DOI: 10.24352/UB.OVGU-2017-015

TECHNISCHE MECHANIK, 36, 1-2, (2016), 132 – 141
submitted: July 10, 2015

Mathematical Models for Thin Piezoelectric Interphases Including Thermal Effects

M. Serpilli

We study the thermo-electromechanical behavior of a thin interphase, constituted by a piezoelectric anisotropic thin layer, embedded between two generic three-dimensional piezoelectric bodies by means of the asymptotic analysis including also thermal effects. After defining a small real dimensionless parameter ε, which will tend to zero, we characterize two different limit models and their associated limit problems, the so-called weak and strong thermo-piezoelectric interface models, respectively. Moreover, we identify the non classical thermo-electromechanical transmission conditions at the interface between the two three-dimensional bodies.

1 Introduction

In recent years the conception and use of smart materials have undergone a major development in all fields of aeronautical, mechanical and civil engineering. Smart materials, such as piezoelectric materials, are often integrated within the structure in different configurations: for instance, piezoelectric transducers can be embedded or glued onto the structural members to be controlled. Moreover, the same piezoelectric actuators are often obtained by alternating different thin layers of material with highly contrasted thermo-electromechanical properties. This generates different types of complex multimaterial assemblies, in which each phase interacts with the others. An extensive list of references on the subject can be found in the following cited papers for what concerns with piezoelectric interphases/interfaces problems using classical variational tools: see, for instance, Benveniste (2006, 2009) for curved thin interphases in conduction phenomena.

The successful application of the asymptotic methods to obtain a mathematical justification of thin structure models in the field of linear and non linear elasticity (see, e.g., Ciarlet (1997)) and in piezoelectricity, taking into account both sensor and actuator functions and the influence of temperature (see, e.g. Blanchard and Francfort (1987); Figueiredo and Franco Leal (2005); Weller and Licht (2010); Miara and Suarez (2013)) has stimulated the research toward a rational simplification of the modeling of complex structures obtained joining elements of different dimensions and/or materials of highly contrasted properties. Thin interphases represent one of the most peculiar bonded joint between two media. Within the theory of elasticity, the asymptotic analysis of a thin elastic interphase between two elastic materials has been deeply investigated through the years, by varying the rigidity ratios between the thin inclusion and the surrounding materials and by considering different geometry features. For instance, it is worth mentioning the contributions by Geymonat et al. (1999), Krasucki et al. (2004), the works by Lebon and Rizzoni (2010, 2011) for the case of thin interfaces with similar and hard rigidities, and, also, the works by Bessoud et al. (2009, 2008, 2011) in which the authors studied the case of plate-like and shell-like elastic inclusions with high rigidity in a rigorous functional framework.

This work is conceived as a generalization of the previous work by Serpilli (2015) on asymptotic weak and strong piezoelectric interface models, taking into account the effect of temperature. In the present work we identify two different interface limit models of an assembly constituted by a thin thermo-piezoelectric layer inserted between two generic thermo-piezoelectric bodies by means of an asymptotic analysis. By defining a small real parameter ε, associated with the thickness and the thermo-electromechanical properties of the middle layer, we perform an asymptotic analysis by letting ε tend to zero. We analyze two different situations by varying the thermo-electromechanical stiffnesses ratios between the middle layer and the adherents. The first case corresponds to the so-called *weak* thermo-piezoelectric interface model, where the thermo-electromechanical coefficients of the intermediate domain have order of magnitude ε with respect to those of the surrounding bodies: in this case, the intermediate layer is considered to be soft, from a mechanical point of view, and with small thermo-electric conductivity properties with respect to the upper and lower bodies. The second case of study is the so-called *strong* thermo-piezoelectric interface model, where the thermo-electromechanical rigidities have order of magnitude $\frac{1}{\varepsilon}$: in this case, the middle layer is rigid and with high thermo-electric conductivities. Within the reduced models, the interphase is replaced by a material surface whose energy, in both cases, is the limit of the interphase energy. This surface energy is then translated in ad hoc transmission conditions at the interface.

2 The physical problem

In the sequel, Greek indices range in the set $\{1, 2\}$, Latin indices range in the set $\{1, 2, 3\}$, and the Einstein's summation convention with respect to the repeated indices is adopted. We also introduce the following notation for the scalar product:

$$\mathbf{a} \cdot \mathbf{b} := a_i b_i, \quad \text{for all vectors } \mathbf{a} = (a_i) \text{ and } \mathbf{b} = (b_i).$$

Let us consider a three-dimensional Euclidian space identified by $\mathbb{R}^3$ and such that the three vectors $\mathbf{e}_i$ form an orthonormal basis. Let Ω^+ and Ω^- be two disjoint open domains with smooth boundaries $\partial\Omega^+$ and $\partial\Omega^-$. Let $\omega := \{\partial\Omega^+ \cap \partial\Omega^-\}^\circ$ be the interior of the common part of the boundaries which is assumed to be a non empty domain in $\mathbb{R}^2$ having a positive two-dimensional measure. We consider the assembly constituted by two solids bonded together by an intermediate thin plate-like body $\Omega^{m,\varepsilon}$ of thickness $2h^\varepsilon$, where $0 < \varepsilon < 1$ is a dimensionless small real parameter which will tend to zero. We suppose that the thickness h^ε of the middle layer depends linearly on ε, so that $h^\varepsilon = \varepsilon h$.

