoviscoplasticity for Finite Deformations

By means of the basic elements introduced above, a rheological network of finite thermoviscoplasticity can be assembled - see Figure 8 - as a generalization of the specific uniaxial model proposed by Bröcker and Matzenmiller

[6]This property will be verified later on.

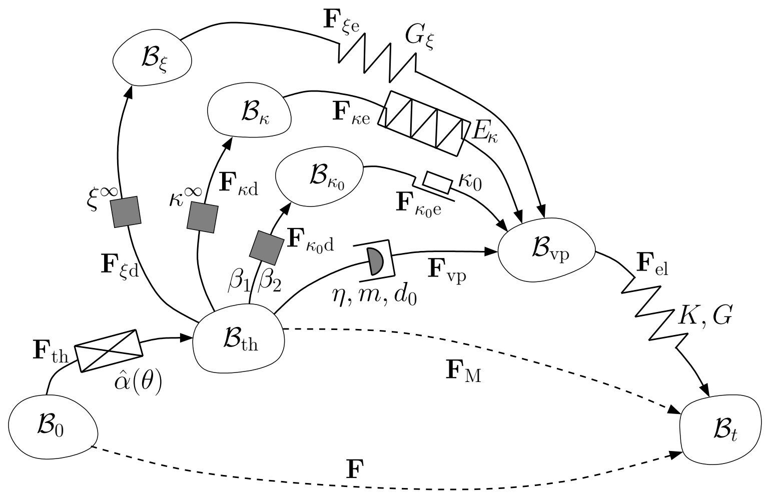

Figure 8: Finite rheological network of thermoviscoplasticity

(2013), representing nonlinear isotropic and kinematic hardening as well as an improved description of energy storage and dissipation during viscoplastic deformations due to a process-dependent energy storage in the friction body analogous to the one-dimensional counterpart.

The general structure of the rheological network has been adopted from the material models of finite thermoviscoplasticity of Lion (2000), Shutov and Kreißig (2008). However, the model at hand provides some additional features and intermediate configurations associated to the initial yield stress κ_0 and the stress of isotropic hardening κ. The rheological network consists of the thermal strain element with the thermal deformation gradient $\mathbf{F}_{\rm th}$ on the left hand side, the complete viscoplastic model setting including seven individual elements associated with the viscoplastic deformation gradient $\mathbf{F}_{\rm vp}$ in the center as well as the elastic spring with the elastic deformation gradient $\mathbf{F}_{\rm el}$ on the right hand side. This means that the thermal, the viscoplastic and the elastic model contributions are arranged in a threefold series connection between the initial $\mathcal{B}_0$ and the current configuration $\mathcal{B}_{\rm t}$ - see Figure 8. The viscoplastic arrangement itself provides four different chains of basic bodies connected in parallel, each one representing a specific phenomenon of the entire viscoplastic behavior. On top, an elastic spring is arranged in series connection to a dissipative strain element allowing for nonlinear kinematic hardening. Below, the hardening element and a dissipative strain component are placed in series arrangement for modeling of nonlinear isotropic hardening. The next chain consists of the friction body with energy storage and another dissipative strain element in order to effect the initial yield threshold and an improved behavior of energy storage and dissipation during viscoplastic loading. In the lowest chain of the viscoplastic network setting, a nonlinear dashpot is placed to account for strain rate effects.

4.1 Kinematics

The fundamental kinematical relations of the thermoviscoplastic model may directly be taken from the rheological network in Figure 8, utilizing the network rules (32) and (29) of the connection principles of series and parallel arrangement of basic element. The total deformation gradient $\mathbf{F}$ of the network is decomposed multiplicatively into the mechanical and the thermal fractions with (25) as

$$\mathbf{F} = \mathbf{F}_{\rm M}\mathbf{F}_{\rm th} \quad , \qquad \mathbf{F}_{\rm th} = \hat{\varphi}(\theta)\,\mathbf{1}\,, \tag{69}$$

which in turn implies the thermal intermediate configuration $\mathcal{B}_{\rm th}$. As well, the mechanical contribution $\mathbf{F}_{\rm M}$ is split multiplicatively into the elastic and the viscoplastic parts

$$\mathbf{F}_{\rm M} = \mathbf{F}_{\rm el}\mathbf{F}_{\rm vp}\,. \tag{70}$$

Hence, a viscoplastic intermediate configuration $\mathcal{B}_{\rm vp}$ is obtained. Plastic incompressibility is assumed as usually for metal plasticity in absence of damage, i.e. the relation

$$\det(\mathbf{F}_{\rm vp}) = 1 \tag{71}$$

holds. Hence, $\mathbf{F}_{\mathrm{vp}}$ turns out as a unimodular tensor $\mathbf{F}_{\mathrm{vp}} = \widehat{\mathbf{F}}_{\mathrm{vp}}$ - cf. $(7)_2$, which implies the properties that $\underset{\mathrm{vp}}{\mathbf{L}}_{\mathrm{vp}}$ and $\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}$ are deviators[7]

$$\underset{\mathrm{vp}}{\mathbf{L}}_{\mathrm{vp}} = \underset{\mathrm{vp}}{\mathbf{L}}^{\mathsf{D}}_{\mathrm{vp}} \quad , \qquad \underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}} = \underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\mathrm{vp}} \tag{72}$$

and, furthermore, the associated right CAUCHY-GREEN type tensor is unimodular as well

$$\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}} = \mathbf{F}^{\mathsf{T}}_{\mathrm{vp}}\mathbf{F}_{\mathrm{vp}} \quad , \qquad \det\left(\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}}\right) = 1\,. \tag{73}$$

The viscoplastic setting of the rheological model in Figure 8 provides a threefold multiplicative decomposition of the viscoplastic part of the deformation gradient

$$\mathbf{F}_{\mathrm{vp}} = \mathbf{F}_{\xi\mathrm{e}}\mathbf{F}_{\xi\mathrm{d}} = \mathbf{F}_{\kappa\mathrm{e}}\mathbf{F}_{\kappa\mathrm{d}} = \mathbf{F}_{\kappa_0\mathrm{e}}\mathbf{F}_{\kappa_0\mathrm{d}} \tag{74}$$

according to the three chains comprising the series connections of dissipative strain elements and the spring, the hardening body and the friction element, respectively. The multiple split (74) yields three additional intermediate configuration $\mathcal{B}_\xi$, $\mathcal{B}_\kappa$ and $\mathcal{B}_{\kappa_0}$, which are associated to kinematic (index ξ) and isotropic hardening (index κ) as well as to the process-dependent energy storage in the friction element (index κ_0). Similar to property (71), it is reasonably assumed that

$$\det(\mathbf{F}_{\xi\mathrm{e}}) = \det(\mathbf{F}_{\xi\mathrm{d}}) = \det(\mathbf{F}_{\kappa\mathrm{e}}) = \det(\mathbf{F}_{\kappa\mathrm{d}}) = \det(\mathbf{F}_{\kappa_0\mathrm{e}}) = \det(\mathbf{F}_{\kappa_0\mathrm{d}}) = 1 \tag{75}$$

holds for the decomposition (74), i.e. all the partial mappings are volume preserving. According to (48) up to (51), the relations

$$\underset{\mathrm{vp}}{\mathbf{L}}^{\mathsf{D}}_{\mathrm{vp}} = \underset{\mathrm{vp}}{\mathbf{L}}^{\mathsf{D}}_{\gamma\mathrm{e}} + \underset{\mathrm{vp}}{\mathbf{L}}^{\mathsf{D}}_{\gamma\mathrm{d}} \quad , \qquad \underset{\mathrm{vp}}{\mathbf{L}}^{\mathsf{D}}_{\gamma\mathrm{d}} = \mathbf{F}_{\gamma\mathrm{e}}\,\underset{\gamma}{\mathbf{L}}^{\mathsf{D}}_{\gamma\mathrm{d}}\,\mathbf{F}^{-1}_{\gamma\mathrm{e}} \quad , \qquad \underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\mathrm{vp}} = \underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\gamma\mathrm{e}} + \underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\gamma\mathrm{d}} \tag{76}$$

with the velocity gradients

$$\underset{\mathrm{vp}}{\mathbf{L}}_{\gamma\mathrm{e}} = \dot{\mathbf{F}}_{\gamma\mathrm{e}}\mathbf{F}^{-1}_{\gamma\mathrm{e}} = \underset{\mathrm{vp}}{\mathbf{L}}^{\mathsf{D}}_{\gamma\mathrm{e}} \quad , \qquad \underset{\gamma}{\mathbf{L}}_{\gamma\mathrm{d}} = \dot{\mathbf{F}}_{\gamma\mathrm{d}}\mathbf{F}^{-1}_{\gamma\mathrm{d}} = \underset{\gamma}{\mathbf{L}}^{\mathsf{D}}_{\gamma\mathrm{d}} \tag{77}$$

are valid on the viscoplastic intermediate configuration $\mathcal{B}_{\mathrm{vp}}$, where the subscripts $\gamma = \kappa_0$, $\gamma = \kappa$ and $\gamma = \xi$ denote the various deformation measures of the motion from the thermal $\mathcal{B}_{\mathrm{th}}$ to the viscoplastic configuration $\mathcal{B}_{\mathrm{vp}}$.