More precisely, we denote respectively with $\Omega^{\pm,\varepsilon} := \{x^\varepsilon := x \pm \varepsilon h \mathbf{e}_3;\ x \in \Omega^\pm\}$, the translation of Ω^+ (resp. Ω^-) along the direction $\mathbf{e}_3$ (resp. $-\mathbf{e}_3$) of the quantity εh, with $\Omega^{m,\varepsilon} := \omega \times (-\varepsilon h, \varepsilon h)$, the central plate-like domain, and with $\Omega^\varepsilon := \Omega^{+,\varepsilon} \cup \Omega^{m,\varepsilon} \cup \Omega^{-,\varepsilon}$, the reference configuration of the assembly. Moreover, we define with $S^{\pm,\varepsilon} := \omega \times \{\pm\varepsilon h\} = \Omega^{\pm,\varepsilon} \cap \Omega^{m,\varepsilon}$, the upper and lower faces of the intermediate plate-like domain, $\Gamma^{\pm,\varepsilon} := \partial\Omega^{\pm,\varepsilon}/S^{\pm,\varepsilon}$, and $\Gamma^{m,\varepsilon}_{lat} := \partial\omega \times (-\varepsilon h, \varepsilon h)$, its lateral surface, see Figure 1. Let $(\Gamma^\varepsilon_{mD}, \Gamma^\varepsilon_{mN})$, $(\Gamma^\varepsilon_{eD}, \Gamma^\varepsilon_{eN})$

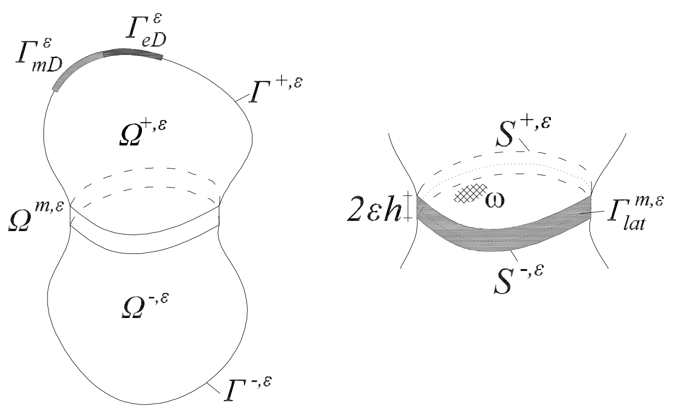

Figure 1: The reference configuration and the geometry of the interphase.

and $(\Gamma^\varepsilon_{tD}, \Gamma^\varepsilon_{tN})$ be three suitable partitions of $\partial\Omega^\varepsilon$, with Γ^ε_{mD}, Γ^ε_{eD} and Γ^ε_{tD} of strictly positive Hausdorff measure. The multimaterial is, on one hand, clamped along Γ^ε_{mD}, at an electrical potential $\varphi^\varepsilon = 0$ on Γ^ε_{eD} and at a certain temperature $\theta^\varepsilon = 0$ on Γ^ε_{tD} and, on the other hand, subject to surface forces g^ε_i on Γ^ε_{mN}, surface electrical charges d^ε on Γ^ε_{eN} and surface heat source w^ε on Γ^ε_{tN}. The assembly is also subject to body forces f^ε_i, electrical loadings ρ^ε_e and an internal heat source j^ε acting in $\Omega^{\pm,\varepsilon}$. We suppose, without loss of generality, that $\Omega^{m,\varepsilon}$ and $\Gamma^{m,\varepsilon}_{lat}$ are both free of mechanical, electrical and thermal charges. We consider the following regularity assumptions for the thermo-electromechanical loads: $f^\varepsilon_i \in L^2(\Omega^{\pm,\varepsilon})$, $\rho^\varepsilon_e \in L^2(\Omega^{\pm,\varepsilon})$, $j^\varepsilon \in L^2(\Omega^{\pm,\varepsilon})$, $g^\varepsilon_i \in L^2(\Gamma^\varepsilon_{mN})$, $w^\varepsilon \in L^2(\Gamma^\varepsilon_{tN})$ and $d^\varepsilon \in L^2(\Gamma^\varepsilon_{eN})$. We finally assume that $\Omega^{\pm,\varepsilon}$ and $\Omega^{m,\varepsilon}$ are constituted by three homogeneous linearly thermo-piezoelectric materials, whose constitutive laws are defined as follows:

$$\begin{cases} \sigma^\varepsilon_{ij}(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \theta^\varepsilon) = C^\varepsilon_{ijk\ell} e^\varepsilon_{k\ell}(\mathbf{u}^\varepsilon) - P^\varepsilon_{kij} E^\varepsilon_k(\varphi^\varepsilon) - X^\varepsilon_{ij}\theta^\varepsilon, \\ D^\varepsilon_i(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \theta^\varepsilon) = P^\varepsilon_{ijk} e^\varepsilon_{jk}(\mathbf{u}^\varepsilon) + H^\varepsilon_{ij} E^\varepsilon_j(\varphi^\varepsilon) + p^\varepsilon_i \theta^\varepsilon, \\ q^\varepsilon_i(\theta^\varepsilon) = -K^\varepsilon_{ij} \partial^\varepsilon_j \theta^\varepsilon, \end{cases} \tag{1}$$