4.2 Equilibrium of Stresses and Free Energy

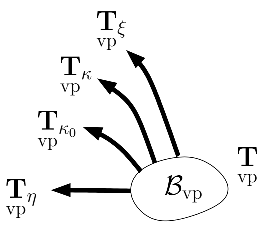

Figure 9: Sketch of equilibrium of stresses on viscoplastic intermediate configuration $\mathcal{B}_{\mathrm{vp}}$

The equilibrium of stresses is considered on the viscoplastic intermediate configuration $\mathcal{B}_{\mathrm{vp}}$ of the rheological model in Figure 8. The total stress $\underset{\mathrm{vp}}{\mathbf{T}} = \mathbf{F}^{-1}_{\mathrm{el}}\underset{\mathrm{t}}{\mathbf{T}}\mathbf{F}^{-\mathsf{T}}_{\mathrm{el}}$ on $\mathcal{B}_{\mathrm{vp}}$ results from the individual contributions of all four parallel chains of the viscoplastic network according to (31), i.e. from the nonlinear dashpot $\underset{\mathrm{vp}}{\mathbf{T}}_\eta$, the friction body $\underset{\mathrm{vp}}{\mathbf{T}}_{\kappa_0}$, the hardening element $\underset{\mathrm{vp}}{\mathbf{T}}_\kappa$ and the spring of kinematic hardening $\underset{\mathrm{vp}}{\mathbf{T}}_\xi$ as - see Figure 9 -

$$\underset{\mathrm{vp}}{\mathbf{T}} = \underset{\mathrm{vp}}{\mathbf{T}}_\eta + \underset{\mathrm{vp}}{\mathbf{T}}_{\kappa_0} + \underset{\mathrm{vp}}{\mathbf{T}}_\kappa + \underset{\mathrm{vp}}{\mathbf{T}}_\xi\,. \tag{78}$$

---

[7]For the unimodular deformation gradient $\widehat{\mathbf{F}} = \left(\det\mathbf{F}\right)^{-\frac{1}{3}}\mathbf{F}$ and with the relations $\dot{\widehat{\mathbf{F}}} = \left(\det\mathbf{F}\right)^{-\frac{1}{3}}\left(\dot{\mathbf{F}} - \frac{1}{3}(\mathbf{1}\cdot\underset{\mathrm{t}}{\mathbf{L}})\mathbf{F}\right)$ and $\widehat{\mathbf{F}}^{-1} = \left(\det\mathbf{F}\right)^{\frac{1}{3}}\mathbf{F}^{-1}$, it follows that $\underset{\mathrm{t}}{\widehat{\mathbf{L}}} = \dot{\widehat{\mathbf{F}}}\widehat{\mathbf{F}}^{-1} = \underset{\mathrm{t}}{\mathbf{L}}^{\mathsf{D}}$ holds.

In terms of the MANDEL stress tensor - cf. eq. (56), the equilibrium of stresses (78) is

$$\underset{\text{vp}}{\mathbf{M}} = \underset{\text{vp}}{\mathbf{C}}_{\text{el}} \underset{\text{vp}}{\mathbf{T}} = \underset{\text{vp}}{\mathbf{M}}_{\eta} + \underset{\text{vp}}{\mathbf{M}}_{\kappa_0} + \underset{\text{vp}}{\mathbf{M}}_{\kappa} + \underset{\text{vp}}{\mathbf{M}}_{\xi} \tag{79}$$

with the elastic CAUCHY-GREEN type tensor $\underset{\text{vp}}{\mathbf{C}}_{\text{el}} = \mathbf{F}_{\text{el}}^{\mathsf{T}}\mathbf{F}_{\text{el}}$ and the partial stress quantities

$$\underset{\text{vp}}{\mathbf{M}}_{\eta} = \underset{\text{vp}}{\mathbf{C}}_{\text{el}} \underset{\text{vp}}{\mathbf{T}}_{\eta} \quad , \qquad \underset{\text{vp}}{\mathbf{M}}_{\kappa_0} = \underset{\text{vp}}{\mathbf{C}}_{\text{el}} \underset{\text{vp}}{\mathbf{T}}_{\kappa_0} \quad , \qquad \underset{\text{vp}}{\mathbf{M}}_{\kappa} = \underset{\text{vp}}{\mathbf{C}}_{\text{el}} \underset{\text{vp}}{\mathbf{T}}_{\kappa} \quad , \qquad \underset{\text{vp}}{\mathbf{M}}_{\xi} = \underset{\text{vp}}{\mathbf{C}}_{\text{el}} \underset{\text{vp}}{\mathbf{T}}_{\xi} \,. \tag{80}$$

The total free energy is obtained from the contributions ψ_{th}, ψ_{el}, ψ_{ξ}, ψ_{κ} and ψ_{κ_0} of the basic elements with the capability of energy storage according to their specific position in the rheological network, i.e. from the thermal strain element ψ_{th}, the elastic spring ψ_{el}, the spring of kinematic hardening ψ_{ξ}, the hardening element ψ_{κ} as well as the friction body ψ_{κ_0} according to the relations (28), (15), (21) and (19) as

$$\psi = \psi_{\text{th}}\big(\theta\big) + \psi_{\text{el}}\Big(\underset{\text{vp}}{\mathbf{E}}_{\text{el}}\Big) + \psi_{\xi}\Big(\underset{\xi}{\mathbf{E}}_{\xi\text{e}}\Big) + \psi_{\kappa}\Big(\bar{\varepsilon}_{\kappa\text{e}}\Big) + \psi_{\kappa_0}\Big(\bar{\varepsilon}_{\kappa_0\text{e}}\Big) \tag{81}$$

$$= \hat{Q}(\theta) + \frac{1}{2\rho_{\text{R}}}\left(K\,(J_{\text{el}}-1)^2 + G\,\big(\mathbf{1}\cdot\underset{\text{vp}}{\widehat{\mathbf{C}}}_{\text{el}} - 3\big) + G_{\xi}\,\big(\mathbf{1}\cdot\underset{\xi}{\widehat{\mathbf{C}}}_{\xi\text{e}} - 3\big) + E_{\kappa}\,\bar{\varepsilon}_{\kappa\text{e}}^2 + 2\,\kappa_0\,\bar{\varepsilon}_{\kappa_0\text{e}}\right) \tag{82}$$

with the strain tensors of GREEN type

$$\underset{\text{vp}}{\mathbf{E}}_{\text{el}} = \frac{1}{2}\big(\mathbf{F}_{\text{el}}^{\mathsf{T}}\mathbf{F}_{\text{el}} - \mathbf{1}\big) = \frac{1}{2}\big(\underset{\text{vp}}{\mathbf{C}}_{\text{el}} - \mathbf{1}\big) \quad , \qquad \underset{\xi}{\mathbf{E}}_{\xi\text{e}} = \frac{1}{2}\big(\mathbf{F}_{\xi\text{e}}^{\mathsf{T}}\mathbf{F}_{\xi\text{e}} - \mathbf{1}\big) = \frac{1}{2}\big(\underset{\xi}{\mathbf{C}}_{\xi\text{e}} - \mathbf{1}\big) \tag{83}$$

and the determinant $J_{\text{el}} = \det\mathbf{F}_{\text{el}}$ and $J_{\xi\text{e}} = \det\mathbf{F}_{\xi\text{e}} = 1$ of both springs as well as with the arclength of the friction and the hardening body according to the definition (14)

$$\bar{\varepsilon}_{\kappa_0\text{e}} = \sqrt{\frac{2}{3}}\int_0^t \|\underset{\text{vp}}{\mathbf{D}}_{\kappa_0\text{e}}\|\,\mathrm{d}\tau = \int_0^t \dot{\bar{\varepsilon}}_{\kappa_0\text{e}}\,\mathrm{d}\tau \quad , \qquad \dot{\bar{\varepsilon}}_{\kappa_0\text{e}} = \sqrt{\frac{2}{3}}\,\|\underset{\text{vp}}{\mathbf{D}}_{\kappa_0\text{e}}\|\,, \tag{84}$$

$$\bar{\varepsilon}_{\kappa\text{e}} = \sqrt{\frac{2}{3}}\int_0^t \|\underset{\text{vp}}{\mathbf{D}}_{\kappa\text{e}}\|\,\mathrm{d}\tau = \int_0^t \dot{\bar{\varepsilon}}_{\kappa\text{e}}\,\mathrm{d}\tau \quad , \qquad \dot{\bar{\varepsilon}}_{\kappa\text{e}} = \sqrt{\frac{2}{3}}\,\|\underset{\text{vp}}{\mathbf{D}}_{\kappa\text{e}}\|\,. \tag{85}$$

Hence, the free energy (81) may generally be summarized as a function of its arguments

$$\psi = \psi\left(\underset{\text{vp}}{\mathbf{E}}_{\text{el}}, \theta, \underset{\xi}{\mathbf{E}}_{\xi\text{e}}, \bar{\varepsilon}_{\kappa\text{e}}, \bar{\varepsilon}_{\kappa_0\text{e}}, \vec{g}_{\text{R}}\right)\,. \tag{86}$$

Note, the temperature gradient $\vec{g}_{\text{R}}$ with respect to the material coordinates

$$\vec{g}_{\text{R}} = \frac{\partial\theta}{\partial\vec{X}} \tag{87}$$

has been added to the list of arguments of the free energy ψ as usually done - see e.g. Haupt (2002), p. 514, since the heat flux $\vec{q}_{\text{R}}$ has to be a function of $\vec{g}_{\text{R}}$. The rate of the free energy (86) results with $(84)_3$ and $(85)_3$ as

$$\dot{\psi} = \frac{\partial\psi}{\partial\underset{\text{vp}}{\mathbf{E}}_{\text{el}}}\cdot\underset{\text{vp}}{\dot{\mathbf{E}}}_{\text{el}} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\underset{\xi}{\mathbf{E}}_{\xi\text{e}}}\cdot\underset{\xi}{\dot{\mathbf{E}}}_{\xi\text{e}} + \sqrt{\frac{2}{3}}\,\frac{\partial\psi}{\partial\bar{\varepsilon}_{\kappa\text{e}}}\,\|\underset{\text{vp}}{\mathbf{D}}_{\kappa\text{e}}\| + \sqrt{\frac{2}{3}}\,\frac{\partial\psi}{\partial\bar{\varepsilon}_{\kappa_0\text{e}}}\,\|\underset{\text{vp}}{\mathbf{D}}_{\kappa_0\text{e}}\| + \frac{\partial\psi}{\partial\vec{g}_{\text{R}}}\cdot\dot{\vec{g}}_{\text{R}}\,. \tag{88}$$

4.3 Decomposition of Stress Power

The total internal stress power l decomposes according to (52), $(53)_3$ and $(55)_2$ as - see Lion (2000) -

$$l = \frac{1}{\rho_{\text{R}}}\underset{0}{\mathbf{T}}\cdot\underset{0}{\dot{\mathbf{E}}} = \frac{1}{\rho_{\text{R}}}\left(\underset{\text{th}}{\mathbf{T}}\cdot\underset{\text{th}}{\dot{\mathbf{E}}}_{\text{M}} + \underset{\text{th}}{\mathbf{C}}_{\text{M}}\,\underset{\text{th}}{\mathbf{T}}\cdot\underset{\text{th}}{\mathbf{L}}_{\text{th}}\right) = l_{\text{M}} + l_{\text{th}} \tag{89}$$

into the mechanical l_{M} and the thermal contribution l_{th}. The mechanical stress power in turn splits according to (52), $(53)_3$, $(55)_3$ and (72) as[8]

$$l_{\text{M}} = \frac{1}{\rho_{\text{R}}}\underset{\text{th}}{\mathbf{T}}\cdot\underset{\text{th}}{\dot{\mathbf{E}}}_{\text{M}} = \frac{1}{\rho_{\text{R}}}\left(\underset{\text{vp}}{\mathbf{T}}\cdot\underset{\text{vp}}{\dot{\mathbf{E}}}_{\text{el}} + \underset{\text{vp}}{\mathbf{M}}\cdot\underset{\text{vp}}{\mathbf{L}}_{\text{vp}}^{\mathsf{D}}\right) = l_{\text{el}} + l_{\text{vp}} \tag{90}$$

[8]Note that $\underset{\text{vp}}{\mathbf{L}}_{\text{vp}}$ is naturally deviatoric due to the assumption of plastic incompressibility in (71). However, in order to visualize this important property, the superscript $(.)^{\mathsf{D}}$ is additionally appended to the measure $\underset{\text{vp}}{\mathbf{L}}_{\text{vp}}$ as well as to its decomposed fractions in (94) and (95).

- see Lion (2000) - into the elastic l_{el} and the viscoplastic fraction l_{vp} of the total stress power l. Hence, the total stress $\underset{\mathrm{vp}}{\mathbf{T}}$ is the natural stress measure of the elastic stress power l_{el}, whereas the MANDEL stress state $\underset{\mathrm{vp}}{\mathbf{M}}$ is naturally associated to the viscoplastic stress power l_{vp}. Notice that the volumetric part of the total MANDEL stress $\underset{\mathrm{vp}}{\mathbf{M}}$ does not lead to any contribution in the viscoplastic stress power $\underset{\mathrm{vp}}{\mathbf{M}} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}}$, since $\underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}}$ is a deviator. Moreover, the viscoplastic stress power l_{vp} is decomposed using the equilibrium relation (79) as

$$l_{\mathrm{vp}} = \frac{1}{\rho_{\mathrm{R}}} \underset{\mathrm{vp}}{\mathbf{M}} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}} = \frac{1}{\rho_{\mathrm{R}}} \left(\underset{\mathrm{vp}}{\mathbf{M}}_{\eta} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\xi} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}} \right) . \tag{91}$$

The MANDEL stress tensor is supposed to be symmetric:

$$\underset{\mathrm{vp}}{\mathbf{M}} = \underset{\mathrm{vp}}{\mathbf{M}}{}^{\mathsf{T}} , \tag{92}$$

which means that $\underset{\mathrm{vp}}{\mathbf{C}}_{\mathrm{el}}$ and $\underset{\mathrm{vp}}{\mathbf{T}}$ have to be coaxial to each other. The property (92) seems somehow to be an arbitrary assumption, but the symmetry will be proven later on. The relation (92) implies that for reasons of plausibility, the partial stresses of the rheological model in (79) have to be symmetric as well[9]

$$\underset{\mathrm{vp}}{\mathbf{M}}_{\eta} = \underset{\mathrm{vp}}{\mathbf{M}}{}^{\mathsf{T}}_{\eta} \quad , \qquad \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0} = \underset{\mathrm{vp}}{\mathbf{M}}{}^{\mathsf{T}}_{\kappa_0} \quad , \qquad \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa} = \underset{\mathrm{vp}}{\mathbf{M}}{}^{\mathsf{T}}_{\kappa} \quad , \qquad \underset{\mathrm{vp}}{\mathbf{M}}_{\xi} = \underset{\mathrm{vp}}{\mathbf{M}}{}^{\mathsf{T}}_{\xi} . \tag{93}$$

According to $(93)_{2,3}$ and $(76)_3$, the relations

$$\underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}} = \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa_0\mathrm{e}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa_0\mathrm{d}} \quad , \qquad \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}} = \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa\mathrm{e}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa\mathrm{d}} \tag{94}$$

hold for the parallel chains of the viscoplastic network contribution with the intermediate configurations $\mathcal{B}_{\kappa_0}$ and $\mathcal{B}_{\kappa}$. Furthermore, with $(52)_4$, $(76)_1$, $(93)_4$, $(53)_3$ and $(55)_2$, the decomposition

$$\underset{\mathrm{vp}}{\mathbf{M}}_{\xi} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}} = \underset{\mathrm{vp}}{\mathbf{M}}{}^{\mathsf{D}}_{\xi} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}} = \underset{\mathrm{vp}}{\mathbf{M}}{}^{\mathsf{D}}_{\xi} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\xi\mathrm{e}} + \underset{\mathrm{vp}}{\mathbf{M}}{}^{\mathsf{D}}_{\xi} \cdot \underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\xi\mathrm{d}} = \underset{\xi}{\mathbf{M}}{}^{*}_{\xi} \cdot \underset{\xi}{\dot{\mathbf{E}}}_{\xi\mathrm{e}} + \underset{\xi}{\mathbf{C}}_{\xi\mathrm{e}} \underset{\xi}{\mathbf{M}}{}^{*}_{\xi} \cdot \underset{\xi}{\mathbf{L}}{}^{\mathsf{D}}_{\xi\mathrm{d}} \tag{95}$$

is obtained[10] for the intermediate configuration $\mathcal{B}_{\xi}$ with the transformed MANDEL type stress measure of kinematic hardening

$$\underset{\xi}{\mathbf{M}}{}^{*}_{\xi} = \mathbf{F}^{-1}_{\xi\mathrm{e}} \underset{\mathrm{vp}}{\mathbf{M}}{}^{\mathsf{D}}_{\xi} \mathbf{F}^{-\mathsf{T}}_{\xi\mathrm{e}} . \tag{96}$$

With (90), (91), (94), (95), (24) and (23), the total stress power (89) results as

$$\begin{aligned} l = \frac{1}{\rho_{\mathrm{R}}} \Big(& \underset{\mathrm{vp}}{\mathbf{T}} \cdot \underset{\mathrm{vp}}{\dot{\mathbf{E}}}_{\mathrm{el}} + \underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{M}} \underset{\mathrm{th}}{\mathbf{T}} \cdot \underset{\mathrm{th}}{\hat{\boldsymbol{\alpha}}}_{\mathrm{th}} \, \dot{\theta} + \underset{\mathrm{vp}}{\mathbf{M}}_{\eta} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\mathrm{vp}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa_0\mathrm{e}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa_0\mathrm{d}} \\ & + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa\mathrm{e}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa} \cdot \underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa\mathrm{d}} + \underset{\xi}{\mathbf{M}}{}^{*}_{\xi} \cdot \underset{\xi}{\dot{\mathbf{E}}}_{\xi\mathrm{e}} + \underset{\xi}{\mathbf{C}}_{\xi\mathrm{e}} \underset{\xi}{\mathbf{M}}{}^{*}_{\xi} \cdot \underset{\xi}{\mathbf{L}}{}^{\mathsf{D}}_{\xi\mathrm{d}} \Big) . \end{aligned} \tag{97}$$

Notice that each individual summand in (97) represents the stress power associated with exactly one of the basic elements of the rheological model in Figure 8.