where $(\sigma^\varepsilon_{ij})$ is the classical Cauchy stress tensor, $(e^\varepsilon_{ij}(\mathbf{u}^\varepsilon)) := \left(\frac{1}{2}(\partial^\varepsilon_i u^\varepsilon_j + \partial^\varepsilon_j u^\varepsilon_i)\right)$ is the linearized strain tensor, (D^ε_i) is the electrical displacement field, (q^ε_i) is the heat flow vector, φ^ε is the electrical potential and $E^\varepsilon_i(\varphi^\varepsilon) := -\partial^\varepsilon_i \varphi^\varepsilon$ its associated electrical field. $(C^\varepsilon_{ijk\ell})$, (P^ε_{ijk}), (H^ε_{ij}), (X^ε_{ij}), (p^ε_i) and (K^ε_{ij}) represent, respectively, the

classical fourth order elasticity tensor, the third order piezoelectric coupling tensor, the second order dielectric tensor, the second order thermal stress tensor, the pyroelectric vector and the second order thermal conductivity tensor related to $\Omega^{\pm,\varepsilon}$ and $\Omega^{m,\varepsilon}$.
Tensors $(C^{\varepsilon}_{ijk\ell})$, (H^{ε}_{ij}), (X^{ε}_{ij}) and (K^{ε}_{ij}) satisfy the following coercivity properties: for any symmetric matrix field (b_{ij}), there exists a constant $c > 0$ such that $C^{\varepsilon}_{ijk\ell}b_{k\ell}b_{ij} \geqslant c\sum_{i,j}|b_{ij}|^2$; for any vector field (a_i), there exist constants $c_1, c_2, c_3 > 0$ such that $H^{\varepsilon}_{ij}a_ja_i \geqslant c_1\sum_i |a_i|^2$, $X^{\varepsilon}_{ij}a_ja_i \geqslant c_2\sum_i |a_i|^2$ and $K^{\varepsilon}_{ij}a_ja_i \geqslant c_3\sum_i |a_i|^2$. Moreover, we have the symmetries $C^{\varepsilon}_{ijk\ell} = C^{\varepsilon}_{k\ell ij} = C^{\varepsilon}_{jik\ell}$, $H^{\varepsilon}_{ij} = H^{\varepsilon}_{ji}$, $P^{\varepsilon}_{kji} = P^{\varepsilon}_{kij}$, $X^{\varepsilon}_{ij} = X^{\varepsilon}_{ji}$ and $K^{\varepsilon}_{ij} = K^{\varepsilon}_{ji}$.

Let $\Sigma^{\varepsilon} \subset \partial\Omega^{\varepsilon}$, we introduce the functional spaces

$$V(\Omega^{\varepsilon}, \Sigma^{\varepsilon}) := \{v^{\varepsilon} \in H^1(\Omega^{\varepsilon});\ v^{\varepsilon} = 0 \text{ on } \Sigma^{\varepsilon}\},\ \ \mathbf{V}(\Omega^{\varepsilon}, \Sigma^{\varepsilon}) := [V(\Omega^{\varepsilon}, \Sigma^{\varepsilon})]^3.$$

The thermo-electromechanical state at the equilibrium is determined by the triplet $s^{\varepsilon} := (\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}, \theta^{\varepsilon})$. The physical variational problem defined over the variable domain Ω^{ε} reads as follows:

$$\left\{\begin{array}{l} \text{Find } s^{\varepsilon} \in \mathbf{V}(\Omega^{\varepsilon}, \Gamma^{\varepsilon}_{mD}) \times V(\Omega^{\varepsilon}, \Gamma^{\varepsilon}_{eD}) \times V(\Omega^{\varepsilon}, \Gamma^{\varepsilon}_{tD}) \text{ such that} \\ A^{-,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) + A^{+,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) + A^{m,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) = L^{\varepsilon}(r^{\varepsilon}), \end{array}\right. \tag{2}$$

for all $r^{\varepsilon} = (\mathbf{v}^{\varepsilon}, \psi^{\varepsilon}, \eta^{\varepsilon}) \in \mathbf{V}(\Omega^{\varepsilon}, \Gamma^{\varepsilon}_{mD}) \times V(\Omega^{\varepsilon}, \Gamma^{\varepsilon}_{eD}) \times V(\Omega^{\varepsilon}, \Gamma^{\varepsilon}_{tD})$, where the bilinear forms $A^{\pm,\varepsilon}(\cdot,\cdot)$ and $A^{m,\varepsilon}(\cdot,\cdot)$, and the linear form $L^{\varepsilon}(\cdot)$ are defined by

$$\begin{aligned} A^{\pm,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) := \int_{\Omega^{\pm,\varepsilon}} &\Big\{ C^{\pm,\varepsilon}_{ijk\ell} e^{\varepsilon}_{k\ell}(\mathbf{u}^{\varepsilon}) e^{\varepsilon}_{ij}(\mathbf{v}^{\varepsilon}) + H^{\pm,\varepsilon}_{ij} E^{\varepsilon}_j(\varphi^{\varepsilon}) E^{\varepsilon}_i(\psi^{\varepsilon}) + K^{\pm,\varepsilon}_{ij} \partial^{\varepsilon}_j \theta^{\varepsilon} \partial^{\varepsilon}_i \eta^{\varepsilon} + \\ &+ P^{\pm,\varepsilon}_{ihk}(E^{\varepsilon}_i(\psi^{\varepsilon}) e^{\varepsilon}_{hk}(\mathbf{u}^{\varepsilon}) - E^{\varepsilon}_i(\varphi^{\varepsilon}) e^{\varepsilon}_{hk}(\mathbf{v}^{\varepsilon})) - X^{\pm,\varepsilon}_{ij} e^{\varepsilon}_{ij}(\mathbf{v}^{\varepsilon})\theta^{\varepsilon} - p^{\pm,\varepsilon}_i \partial^{\varepsilon}_i \psi^{\varepsilon} \theta^{\varepsilon} \Big\}\, dx^{\varepsilon}, \\ A^{m,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) := \int_{\Omega^{m,\varepsilon}} &\Big\{ C^{m,\varepsilon}_{ijk\ell} e^{\varepsilon}_{k\ell}(\mathbf{u}^{\varepsilon}) e^{\varepsilon}_{ij}(\mathbf{v}^{\varepsilon}) + H^{m,\varepsilon}_{ij} E^{\varepsilon}_j(\varphi^{\varepsilon}) E^{\varepsilon}_i(\psi^{\varepsilon}) + K^{m,\varepsilon}_{ij} \partial^{\varepsilon}_j \theta^{\varepsilon} \partial^{\varepsilon}_i \eta^{\varepsilon} + \\ &+ P^{m,\varepsilon}_{ihk}(E^{\varepsilon}_i(\psi^{\varepsilon}) e^{\varepsilon}_{hk}(\mathbf{u}^{\varepsilon}) - E^{\varepsilon}_i(\varphi^{\varepsilon}) e^{\varepsilon}_{hk}(\mathbf{v}^{\varepsilon})) - X^{m,\varepsilon}_{ij} e^{\varepsilon}_{ij}(\mathbf{v}^{\varepsilon})\theta^{\varepsilon} - p^{m,\varepsilon}_i \partial^{\varepsilon}_i \psi^{\varepsilon} \theta^{\varepsilon} \Big\}\, dx^{\varepsilon}, \\ L^{\varepsilon}(r^{\varepsilon}) := \int_{\Omega^{\pm,\varepsilon}} & (f^{\varepsilon}_i v^{\varepsilon}_i + \rho^{\varepsilon}_e \psi^{\varepsilon} + j^{\varepsilon}\eta^{\varepsilon}) dx^{\varepsilon} + \int_{\Gamma^{\varepsilon}_{mN}} g^{\varepsilon}_i v^{\varepsilon}_i d\Gamma^{\varepsilon} - \int_{\Gamma^{\varepsilon}_{eN}} d^{\varepsilon}\psi^{\varepsilon} d\Gamma^{\varepsilon} - \int_{\Gamma^{\varepsilon}_{tN}} w^{\varepsilon}\eta^{\varepsilon} d\Gamma^{\varepsilon}. \end{aligned}$$

By virtue of the $\mathbf{V}(\Omega^{\varepsilon}, \Gamma^{\varepsilon}_{mD}) \times V(\Omega^{\varepsilon}, \Gamma^{\varepsilon}_{eD}) \times V(\Omega^{\varepsilon}, \Gamma^{\varepsilon}_{tD})$-coercivity of the bilinear forms and thanks to the Lax-Milgram lemma, problem (2) admits one and only one solution.

3 The asymptotic expansion method

In order to study the asymptotic behavior of the solution of problem (2) when ε tends to zero, we rewrite the problem on a fixed domain Ω independent of ε. By using the approach of Ciarlet (1997) we consider the bijection $\pi^{\varepsilon} : x \in \overline{\Omega} \mapsto x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$ given by

$$\left\{\begin{array}{ll} \pi^{\varepsilon}(x_1, x_2, x_3) = (x_1, x_2, x_3 - h(1-\varepsilon)), & \text{for all } x \in \overline{\Omega}^{+}_{tr}, \\ \pi^{\varepsilon}(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3), & \text{for all } x \in \overline{\Omega}^{m}, \\ \pi^{\varepsilon}(x_1, x_2, x_3) = (x_1, x_2, x_3 + h(1-\varepsilon)), & \text{for all } x \in \overline{\Omega}^{-}_{tr}, \end{array}\right.$$

where $\Omega^{\pm}_{tr} := \{x \pm h\mathbf{e}_3,\ x \in \Omega^{\pm}\}$, $\Omega^m := \omega \times (-h, h)$ and $S^{\pm} := \omega \times \{\pm h\}$. In order to simplify the notation, we identify $\Omega^{\pm}_{tr}$ with $\Omega^{\pm}$, and $\overline{\Omega}$ with $\overline{\Omega}^{\pm} \cup \overline{\Omega}^{m}$. Likewise, we note $\Gamma_{\pm} := \partial\Omega^{\pm}/S^{\pm}$, $\Gamma^m_{lat} := \partial\omega \times (-h, h)$, $(\Gamma_{mD}, \Gamma_{mN})$, $(\Gamma_{eD}, \Gamma_{eN})$ and $(\Gamma_{tD}, \Gamma_{tN})$, the partitions of $\partial\Omega := \Gamma^{\pm} \cup \Gamma^m_{lat}$.
Consequently,

$$\partial^{\varepsilon}_{\alpha} = \partial_{\alpha} \text{ and } \partial^{\varepsilon}_3 = \frac{1}{\varepsilon}\partial_3 \ \text{ in } \Omega^m.$$

In the sequel, only if necessary, we will note, respectively, with $(\mathbf{v}^{\pm}, \psi^{\pm}, \eta^{\pm})$ and $(\mathbf{v}^m, \psi^m, \eta^m)$, the restrictions of functions $(\mathbf{v}, \psi, \eta)$ to $\Omega^{\pm}$ and Ω^m.