4.4 Evaluation of Internal Dissipation

The inequality of the internal dissipation δ results from the balance equations for energy and entropy in combination with the second law of thermodynamics - see Haupt (2002), p. 511 - as

$$\delta = -\dot{\psi} - \dot{\theta} s + \frac{1}{\rho_{\mathrm{R}}} \underset{0}{\mathbf{T}} \cdot \underset{0}{\dot{\mathbf{E}}} - \frac{1}{\rho_{\mathrm{R}} \theta} \, \vec{q}_{\mathrm{R}} \cdot \vec{g}_{\mathrm{R}} \geq 0 \tag{98}$$

and has to be satisfied by the constitutive model. In (98), s denotes the entropy, besides all the other quantities already known from above. Inserting the rate of the free energy (88) and the decomposed stress power (97) into

[9]This means, $\underset{\mathrm{vp}}{\mathbf{L}}{}^{\mathsf{D}}_{\mathrm{vp}}$ may be replaced by its symmetric fraction $\underset{\mathrm{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\mathrm{vp}}$ in (91).

[10]The rate $\underset{\mathrm{vp}}{\mathbf{D}}_{\xi\mathrm{e}}$ is naturally deviatoric, which means $\mathbf{1} \cdot \underset{\mathrm{vp}}{\mathbf{D}}_{\xi\mathrm{e}} = 0$ holds. However, this property is lost for $\underset{\xi}{\dot{\mathbf{E}}}_{\xi\mathrm{e}}$ due to the transformation $\underset{\xi}{\dot{\mathbf{E}}}_{\xi\mathrm{e}} = \mathbf{F}^{\mathsf{T}}_{\xi\mathrm{e}} \underset{\mathrm{vp}}{\mathbf{D}}_{\xi\mathrm{e}} \mathbf{F}_{\xi\mathrm{e}}$, i.e. $\underset{\xi}{\dot{\mathbf{E}}}_{\xi\mathrm{e}}$ is not a deviator. Similarly, the pull-backward operation (96) yields a non-deviatoric stress measure $\underset{\xi}{\mathbf{M}}{}^{*}_{\xi}$ on $\mathcal{B}_{\xi}$.

(98) yields for the internal dissipation[11]:

$$\begin{aligned}\delta = & \left(\frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{vp}}{\mathbf{T}} - \frac{\partial\psi}{\partial\underset{\mathrm{vp}}{\mathbf{E}}_{\mathrm{el}}}\right)\cdot\underset{\mathrm{vp}}{\dot{\mathbf{E}}}_{\mathrm{el}} + \left(\frac{1}{\rho_{\mathrm{R}}}\underset{\xi}{\mathbf{M}}^{*}_{\xi} - \frac{\partial\psi}{\partial\underset{\xi}{\mathbf{E}}_{\xi\mathrm{e}}}\right)\cdot\underset{\xi}{\dot{\mathbf{E}}}_{\xi\mathrm{e}} - \left(s + \frac{\partial\psi}{\partial\theta} - \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{M}}\,\underset{\mathrm{th}}{\mathbf{T}}\cdot\underset{\mathrm{th}}{\hat{\boldsymbol{\alpha}}}_{\mathrm{th}}\right)\dot{\theta} - \frac{\partial\psi}{\partial\vec{g}_{\mathrm{R}}}\cdot\dot{\vec{g}}_{\mathrm{R}} \\ & + \left(\frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{vp}}{\mathbf{M}}_{\kappa}\cdot\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{e}}\right) - \sqrt{\frac{2}{3}}\,\frac{\partial\psi}{\partial\bar{\varepsilon}_{\kappa\mathrm{e}}}\right)\|\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa\mathrm{e}}\| + \left(\frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0}\cdot\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{e}}\right) - \sqrt{\frac{2}{3}}\,\frac{\partial\psi}{\partial\bar{\varepsilon}_{\kappa_0\mathrm{e}}}\right)\|\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa_0\mathrm{e}}\| \\ & - \frac{1}{\rho_{\mathrm{R}}\theta}\,\vec{q}_{\mathrm{R}}\cdot\vec{g}_{\mathrm{R}} + \frac{1}{\rho_{\mathrm{R}}}\left(\underset{\mathrm{vp}}{\mathbf{M}}_{\eta}\cdot\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\mathrm{vp}} + \underset{\xi}{\mathbf{C}}_{\xi\mathrm{e}}\,\underset{\xi}{\mathbf{M}}^{*}_{\xi}\cdot\underset{\xi}{\mathbf{L}}^{\mathsf{D}}_{\xi\mathrm{d}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa}\cdot\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{d}} + \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0}\cdot\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{d}}\right).\end{aligned} \tag{99}$$

For arbitrary rates $\underset{\mathrm{vp}}{\dot{\mathbf{E}}}_{\mathrm{el}}$, $\underset{\xi}{\dot{\mathbf{E}}}_{\xi\mathrm{e}}$, $\dot{\theta}$, $\dot{\vec{g}}_{\mathrm{R}}$, $\|\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa\mathrm{e}}\|$, $\|\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa_0\mathrm{e}}\|$ and an arbitrary temperature gradient $\vec{g}_{\mathrm{R}}$, it follows according to the standard argumentation of Coleman and Gurtin (1967) that the factors ahead of these rates have to vanish for any admissible thermomechanical process. Hence, the potential relations

$$\underset{\mathrm{vp}}{\mathbf{T}} = \rho_{\mathrm{R}}\frac{\partial\psi}{\partial\underset{\mathrm{vp}}{\mathbf{E}}_{\mathrm{el}}} \quad , \qquad \underset{\xi}{\mathbf{M}}^{*}_{\xi} = \rho_{\mathrm{R}}\frac{\partial\psi}{\partial\underset{\xi}{\mathbf{E}}_{\xi\mathrm{e}}} \tag{100}$$

and

$$s = \frac{1}{\rho_{\mathrm{R}}}\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{M}}\,\underset{\mathrm{th}}{\mathbf{T}}\cdot\underset{\mathrm{th}}{\hat{\boldsymbol{\alpha}}}_{\mathrm{th}} - \frac{\partial\psi}{\partial\theta} \tag{101}$$

are obtained for the total stress $\underset{\mathrm{vp}}{\mathbf{T}}$ on $\mathcal{B}_{\mathrm{vp}}$, the MANDEL type stress tensor $\underset{\xi}{\mathbf{M}}^{*}_{\xi}$ of kinematic hardening on $\mathcal{B}_{\xi}$ as well as for the entropy s. The relations (100) reveal that $\underset{\mathrm{vp}}{\mathbf{T}}$ and $\underset{\mathrm{vp}}{\mathbf{C}}_{\mathrm{el}}$ as well as $\underset{\xi}{\mathbf{M}}^{*}_{\xi}$ and $\underset{\xi}{\mathbf{C}}_{\xi\mathrm{e}}$ are coaxial to each other for the free energy (82). Hence, the symmetry assumption (92) is verified. Furthermore, (99) yields the property

$$\frac{\partial\psi}{\partial\vec{g}_{\mathrm{R}}} = \vec{0}\,, \tag{102}$$

i.e. the free energy must be independent of the temperature gradient $\vec{g}_{\mathrm{R}}$:

$$\psi = \psi\left(\underset{\mathrm{vp}}{\mathbf{E}}_{\mathrm{el}}, \theta, \underset{\xi}{\mathbf{E}}_{\xi\mathrm{e}}, \bar{\varepsilon}_{\kappa\mathrm{e}}, \bar{\varepsilon}_{\kappa_0\mathrm{e}}\right). \tag{103}$$

Moreover, the fifth and the sixth summand of (99) provide the relations

$$\underset{\mathrm{vp}}{\mathbf{M}}_{\kappa}\cdot\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{e}}\right) - \sqrt{\frac{2}{3}}\,\rho_{\mathrm{R}}\frac{\partial\psi}{\partial\bar{\varepsilon}_{\kappa\mathrm{e}}} = 0 \quad , \qquad \underset{\mathrm{vp}}{\mathbf{M}}_{\kappa_0}\cdot\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{e}}\right) - \sqrt{\frac{2}{3}}\,\rho_{\mathrm{R}}\frac{\partial\psi}{\partial\bar{\varepsilon}_{\kappa_0\mathrm{e}}} = 0\,, \tag{104}$$

which may be rearranged by means of the properties $\mathbf{A}\cdot\mathbf{B}^{\mathsf{D}} = \mathbf{A}^{\mathsf{D}}\cdot\mathbf{B}^{\mathsf{D}}$ and $(11)_2$ according to

$$\left(\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}}_{\kappa} - \sqrt{\frac{2}{3}}\,\rho_{\mathrm{R}}\frac{\partial\psi}{\partial\bar{\varepsilon}_{\kappa\mathrm{e}}}\,\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{e}}\right)\right)\cdot\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{e}}\right) = 0 \tag{105}$$

and

$$\left(\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}}_{\kappa_0} - \sqrt{\frac{2}{3}}\,\rho_{\mathrm{R}}\frac{\partial\psi}{\partial\bar{\varepsilon}_{\kappa_0\mathrm{e}}}\,\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{e}}\right)\right)\cdot\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{e}}\right) = 0\,. \tag{106}$$