With the unknown thermo-electromechanical state $s^{\varepsilon} = (\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}, \theta^{\varepsilon})$, we associate the scaled unknown thermo-electromechanical state $s(\varepsilon) := (\mathbf{u}(\varepsilon), \varphi(\varepsilon), \theta(\varepsilon))$ defined by:

$$u^{\varepsilon}_i(x^{\varepsilon}) = u_i(\varepsilon)(x),\ \ \varphi^{\varepsilon}(x^{\varepsilon}) = \varphi(\varepsilon)(x),\ \ \theta^{\varepsilon}(x^{\varepsilon}) = \theta(\varepsilon)(x) \quad \text{for all } x^{\varepsilon} = \pi^{\varepsilon}x \in \overline{\Omega}^{\varepsilon}.$$

We likewise associate with any test functions $r^{\varepsilon} = (\mathbf{v}^{\varepsilon}, \psi^{\varepsilon}, \eta^{\varepsilon})$, the scaled test functions $r = (\mathbf{v}, \psi, \eta)$, defined by the scalings:

$$v^{\varepsilon}_i(x^{\varepsilon}) = v_i(x),\ \ \psi^{\varepsilon}(x^{\varepsilon}) = \psi(x),\ \ \eta^{\varepsilon}(x^{\varepsilon}) = \eta(x) \quad \text{for all } x^{\varepsilon} = \pi^{\varepsilon}x \in \overline{\Omega}^{\varepsilon}.$$

We suppose that the thermo-electromechanical coefficients of $\Omega^{\pm}$ are independent of ε, so that

$$C^{\pm,\varepsilon}_{ijk\ell} := C^{\pm}_{ijk\ell},\;\; H^{\pm,\varepsilon}_{ij} := H^{\pm}_{ij},\;\; P^{\pm,\varepsilon}_{ijk} := P^{\pm}_{ijk},\;\; X^{\pm,\varepsilon}_{ij} := X^{\pm}_{ij},\;\; K^{\pm,\varepsilon}_{ij} := K^{\pm}_{ij},\;\; p^{\pm,\varepsilon}_{i} := p^{\pm}_{i},$$

while the thermo-electromechanical coefficients of Ω^m have the following dependence with respect to ε

$$C^{m,\varepsilon}_{ijk\ell} := \varepsilon^p C^{m}_{ijk\ell},\;\; H^{m,\varepsilon}_{ij} := \varepsilon^p H^{m}_{ij},\;\; P^{m,\varepsilon}_{ijk} := \varepsilon^p P^{m}_{ijk},\;\; X^{m,\varepsilon}_{ij} := \varepsilon^p X^{m}_{ij},\;\; K^{m,\varepsilon}_{ij} := \varepsilon^p K^{m}_{ij},\;\; p^{m,\varepsilon}_{i} := \varepsilon^p p^{m}_{i},$$

with $p \in \{-1, 1\}$. Two different limit behaviors will be characterized according to the choice of the exponent p: in the case of $p = -1$, we derive a model for a strong thermo-piezoelectric interface; by choosing $p = 1$, we deduce a model for a weak thermo-piezoelectric interface.

We also make the following assumptions on the applied mechanical, electrical and thermal loads:

$$\begin{array}{ll} f^\varepsilon_i(x^\varepsilon) = f_i(x) \text{ and } g^\varepsilon_i(x^\varepsilon) = g_i(x) & \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{\pm,\varepsilon}, \\ \rho^\varepsilon_e(x^\varepsilon) = \rho_e(x) \text{ and } d^\varepsilon(x^\varepsilon) = d(x) & \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{\pm,\varepsilon}, \\ j^\varepsilon(x^\varepsilon) = j(x) \text{ and } w^\varepsilon(x^\varepsilon) = w(x) & \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{\pm,\varepsilon}, \end{array}$$

where functions $f_i \in L^2(\Omega^{\pm})$, $\rho_e \in L^2(\Omega^{\pm})$, $j \in L^2(\Omega^{\pm})$, $g_i \in L^2(\Gamma_{mN})$, $d \in L^2(\Gamma_{eN})$ and $w \in L^2(\Gamma_{tN})$ are independent of ε. Thus $L^\varepsilon(r^\varepsilon) = L(r)$.

According to the previous hypothesis, problem (2) can be reformulated on a fixed domain Ω independent of ε. Thus we obtain the following scaled problem:

$$\left\{ \begin{array}{l} \text{Find } s(\varepsilon) \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD}) \times V(\Omega, \Gamma_{tD}) \text{ such that} \\ A^-(s(\varepsilon), r) + A^+(s(\varepsilon), r) + A^{m,p}(\varepsilon)(s(\varepsilon), r) = L(r), \end{array} \right. \tag{3}$$

for all $r \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD}) \times V(\Omega, \Gamma_{tD})$, $p \in \{-1, 1\}$, where

$$\begin{aligned} A^{\pm}(s(\varepsilon), r) := \int_{\Omega^{\pm}} \Big\{ & C^{\pm}_{ijk\ell} e_{k\ell}(\mathbf{u}(\varepsilon)) e_{ij}(\mathbf{v}) + H^{\pm}_{ij} \partial_j \varphi(\varepsilon) \partial_i \psi + K^{\pm}_{ij} \partial_j \theta(\varepsilon) \partial_i \eta + \\ & + P^{\pm}_{ihk} (\partial_i \varphi(\varepsilon) e_{hk}(\mathbf{v}) - \partial_i \psi e_{hk}(\mathbf{u}(\varepsilon))) - X^{\pm}_{ij} e_{ij}(\mathbf{v}) \theta(\varepsilon) - p^{\pm}_i \partial_i \psi \theta(\varepsilon) \Big\}\, dx, \\ A^{m,p}(\varepsilon)(s(\varepsilon), r) := {} & \varepsilon^{p-1} a^m(s(\varepsilon), r) + \varepsilon^p b^m(s(\varepsilon), r) + \varepsilon^{p+1} c^m(s(\varepsilon), r), \end{aligned}$$

with

$$\begin{aligned} a^m(s(\varepsilon), r) := \int_{\Omega^m} \{ & C^m_{i3j3} \partial_3 u_i(\varepsilon) \partial_3 v_j + H^m_{33} \partial_3 \varphi(\varepsilon) \partial_3 \psi + K^m_{33} \partial_3 \theta(\varepsilon) \partial_3 \eta + \\ & + P^m_{3i3} (\partial_3 \varphi(\varepsilon) \partial_3 v_i - \partial_3 \psi \partial_3 u_i(\varepsilon)) \}\, dx, \\ b^m(s(\varepsilon), r) := \int_{\Omega^m} \{ & C^m_{i3j\alpha} (\partial_3 u_i(\varepsilon) \partial_\alpha v_j + \partial_\alpha u_j(\varepsilon) \partial_3 v_i) + H^m_{\alpha 3} (\partial_3 \varphi(\varepsilon) \partial_\alpha \psi + \partial_3 \psi \partial_\alpha \varphi(\varepsilon)) + \\ & + K^m_{\alpha 3} (\partial_3 \theta(\varepsilon) \partial_\alpha \eta + \partial_3 \eta \partial_\alpha \theta(\varepsilon)) + P^m_{3\alpha i} (\partial_3 \varphi(\varepsilon) \partial_\alpha v_i - \partial_3 \psi \partial_\alpha u_i(\varepsilon)) + \\ & + P^m_{\alpha i 3} (\partial_\alpha \varphi(\varepsilon) \partial_3 v_i - \partial_\alpha \psi \partial_3 u_i(\varepsilon)) - X^m_{i3} \partial_3 v_i \theta(\varepsilon) - p^m_3 \partial_3 \psi \theta(\varepsilon) \}\, dx, \\ c^m(s(\varepsilon), r) := \int_{\Omega^m} \{ & C^m_{i\alpha j\beta} \partial_\alpha u_i(\varepsilon) \partial_\beta v_j + H^m_{\alpha\beta} \partial_\alpha \varphi(\varepsilon) \partial_\beta \psi + K^m_{\alpha\beta} \partial_\alpha \theta(\varepsilon) \partial_\beta \eta \\ & + P^m_{\alpha\beta i} (\partial_\alpha \varphi(\varepsilon) \partial_\beta v_i - \partial_\alpha \psi \partial_\beta u_i(\varepsilon)) - X^m_{i\alpha} \partial_\alpha v_i \theta(\varepsilon) - p^m_\alpha \partial_\alpha \psi \theta(\varepsilon) \Big\}\, dx. \end{aligned}$$

We can now perform an asymptotic analysis of the rescaled problem (3). Since the rescaled problem (3) has a polynomial structure with respect to the small parameter ε, we can look for the solution $s(\varepsilon) = (\mathbf{u}(\varepsilon), \varphi(\varepsilon), \theta(\varepsilon))$ of the problem as a series of powers of ε:

$$s(\varepsilon) = s^0 + \varepsilon s^1 + \varepsilon^2 s^2 + \ldots \;\; \Rightarrow \;\; \left\{ \begin{array}{l} \mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \varepsilon^2 \mathbf{u}^2 + \ldots \\ \varphi(\varepsilon) = \varphi^0 + \varepsilon \varphi^1 + \varepsilon^2 \varphi^2 + \ldots \\ \theta(\varepsilon) = \theta^0 + \varepsilon \theta^1 + \varepsilon^2 \theta^2 + \ldots \end{array} \right. , \tag{4}$$

with $s^q = (\mathbf{u}^q, \varphi^q, \theta^q) \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD}) \times V(\Omega, \Gamma_{tD})$, $q \geqslant 0$. By substituting (4) into the rescaled problem (3), and by identifying the terms with identical power of ε, we obtain, as customary, a set of variational problems to be solved in order to characterize the limit thermo-electromechanical state $s^0 = (\mathbf{u}^0, \varphi^0, \theta^0)$ and its associated limit problem, for $p \in \{-1, 1\}$.

4 The case of $p = 1$: the *weak* thermo-piezoelectric interface

In this Section we characterize the limit model for a weak thermo-piezoelectric interface. By choosing $p = 1$, we obtain the following set of variational problems:

$$\begin{array}{ll} \mathcal{P}_0^1: & A^+(s^0, r) + A^-(s^0, r) + a^m(s^0, r) = L(r), \\ \mathcal{P}_1^1: & A^+(s^1, r) + A^-(s^1, r) + a^m(s^1, r) + b^m(s^0, r) = 0, \\ \mathcal{P}_q^1: & A^+(s^q, r) + A^-(s^q, r) + a^m(s^q, r) + b^m(s^{q-1}, r) + c^m(s^{q-2}, r) = 0, \ q \geqslant 2. \end{array} \tag{5}$$