Both equations (105) and (106) must be satisfied for arbitrary rates $\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{e}} \neq \mathbf{0}$ and $\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{e}} \neq \mathbf{0}$, leading to the additional potential relations

$$\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}}_{\kappa} = \sqrt{\frac{2}{3}}\,\rho_{\mathrm{R}}\frac{\partial\psi}{\partial\,\bar{\varepsilon}_{\kappa\mathrm{e}}}\,\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{e}}\right) = \sqrt{\frac{2}{3}}\,\kappa\,\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{e}}\right) \quad , \qquad \kappa = \rho_{\mathrm{R}}\frac{\partial\psi}{\partial\,\bar{\varepsilon}_{\kappa\mathrm{e}}} \qquad \text{for} \quad \underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{e}} \neq \mathbf{0} \tag{107}$$

and

$$\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}}_{\kappa_0} = \sqrt{\frac{2}{3}}\,\rho_{\mathrm{R}}\frac{\partial\psi}{\partial\,\bar{\varepsilon}_{\kappa_0\mathrm{e}}}\,\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{e}}\right) = \sqrt{\frac{2}{3}}\,\kappa_0\,\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{e}}\right) \quad , \qquad \kappa_0 = \rho_{\mathrm{R}}\frac{\partial\psi}{\partial\,\bar{\varepsilon}_{\kappa_0\mathrm{e}}} \qquad \text{for} \quad \underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{e}} \neq \mathbf{0} \tag{108}$$

[11] Note, $\underset{\mathrm{vp}}{\mathbf{L}}_{\kappa\mathrm{e}}$ and $\underset{\mathrm{vp}}{\mathbf{L}}_{\kappa_0\mathrm{e}}$ are deviatoric, since $\underset{\mathrm{vp}}{\mathbf{L}}_{\mathrm{vp}}$ is a deviator - see (72). Hence, the properties $\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa\mathrm{e}} = \underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa\mathrm{e}}$ and $\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa_0\mathrm{e}} = \underset{\mathrm{vp}}{\mathbf{D}}^{\mathsf{D}}_{\kappa_0\mathrm{e}}$ hold.

for the stress state in the hardening and the friction body in terms of the quantities of the viscoplastic intermediate configuration $\mathcal{B}_{\rm vp}$. The scalar κ denotes the internal stress in the isotropic hardening part. Both variables κ and κ_0 in $(107)_3$ and $(108)_3$ are strictly non-negative, if the associated energy contributions of the hardening and friction body ψ_κ and ψ_{κ_0} in (81) meet the following demands: ψ_κ and ψ_{κ_0} have to be convex functions of their arguments $\bar{\varepsilon}_{\kappa\rm e}$ and $\bar{\varepsilon}_{\kappa_0\rm e}$ with the initial states $\psi_\kappa(\bar{\varepsilon}_{\kappa\rm e}{=}0) = 0$ and $\psi_{\kappa_0}(\bar{\varepsilon}_{\kappa_0\rm e}{=}0) = 0$. Furthermore, the properties $\psi_\kappa(\bar{\varepsilon}_{\kappa\rm e}{>}0) > 0$ and $\psi_{\kappa_0}(\bar{\varepsilon}_{\kappa_0\rm e}{>}0) > 0$ must be fulfilled during loading. Thus, the internal dissipation (99) is reduced to the residual inequality

$$\delta = -\frac{1}{\rho_{\rm R}\theta}\,\vec{q}_{\rm R}\cdot\vec{g}_{\rm R} + \frac{1}{\rho_{\rm R}}\left(\underset{\rm vp}{\mathbf{M}}_\eta \cdot \underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp} + \underset{\xi}{\mathbf{C}}_{\xi\rm e}\,\underset{\xi}{\mathbf{M}}^*_\xi \cdot \underset{\xi}{\mathbf{D}}^{\sf D}_{\xi\rm d} + \underset{\rm vp}{\mathbf{M}}_\kappa \cdot \underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d} + \underset{\rm vp}{\mathbf{M}}_{\kappa_0} \cdot \underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm d}\right) = \delta_{\rm th} + \delta_{\rm M} \geq 0\,, \quad (109)$$

and separated into its thermal $\delta_{\rm th}$ and mechanical contributions $\delta_{\rm M}$. The thermal dissipation

$$\delta_{\rm th} = -\frac{1}{\rho_{\rm R}\theta}\,\vec{q}_{\rm R}\cdot\vec{g}_{\rm R} \geq 0 \quad (110)$$

is usually satisfied by means of FOURIER's model of heat conduction

$$\vec{q}_{\rm R} = -\mathbf{K}\,\vec{g}_{\rm R} \quad (111)$$

with $\mathbf{K}$ representing a positive definit tensor in general or with $\mathbf{K} = k\,\mathbf{1}$ in the case of isotropic heat conduction. By using the property $\underset{\xi}{\mathbf{C}}_{\xi\rm e}\,\underset{\xi}{\mathbf{M}}^*_\xi = \big(\underset{\xi}{\mathbf{C}}_{\xi\rm e}\,\underset{\xi}{\mathbf{M}}_\xi\big)^{\sf D}$, the mechanical dissipation results as

$$\delta_{\rm M} = \frac{1}{\rho_{\rm R}}\left(\underset{\rm vp}{\mathbf{M}}_\eta \cdot \underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp} + \underset{\xi}{\mathbf{C}}_{\xi\rm e}\,\underset{\xi}{\mathbf{M}}_\xi \cdot \underset{\xi}{\mathbf{D}}^{\sf D}_{\xi\rm d} + \underset{\rm vp}{\mathbf{M}}_\kappa \cdot \underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d} + \underset{\rm vp}{\mathbf{M}}_{\kappa_0} \cdot \underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm d}\right) \geq 0\,. \quad (112)$$

In order to ensure $\delta_{\rm M} \geq 0$ in any case, the following sufficient condition may be specified for the nonlinear dashpot

$$\mathsf{N}\left(\underset{\rm vp}{\mathbf{M}}^{\sf D}_\eta\right) = \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\right) \quad (113)$$

as well as for the three dissipative strain elements

$$\mathsf{N}\left(\underset{\xi}{\mathbf{D}}^{\sf D}_{\xi\rm d}\right) = \mathsf{N}\left(\big(\underset{\xi}{\mathbf{C}}_{\xi\rm e}\,\underset{\xi}{\mathbf{M}}_\xi\big)^{\sf D}\right) \quad , \quad \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d}\right) = \mathsf{N}\left(\underset{\rm vp}{\mathbf{M}}^{\sf D}_\kappa\right) \quad , \quad \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm d}\right) = \mathsf{N}\left(\underset{\rm vp}{\mathbf{M}}^{\sf D}_{\kappa_0}\right)\,. \quad (114)$$

Satisfying (113) and (114) and using (9) as well as $(11)_2$, the mechanical dissipation (112) is always non-negative and results as

$$\delta_{\rm M} = \frac{1}{\rho_{\rm R}}\left(\|\underset{\rm vp}{\mathbf{M}}^{\sf D}_\eta\|\,\|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\| + \|\big(\underset{\xi}{\mathbf{C}}_{\xi\rm e}\,\underset{\xi}{\mathbf{M}}_\xi\big)^{\sf D}\|\,\|\underset{\xi}{\mathbf{D}}^{\sf D}_{\xi\rm d}\| + \|\underset{\rm vp}{\mathbf{M}}^{\sf D}_\kappa\|\,\|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d}\| + \|\underset{\rm vp}{\mathbf{M}}^{\sf D}_{\kappa_0}\|\,\|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm d}\|\right) \geq 0\,. \quad (115)$$

According to $(114)_{2,3}$, (107) and (108), the relations

$$\mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d}\right) = \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm e}\right) \quad , \quad \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm d}\right) = \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm e}\right) \quad (116)$$

are obtained for the directions of the internal rates of deformation. Moreover, the decomposition (49) of $\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}$ on $\mathcal{B}_{\rm vp}$ leads with (9) and (116) according to

$$\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp} = \|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\|\,\mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\right) = \|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm e}\|\,\mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm e}\right) + \|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d}\|\,\mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d}\right) = \left(\|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm e}\| + \|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d}\|\right)\mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm e}\right) \quad (117)$$

to the important properties

$$\|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\| = \|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm e}\| + \|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d}\| \quad , \quad \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\right) = \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm e}\right) = \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa\rm d}\right) \quad (118)$$

as well as

$$\|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\| = \|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm e}\| + \|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm d}\| \quad , \quad \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\right) = \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm e}\right) = \mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\kappa_0\rm d}\right)\,. \quad (119)$$

This means that all the directions of the internal rates of deformation on the viscoplastic intermediate configuration $\mathcal{B}_{\rm vp}$ - resulting from the parallel chains of the friction and the hardening element in the viscoplastic network setting - are equal to the direction of the viscoplastic flow $\mathsf{N}\left(\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\right)$. Furthermore, the norm $\|\underset{\rm vp}{\mathbf{D}}^{\sf D}_{\rm vp}\|$ splits additively into the norm of the hardening body and the one of the dissipative strain element - see eq. $(118)_1$ - as well as similarly into the norms of the friction component and the associated dissipative strain element in $(119)_1$.