The first problem $\mathcal{P}_0^1$ of (5) represents the so-called limit problem, which reads

$$\left\{ \begin{array}{l} \text{Find } s^0 = (\mathbf{u}^0, \varphi^0, \theta^0) \in \mathbf{W}(\Omega, \Gamma_{mD}) \times W(\Omega, \Gamma_{eD}) \times W(\Omega, \Gamma_{tD}) \text{ such that} \\ A^-(s^0, r) + A^+(s^0, r) + a^m(s^0, r) = L(r), \end{array} \right. \tag{6}$$

for all $r \in \mathbf{W}(\Omega, \Gamma_{mD}) \times W(\Omega, \Gamma_{eD}) \times W(\Omega, \Gamma_{tD})$, where

$$\begin{array}{l} W(\Omega, \Sigma) := \{v \in L^2(\Omega); \ v^\pm \in H^1(\Omega^\pm), \partial_3 v^m \in L^2(\Omega^m), \ v = 0 \text{ on } \Sigma, \ v^\pm = v^m \text{ on } S^\pm\}, \\ \mathbf{W}(\Omega, \Sigma) := [W(\Omega, \Sigma)]^3. \end{array}$$

The limit problem (6) can be simplified if one considers the structure of the bilinear form $a^m(\cdot, \cdot)$, which involves only the derivatives along the x_3-coordinates. Indeed, by choosing test functions $v_i, \psi, \eta \in \mathcal{D}(\Omega^m)$, one has

$$\int_{\Omega^m} \left\{ (\mathbf{C}\partial_3 \mathbf{u}^{0,m} + \mathbf{P}\partial_3 \varphi^{0,m}) \cdot \partial_3 \mathbf{v} + (H\partial_3 \varphi^{0,m} - \mathbf{P} \cdot \partial_3 \mathbf{u}^{0,m})\partial_3 \psi + K\partial_3 \theta^{0,m} \partial_3 \eta \right\} dx = 0,$$

where $\mathbf{C} := (C^m_{i3j3})$, $\mathbf{P} := (P^m_{3i3})$, $H := H^m_{33}$ and $K := K^m_{33}$ are introduced for the compact notation of the problem. The previous variational equation implies the existence of three constant functions with respect to x_3, namely, $\mathbf{z} = \mathbf{z}(\tilde{x})$, $a = a(\tilde{x})$ and $b = b(\tilde{x})$, with $\tilde{x} = (x_\alpha)$, such that

$$\left\{ \begin{array}{l} \mathbf{C}\partial_3 \mathbf{u}^{0,m} + \mathbf{P}\partial_3 \varphi^{0,m} = \mathbf{z}, \\ H\partial_3 \varphi^{0,m} - \mathbf{P} \cdot \partial_3 \mathbf{u}^{0,m} = a, \\ K\partial_3 \theta^{0,m} = b. \end{array} \right.$$

By solving the linear system above and thanks to the continuity conditions on $x_3 = \pm h$, we can explicitly compute $\mathbf{z}$, a and b as functions of the jumps of the displacement field, electric potential and temperature at the interface between Ω^+ and Ω^-, as follows

$$\mathbf{z} = \frac{1}{2h} \left(\mathbf{C}[\![\mathbf{u}^0]\!] + \mathbf{P}[\![\varphi^0]\!] \right), \quad a = \frac{1}{2h} \left(H[\![\varphi^0]\!] - \mathbf{P} \cdot [\![\mathbf{u}^0]\!] \right), \quad b = \frac{1}{2h} K[\![\theta^0]\!]. \tag{7}$$

This implies that $\partial_3 \mathbf{u}^{0,m} = \frac{[\![\mathbf{u}^0]\!]}{2h}$, $\partial_3 \varphi^{0,m} = \frac{[\![\varphi^0]\!]}{2h}$ and $\partial_3 \theta^{0,m} = \frac{[\![\theta^0]\!]}{2h}$, and thus, $\mathbf{u}^{0,m}$, $\varphi^{0,m}$ and $\theta^{0,m}$ become affine functions of x_3. Indeed, one has

$$\mathbf{u}^{0,m} = \langle \mathbf{u}^0 \rangle + \frac{x_3}{2h}[\![\mathbf{u}^0]\!], \quad \varphi^{0,m} = \langle \varphi^0 \rangle + \frac{x_3}{2h}[\![\varphi^0]\!], \quad \theta^{0,m} = \langle \theta^0 \rangle + \frac{x_3}{2h}[\![\theta^0]\!],$$

where $\langle f \rangle := \frac{f^+ + f^-}{2}$ and $[\![f]\!] := f^+ - f^-$ denote, respectively, the mean value and the jump of the restrictions of f on S^+ and S^-. By using the continuity conditions on S^+ and S^- and after an integration by parts on x_3, we get

$$a^m(s^0, r) = \int_{S^+} (\mathbf{z} \cdot \mathbf{v}^+ + a\psi^+ + b\eta^+) d\Gamma - \int_{S^-} (\mathbf{z} \cdot \mathbf{v}^- + a\psi^- + b\eta^-) d\Gamma.$$