4.5 Yield Function and Direction of Viscoplastic Flow

The deviatoric part of the MANDEL stress tensors[12] in the equilibrium relation (79) is evaluated on $\mathcal{B}_{\mathrm{vp}}$, by using the component stresses (107) and (108) as well as the split in (9) and the condition in (113):

$$\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}} = \|\underset{\mathrm{vp}}{\mathbf{M}}_{\eta}^{\mathsf{D}}\| \, \mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}^{\mathsf{D}}\right) + \sqrt{\frac{2}{3}}\,\kappa_0\,\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa_0\mathrm{e}}^{\mathsf{D}}\right) + \sqrt{\frac{2}{3}}\,\kappa\,\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa\mathrm{e}}^{\mathsf{D}}\right) + \underset{\mathrm{vp}}{\mathbf{M}}_{\xi}^{\mathsf{D}} \,. \tag{120}$$

Rearranging (120) with the relations (9), $(118)_2$ and $(119)_2$ leads to the expression[13]

$$\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}} - \underset{\mathrm{vp}}{\mathbf{M}}_{\xi}^{\mathsf{D}} = \|\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}} - \underset{\mathrm{vp}}{\mathbf{M}}_{\xi}^{\mathsf{D}}\| \, \mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}} - \underset{\mathrm{vp}}{\mathbf{M}}_{\xi}^{\mathsf{D}}\right) = \left(\|\underset{\mathrm{vp}}{\mathbf{M}}_{\eta}^{\mathsf{D}}\| + \sqrt{\frac{2}{3}}(\kappa_0 + \kappa)\right) \mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}^{\mathsf{D}}\right) \,. \tag{121}$$

Since all summands in the bracket on the right hand side of (121) may only have non-negative values, equation (121) can be separated into its direction and its norm, which directly gives the direction of evolution for the viscoplastic flow $\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}^{\mathsf{D}}$ according to

$$\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}^{\mathsf{D}}\right) = \mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}} - \underset{\mathrm{vp}}{\mathbf{M}}_{\xi}^{\mathsf{D}}\right) \,. \tag{122}$$

Moreover, the scalar equation

$$\sqrt{\frac{3}{2}}\,\|\underset{\mathrm{vp}}{\mathbf{M}}_{\eta}^{\mathsf{D}}\| = \sqrt{\frac{3}{2}}\,\|\underset{\mathrm{vp}}{\mathbf{M}}^{\mathsf{D}} - \underset{\mathrm{vp}}{\mathbf{M}}_{\xi}^{\mathsf{D}}\| - (\kappa_0 + \kappa) =: F \tag{123}$$

is obtained from (121) by comparison of the coefficients. The relation (123) implies a mathematical contradiction for all model states of $F < 0$, since the stress equilibrium, as evaluated in (120), holds for non-zero viscoplastic flow $\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}^{\mathsf{D}} \neq \mathbf{0}$ only - see footnote 13. This means that viscoplastic strain may evolve solely during processes with $F > 0$ and, furthermore, $F < 0$ indicates the thermoelastic range of the model. Hence, the yield function F turns out in (123) with the case distinction - see also Bröcker and Matzenmiller (2013):

$$F = \begin{cases} \leq 0 & \text{elastic domain} \\ > 0 & \text{viscoplastic domain} \end{cases} \,. \tag{124}$$

4.6 Viscoplastic Flow and Evolution Equations of Dissipative Strain Elements

Viscoplastic Flow

Since the nonlinear dashpot is arranged in a series connection to the elastic spring in the rheological network of Figure 8, the dashpot provides a MANDEL type stress measure[14] on the viscoplastic intermediate configuration $\mathcal{B}_{\mathrm{vp}}$, which turns out according to the definition (17) as

$$\underset{\mathrm{vp}}{\mathbf{M}}_{\eta}^{\mathsf{D}} = \sqrt{\frac{2}{3}}\left(\eta\,\|\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}^{\mathsf{D}}\|\right)^{1/m} d_0\,\mathsf{N}\left(\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}^{\mathsf{D}}\right) \,. \tag{125}$$

Inserting the norm of the dashpot stress (125) into (123) yields the relation

$$\sqrt{\frac{3}{2}}\,\|\underset{\mathrm{vp}}{\mathbf{M}}_{\eta}^{\mathsf{D}}\| = \left(\eta\,\|\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}^{\mathsf{D}}\|\right)^{1/m} d_0 = \langle F \rangle \,, \tag{126}$$

[12] Note, since $\mathbf{F}_{\mathrm{vp}}$ is a unimodular tensor, no spherical stress contribution may arise from the viscoplastic setting of the rheological network. However, the hydrostatic state of stress, resulting from the elastic spring between the configurations $\mathcal{B}_{\mathrm{vp}}$ and $\mathcal{B}_{\mathrm{t}}$, affects the total rheological model in consequence of the equilibrium of stresses (39) in the series connections of the elements.

[13] The stress relations (107) and (108) hold only for $\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa\mathrm{e}}^{\mathsf{D}} \neq \mathbf{0}$ and $\underset{\mathrm{vp}}{\mathbf{D}}_{\kappa_0\mathrm{e}}^{\mathsf{D}} \neq \mathbf{0}$. Hence, in consequence of $(118)_2$ and $(119)_2$, equation (120) is valid only for all states of $\underset{\mathrm{vp}}{\mathbf{D}}_{\mathrm{vp}}^{\mathsf{D}} \neq \mathbf{0}$.

[14] See also the stress relation of the first elastic spring and the conclusions in Section 3.3.

where additionally the Macauley-brackets $\langle x \rangle = (x + |x|)/2$ have been introduced for the yield function F in order to distinguish loading $F > 0$. Rearranging of (126) provides the norm of the viscoplastic flow tensor

$$\|\underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\text{vp}}\| = \frac{1}{\eta} \left\langle \frac{F}{d_0} \right\rangle^m = \lambda \,, \tag{127}$$

which is abbreviated with the symbol λ and denoted as the viscoplastic multiplier. Combining relation (122) and (127) gives the flow rule for the evolution of the viscoplastic strains

$$\underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\text{vp}} = \lambda \, \mathsf{N}\left(\underset{\text{vp}}{\mathbf{M}}{}^{\mathsf{D}} - \underset{\text{vp}}{\mathbf{M}}{}^{\mathsf{D}}_{\xi} \right) \,. \tag{128}$$

According to the definition in (14), the arclength of the viscoplastic strain rate is obtained as

$$\bar{\varepsilon}_{\text{vp}} = \sqrt{\frac{2}{3}} \int_0^t \|\underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\text{vp}}\| \, \mathrm{d}\tau = \int_0^t \dot{\bar{\varepsilon}}_{\text{vp}} \, \mathrm{d}\tau \quad , \qquad \dot{\bar{\varepsilon}}_{\text{vp}} = \sqrt{\frac{2}{3}} \, \lambda \,. \tag{129}$$

Dissipative Strain Element of Kinematic Hardening

Motivated by the sufficient condition $(114)_1$ and the uniaxial model of Bröcker and Matzenmiller (2013), the norm $\|\underset{\xi}{\mathbf{D}}{}^{\mathsf{D}}_{\xi\text{d}}\|$ is chosen for the dissipative strain element in the series connection with the spring of kinematic hardening according to

$$\|\underset{\xi}{\mathbf{D}}{}^{\mathsf{D}}_{\xi\text{d}}\| = \frac{1}{\xi^\infty} \, \| \big(\underset{\xi}{\mathbf{C}}_{\xi\text{e}} \, \underset{\xi}{\mathbf{M}}_{\xi} \big)^{\mathsf{D}} \| \sqrt{\frac{2}{3}} \lambda \tag{130}$$

with ξ^∞ as the saturation value of the norm $\| \big(\underset{\xi}{\mathbf{C}}_{\xi\text{e}} \, \underset{\xi}{\mathbf{M}}_{\xi} \big)^{\mathsf{D}} \|$ and the factor λ driving the change of the internal strain rate. Hence, the evolution equation of $\underset{\xi}{\mathbf{D}}{}^{\mathsf{D}}_{\xi\text{d}}$ is obtained from $(114)_1$ and (130) as

$$\underset{\xi}{\mathbf{D}}{}^{\mathsf{D}}_{\xi\text{d}} = \frac{1}{\xi^\infty} \, \big(\underset{\xi}{\mathbf{C}}_{\xi\text{e}} \, \underset{\xi}{\mathbf{M}}_{\xi} \big)^{\mathsf{D}} \sqrt{\frac{2}{3}} \lambda \,. \tag{131}$$

Dissipative Strain Element of Isotropic Hardening

The ansatz for the norm $\|\underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa\text{d}}\|$ is taken similarly as for the case of the uniaxial model of Bröcker and Matzenmiller (2013):

$$\|\underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa\text{d}}\| = \frac{\kappa}{\kappa^\infty} \, \lambda \,, \tag{132}$$

which yields with (9), $(118)_2$ and (127) to the evolution equation

$$\underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa\text{d}} = \frac{\kappa}{\kappa^\infty} \, \underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\text{vp}} \tag{133}$$

with κ^∞ as the saturation value of the stress of isotropic hardening κ.

Dissipative Strain Element of Friction Body

The norm $\|\underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa_0\text{d}}\|$ for the third dissipative strain element is adopted from the uniaxial model of Bröcker and Matzenmiller (2013) as

$$\|\underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa_0\text{d}}\| = \big(1 - \beta_1 \exp\left(-\beta_2 \, \bar{\varepsilon}_{\text{vp}} \right) \big) \lambda \tag{134}$$

with the arclength $\bar{\varepsilon}_{\text{vp}}$ of the viscoplastic strain rate according to (129). Hence, with (9), (134), $(119)_2$ and (127), the evolution equation of this dissipation body in the series arrangement to the friction element reads as

$$\underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa_0\text{d}} = \big(1 - \beta_1 \exp\left(-\beta_2 \, \bar{\varepsilon}_{\text{vp}} \right) \big) \underset{\text{vp}}{\mathbf{D}}{}^{\mathsf{D}}_{\text{vp}} \,. \tag{135}$$

4.7 Constitutive Relation of Total and Partial Stresses and Mechanical Dissipation

By means of the potential relation $(100)_1$, the total stress state on the viscoplastic intermediate configuration $\mathcal{B}_{\rm vp}$ is calculated from the free energy (82) - see also (16)

$$\underset{\rm vp}{\mathbf{T}} = 2\rho_{\rm R}\frac{\partial\psi}{\partial \underset{\rm vp}{\mathbf{C}}_{\rm el}} = \left(K\,J_{\rm el}(J_{\rm el}-1)\,\mathbf{1} + G\,\underset{\rm vp}{\widehat{\mathbf{C}}}{}^{\mathsf{D}}_{\rm el}\right)\underset{\rm vp}{\mathbf{C}}{}^{-1}_{\rm el}\,. \tag{136}$$

The MANDEL type stress of kinematic hardening becomes with the help of (82) and $(100)_2$ at the intermediate configuration $\mathcal{B}_\xi$

$$\underset{\xi}{\mathbf{M}}{}^{*}_{\xi} = 2\rho_{\rm R}\frac{\partial\psi}{\partial \underset{\xi}{\mathbf{C}}_{\xi\rm e}} = G_\xi\,\underset{\xi}{\mathbf{C}}{}^{\mathsf{D}}_{\xi\rm e}\,\underset{\xi}{\mathbf{C}}{}^{-1}_{\xi\rm e} \tag{137}$$

and the property $\underset{\xi}{\mathbf{C}}{}^{\mathsf{D}}_{\xi\rm e} = \underset{\xi}{\widehat{\mathbf{C}}}{}^{\mathsf{D}}_{\xi\rm e}$ holds. Moreover, the scalar stress of isotropic hardening emerges from the free energy (82) according to $(107)_3$ as

$$\kappa = E_\kappa\,\bar{\varepsilon}_{\kappa\rm e}\,, \tag{138}$$

and its evolution equation is obtained from (138) with $(85)_3$, $(118)_1$, (132) and $(129)_3$ as

$$\dot{\kappa} = E_\kappa\,\dot{\bar{\varepsilon}}_{\kappa\rm e} = E_\kappa\sqrt{\frac{2}{3}}\,\|\underset{\rm vp}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa\rm e}\| = E_\kappa\left(1-\frac{\kappa}{\kappa^\infty}\right)\dot{\bar{\varepsilon}}_{\rm vp}\,, \tag{139}$$

i.e. the structure of (139) is identical to the one of the uniaxial model.

The internal rate $\dot{\bar{\varepsilon}}_{\kappa_0\rm e}$ of energy storage ψ_{κ_0} in the friction body - see (81) - may be determined from $(84)_3$, $(119)_1$, (134) and $(129)_3$:

$$\dot{\bar{\varepsilon}}_{\kappa_0\rm e} = \sqrt{\frac{2}{3}}\,\|\underset{\rm vp}{\mathbf{D}}{}^{\mathsf{D}}_{\kappa_0\rm e}\| = \beta_1\exp\left(-\beta_2\,\bar{\varepsilon}_{\rm vp}\right)\dot{\bar{\varepsilon}}_{\rm vp}\,. \tag{140}$$

The mechanical dissipation (115) leads to the non-negative expression for $\delta_{\rm M}$ by using the relations $(107)_2$, $(108)_2$, (123), (127), $(129)_3$, (130), (132) and (134):

$$\delta_{\rm M} = \frac{1}{\rho_{\rm R}}\left(F + \frac{1}{\xi^\infty}\,\|(\underset{\xi}{\mathbf{C}}_{\xi\rm e}\,\underset{\xi}{\mathbf{M}}_{\xi})^{\mathsf{D}}\|^2 + \frac{\kappa^2}{\kappa^\infty} + \kappa_0\Big(1-\beta_1\exp\left(-\beta_2\,\bar{\varepsilon}_{\rm vp}\right)\Big)\right)\dot{\bar{\varepsilon}}_{\rm vp} \geq 0\,. \tag{141}$$

The set of constitutive equations (136) – (141) is completed with the thermal contribution of the deformation gradient $(69)_2$, the yield function (123), the flow rule (128) together with the viscoplastic multiplier (127) as well as with the evolution equation of the dissipative strain element of kinematic hardening in (131) for the rheological model of thermoviscoplasticity in Figure 8 .

Remark:
All material parameters of the model may in general be non-negative functions of the temperature θ. This yields for each temperature-dependent parameter of the free energy (82) one additional summand in the entropy (101) besides the partial derivative $\frac{\partial\hat{Q}}{\partial\theta}$. However, the validity of the mechanical dissipation (141) and the entire line of argumentation in Section 4 remain unaffected, if some or all the parameters are assumed as temperature dependent.

4.8 Summary of Constitutive Model on Initial $\mathcal{B}_0$ and Thermal Intermediate Configuration $\mathcal{B}_{\rm th}$

By means of some lengthy mathematical rearrangements, the constitutive model can be transferred to the initial $\mathcal{B}_0$ or the thermal intermediate configuration $\mathcal{B}_{\rm th}$. These two configurations only differ from each other due to the isotropic thermal expansion - see $(69)_2$. Hence, the second PIOLA-KIRCHHOFF stress is expressed in terms of the configurations $\mathcal{B}_0$ and $\mathcal{B}_{\rm th}$ according to

$$\underset{0}{\mathbf{T}} = \underset{0}{\mathbf{C}}{}^{-1}\left(K\,J_{\rm el}(J_{\rm el}-1)\,\mathbf{1} + G\,(\underset{0}{\widehat{\mathbf{C}}}\,\underset{\rm th}{\mathbf{C}}{}^{-1}_{\rm vp})^{\mathsf{D}}\right)\quad,\qquad J_{\rm el} = \frac{J}{\hat{\varphi}(\theta)^3} \tag{142}$$

with the temperature dependent function $\hat{\varphi}(\theta)$ specifying the thermal deformation. The yield function F results to

$$F = \sqrt{\frac{3}{2}}\,\Sigma - (\kappa_0 + \kappa) \quad , \qquad \Sigma = \sqrt{\mathrm{tr}\Big(\underset{\mathrm{th}}{\mathbf{\Sigma}}^2\Big)} \tag{143}$$

with the stress type tensor

$$\underset{\mathrm{th}}{\mathbf{\Sigma}} = \big(\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{M}}\,\underset{\mathrm{th}}{\mathbf{T}}\big)^{\mathsf{D}} - \underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}}\,\underset{\mathrm{th}}{\mathbf{M}}_{\xi}^{*} \quad , \qquad \underset{\mathrm{th}}{\mathbf{T}} = \hat{\varphi}(\theta)^2\,\underset{0}{\mathbf{T}} \tag{144}$$

and the property $\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}}\,\underset{\mathrm{th}}{\mathbf{M}}_{\xi}^{*} = \big(\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}}\,\underset{\mathrm{th}}{\mathbf{M}}_{\xi}\big)^{\mathsf{D}}$. The viscoplastic flow rule is obtained as

$$\underset{\mathrm{th}}{\dot{\mathbf{C}}}_{\mathrm{vp}} = \frac{2\,\lambda}{\Sigma}\,\underset{\mathrm{th}}{\mathbf{\Sigma}}\,\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}} \quad , \qquad \lambda = \frac{1}{\eta}\left\langle\frac{F}{d_0}\right\rangle^{m} . \tag{145}$$

The scalar stress of isotropic hardening is governed by the evolution equation

$$\dot{\kappa} = E_{\kappa}\left(1 - \frac{\kappa}{\kappa^{\infty}}\right)\dot{\bar{\varepsilon}}_{\mathrm{vp}} \quad , \qquad \dot{\bar{\varepsilon}}_{\mathrm{vp}} = \sqrt{\frac{2}{3}}\,\lambda\,. \tag{146}$$

The MANDEL type stress tensor of kinematic hardening is calculated as

$$\underset{\mathrm{th}}{\mathbf{M}}_{\xi}^{*} = \mathbf{F}_{\xi\mathrm{d}}^{-1}\underset{\xi}{\mathbf{M}}_{\xi}^{*}\mathbf{F}_{\xi\mathrm{d}}^{-\mathsf{T}} = \mathbf{F}_{\mathrm{vp}}^{-1}\underset{\mathrm{vp}}{\mathbf{M}}_{\xi}^{\mathsf{D}}\mathbf{F}_{\mathrm{vp}}^{-\mathsf{T}} = G_{\xi}\,\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}}^{-1}\big(\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}}\,\underset{\mathrm{th}}{\mathbf{C}}_{\xi\mathrm{d}}^{-1}\big)^{\mathsf{D}} \tag{147}$$

with the flow rule of the associated dissipative strain element

$$\underset{\mathrm{th}}{\dot{\mathbf{C}}}_{\xi\mathrm{d}} = \sqrt{\frac{2}{3}}\,\frac{2\,\lambda}{\xi^{\infty}}\,\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}}\,\underset{\mathrm{th}}{\mathbf{M}}_{\xi}^{*}\,\underset{\mathrm{th}}{\mathbf{C}}_{\xi\mathrm{d}}\,. \tag{148}$$

Besides the capability of thermal expansion and the volumetric stress relation, the mechanical behavior of the constitutive equations summarized above in this section are equivalent to the material model proposed by Shutov and Kreißig (2008). However, in the paper at hand, a novel method has been utilized for deducing the constitutive relations by means of a rheological network of finite deformations as a generalization of the procedure applied by Bröcker and Matzenmiller (2013). Moreover, the resulting mechanical dissipation

$$\delta_{\mathrm{M}} = \frac{1}{\rho_{\mathrm{R}}}\left(F + \kappa_0\Big(1 - \beta_1\exp\left(-\beta_2\,\bar{\varepsilon}_{\mathrm{vp}}\right)\Big) + \frac{\kappa^2}{\kappa^{\infty}} + \frac{1}{\xi^{\infty}}\,\mathrm{tr}\Big(\big(\underset{\mathrm{th}}{\mathbf{C}}_{\mathrm{vp}}\,\underset{\mathrm{th}}{\mathbf{M}}_{\xi}^{*}\big)^2\Big)\right)\dot{\bar{\varepsilon}}_{\mathrm{vp}} \geq 0 \tag{149}$$

differs from the model of Shutov and Kreißig (2008), since a process-dependent energy storage is assumed for the friction body, and hence, the stress power $l_{\kappa_0} = \frac{1}{\rho_{\mathrm{R}}}\kappa_0\dot{\bar{\varepsilon}}_{\mathrm{vp}}$, associated with ideal plasticity, is partly dissipated as heat only - see the second summand of (149).

5 Enhancement for Damage Representation

In continuum damage mechanics, the concept of effective stresses has been established to account for damage evolution in solids - see Lemaitre and Chaboche (1990), Lemaitre and Desmorat (2005). Its main idea is summarized in Figure 10. During a loading process, defects are generated in solid materials. These imperfections in the physical space reduce the area A_0 of the cross section, which effectively transfers the external force F. This fact motivates the transformation of the physical space into an effective one, whose cross section A is reduced for the fraction of the defects A_{def}, i.e. $A = A_0 - A_{\mathrm{def}}$ holds for the effective space. The equilibrium of forces of both spaces gives

$$F = \sigma^{\mathrm{a}} A_0 = \sigma(A_0 - A_{\mathrm{def}}) \tag{150}$$

with σ^{a} representing the actual stress of the physical space with damage and σ denoting the effective stress. By means of the definition of the scalar internal damage variable

$$D = \frac{A_{\mathrm{def}}}{A_0} \qquad D \in [0, 1)\,, \tag{151}$$

with the range of values from $D = 0$ (undamaged state) to $D \to 1$ approaching one (total fracture), eq. (150) may be rearranged according to

$$\sigma = \frac{\sigma^{\mathrm{a}}}{(1 - D)} \qquad \Leftrightarrow \qquad \sigma^{\mathrm{a}} = (1 - D)\,\sigma\,. \tag{152}$$

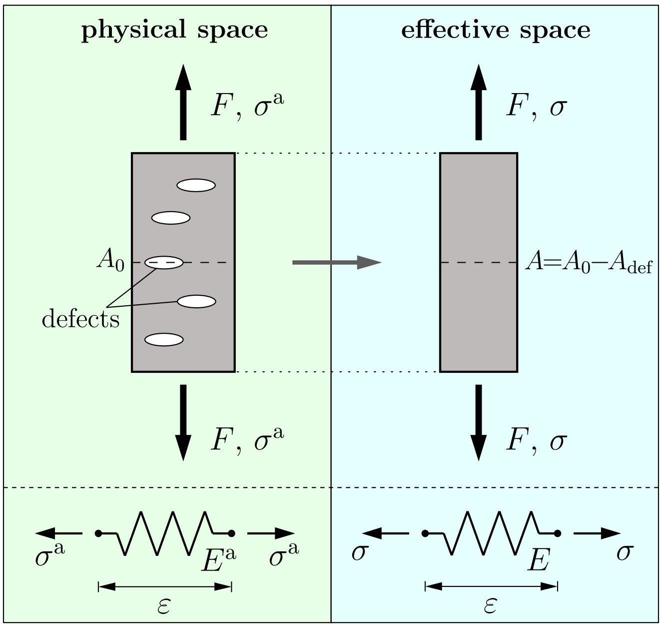

Figure 10: Concept of effective stress and application to rheological models

Thus, the stress in the damaged state σ^{a} is related to the effective counterpart σ by means of the damage or degradation factor $(1-D)$. The damage variable D has to be calculated by a closed form damage criterion or an evolution equation. This may be chosen for example in one of the most simple cases as

$$D = \left\langle \frac{\bar{\varepsilon}-\varepsilon_{\mathrm{c}}}{\varepsilon_{\mathrm{f}}-\varepsilon_{\mathrm{c}}} \right\rangle^{n_D} \qquad \Leftrightarrow \qquad \dot{D} = \frac{n_D}{\varepsilon_{\mathrm{f}}-\varepsilon_{\mathrm{c}}} \left\langle \frac{\bar{\varepsilon}-\varepsilon_{\mathrm{c}}}{\varepsilon_{\mathrm{f}}-\varepsilon_{\mathrm{c}}} \right\rangle^{n_D-1} \dot{\bar{\varepsilon}}\,, \tag{153}$$

in which n_D is a positive exponent, $\varepsilon_{\mathrm{c}} \geq 0$ is the critical threshold of the arclength $\bar{\varepsilon} = \int_0^t |\dot{\varepsilon}|\,\mathrm{d}\tau$, where damage processes are initialized, and $\varepsilon_{\mathrm{f}} > \varepsilon_{\mathrm{c}}$ is the associated state of failure, i.e. $D(\bar{\varepsilon} = \varepsilon_{\mathrm{f}}) = 1$.

The relationship (152) may be formally introduced into the rheological networks as well, leading to damage representation in rheological models - see Bröcker and Matzenmiller (2012). In the framework of continuum mechanics and thermomechanically consistent material modeling, the constitutive equations may be deduced from the rheological network in agreement to the well established concept of effective stresses of continuum damage mechanics (Lemaitre and Chaboche, 1990). It turns out that the strain equivalence holds for the the total and the partial strain measures, i.e. the strain responses of the effective and the damaged state of the model are equivalent. Moreover, in the special case of isothermal conditions, a linear relationship turns out between the actual (damaged) and the effective (undamaged) state of the constitutive model[15]. Hence, the damaged state of the model is calculated by multiplying the total and internal effective stresses with the degradation/damage factor $(1-D)$, i.e. the stress state affected by damage reads

$$\sigma^{\mathrm{a}} = (1-D)\,\sigma \tag{154}$$

in the uniaxial case or

$$\underset{0}{\mathbf{T}}^{\mathrm{a}} = (1-D)\,\underset{0}{\mathbf{T}} \qquad \Leftrightarrow \qquad \underset{\mathrm{vp}}{\mathbf{T}}^{\mathrm{a}} = (1-D)\,\underset{\mathrm{vp}}{\mathbf{T}} \qquad \Leftrightarrow \qquad \underset{\mathrm{t}}{\mathbf{T}}^{\mathrm{a}} = (1-D)\,\underset{\mathrm{t}}{\mathbf{T}} \tag{155}$$

in applications of finite deformation kinematics. Analogously, the internal stresses of hardening and the initial yield limit are obtained according to

$$\underset{\xi}{\mathbf{M}}^{\mathrm{a}} = (1-D)\,\underset{\xi}{\mathbf{M}}_{\xi} \qquad , \qquad \kappa^{\mathrm{a}} = (1-D)\,\kappa \qquad , \qquad \kappa_0^{\mathrm{a}} = (1-D)\,\kappa_0\,. \tag{156}$$

In finite thermoviscoplasticity, the evolution equation of damage $\dot{D}$ may be specified as given in $(153)_2$, with the arclength $\bar{\varepsilon}$ of the total strain substituted with the viscoplastic one $\bar{\varepsilon}_{\mathrm{vp}}$. The mechanical dissipation $\delta_{\mathrm{M}}^{\mathrm{a}}$ for the damaged state of the material results as

$$\delta_{\mathrm{M}}^{\mathrm{a}} = (1-D)\,\delta_{\mathrm{M}} + \psi_{\mathrm{M}}\,\dot{D} \geq 0 \tag{157}$$

[15] In general there is a coupling between the effective and the nominal state of the model due to the temperature dependency of the material parameters and the dissipation of heat during the loading process.

with δ_{M} from (149) and the mechanical contribution

$$\psi_{\mathrm{M}} = \psi_{\mathrm{el}} + \psi_{\xi} + \psi_{\kappa} + \psi_{\kappa_0} \tag{158}$$

of the free energy in the equations (81) and (82). Hence, the first summand of the dissipation power (157) is driven by viscoplastic deformation and the second contribution of (157) is due to damage evolution.

6 Conclusions

The enhanced concept of rheological modeling (Bröcker and Matzenmiller, 2013) is generalized to finite deformations. By assembling the basic bodies into a network, kinematical and kinetic relations of the constitutive model arise using the elementary properties of series and parallel arrangement of elements within the concept of dual variables. Hence, the finite material equations are obtained in an analogous procedure as in the uniaxial case, providing the yield function and the flow rule naturally from the equilibrium of stresses. The resulting constitutive theory is very similar to well-known approaches of viscoplasticity in the literature, however, derived differently as usual and extended to an improved description of energy storage. Moreover, damage evolution is additionally considered for the proposed model.

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