Hence, using expressions (7) and by identifying S^+ and S^- with the interface ω, the limit problem can be reformulated in the following reduced form:

$$\left\{ \begin{array}{l} \text{Find } s^0 \in \tilde{\mathbf{W}}(\Omega, \Gamma_{mD}) \times \tilde{W}(\Omega, \Gamma_{eD}) \times \tilde{W}(\Omega, \Gamma_{tD}) \text{ such that} \\ A^-(s^0, r) + A^+(s^0, r) + \tilde{a}^m(s^0, r) = L(r), \end{array} \right. \tag{8}$$

for all $r \in \tilde{\mathbf{W}}(\Omega, \Gamma_{mD}) \times \tilde{W}(\Omega, \Gamma_{eD}) \times \tilde{W}(\Omega, \Gamma_{tD})$, where

$$\tilde{W}(\Omega, \Sigma) := \{v \in L^2(\Omega); \ v^\pm \in H^1(\Omega^\pm), \ v = 0 \text{ on } \Sigma\}, \quad \tilde{\mathbf{W}}(\Omega, \Sigma) := [\tilde{W}(\Omega, \Sigma)]^3,$$

and,

$$\tilde{a}^m(s^0, r) := \frac{1}{2h} \int_\omega \left\{ \left(\mathbf{C}[\![\mathbf{u}^0]\!] + \mathbf{P}[\![\varphi^0]\!] \right) \cdot [\![\mathbf{v}]\!] + \left(H[\![\varphi^0]\!] - \mathbf{P} \cdot [\![\mathbf{u}^0]\!] \right) [\![\psi]\!] + K[\![\theta^0]\!][\![\eta]\!] \right\} d\tilde{x}.$$

Remark 1. Thanks to the asymptotic analysis, we transform the limit problem onto a coupled thermo-electromechanical interface problem between Ω^+ and Ω^-, with non classical transmission conditions at the interface ω. This problem represents a generalization of the one obtained for weak piezoelectric interfaces in Serpilli (2015) including thermal effects. We rewrite problem (8) in its differential form and we obtain:

Thermal problems in $\Omega^\pm$ **Electrostatic problems in $\Omega^\pm$** **Elasticity problems in $\Omega^\pm$**

$$\begin{cases} \partial_i q_i^\pm = j & \text{in } \Omega^\pm, \\ q_i^\pm n_i = w & \text{on } \Gamma_{tN}, \\ \theta^0 = 0 & \text{on } \Gamma_{tD}, \end{cases} \quad \begin{cases} \partial_i D_i^\pm = \rho_e & \text{in } \Omega^\pm, \\ D_i^\pm n_i = d & \text{on } \Gamma_{eN}, \\ \varphi^0 = 0 & \text{on } \Gamma_{eD}, \end{cases} \quad \begin{cases} -\partial_j \sigma_{ij}^\pm = f_i & \text{in } \Omega^\pm, \\ \sigma_{ij}^\pm n_j = g_i & \text{on } \Gamma_{mN}, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_{mD}, \end{cases}$$

Transmission conditions on ω

$$\begin{cases} \sigma_{i3}^+ = -\frac{1}{2h}\left(C_{i3j3}^m [\![u_j^0]\!] + P_{3i3}^m [\![\varphi^0]\!]\right) & \text{on } \omega, \\ \sigma_{i3}^- = -\frac{1}{2h}\left(C_{i3j3}^m [\![u_j^0]\!] + P_{3i3}^m [\![\varphi^0]\!]\right) & \text{on } \omega, \\ D_3^+ = \frac{1}{2h}\left(H_{33}^m [\![\varphi^0]\!] - P_{3i3}^m [\![u_i^0]\!]\right) & \text{on } \omega, \\ D_3^- = \frac{1}{2h}\left(H_{33}^m [\![\varphi^0]\!] - P_{3i3}^m [\![u_i^0]\!]\right) & \text{on } \omega, \\ q_3^+ = \frac{1}{2h} K_{33} [\![\theta^0]\!] & \text{on } \omega, \\ q_3^- = \frac{1}{2h} K_{33} [\![\theta^0]\!] & \text{on } \omega, \end{cases}$$

which can be rewritten, following Geymonat et al. (1998),

$$\begin{cases} [\![\sigma_{i3}]\!] = 0, \; [\![D_3]\!] = 0, \; [\![q_3]\!] = 0 & \text{on } \omega, \\ \sigma_{i3}^+ + \frac{1}{h}(C_{i3j3}^m u_j^{0,+} + P_{3i3}^m \varphi^{0,+}) = -\sigma_{i3}^- + \frac{1}{h}(C_{i3j3}^m u_j^{0,-} + P_{3i3}^m \varphi^{0,-}) & \text{on } \omega, \\ D_3^+ - \frac{1}{h}(H_{33}^m \varphi^{0,+} - P_{3i3}^m u_i^{0,+}) = -D_3^- - \frac{1}{h}(H_{33}^m \varphi^{0,-} - P_{3i3}^m u_i^{0,-}) & \text{on } \omega, \\ q_3^+ - \frac{1}{h} K_{33}^m \theta^{0,+} = -q_3^- - \frac{1}{h} K_{33}^m \theta^{0,-} & \text{on } \omega. \end{cases}$$

Remark 2. By applying the rescaling method to the constitutive law (1), one can compute the scaled stresses , the scaled electric displacements and the scaled heat flows, as follows

$$\begin{cases} \sigma_{ij}^\pm(\varepsilon) = C_{ijk\ell}^\pm e_{k\ell}(\mathbf{u}(\varepsilon)) + P_{kij}^\pm \partial_k \varphi(\varepsilon) - X_{ij}^\pm \theta(\varepsilon), \\ D_i^\pm(\varepsilon) = P_{ijk}^\pm e_{jk}(\mathbf{u}(\varepsilon)) - H_{ij}^\pm \partial_j \varphi(\varepsilon) + p_i^\pm \theta(\varepsilon), \\ q_i^\pm(\varepsilon) = -K_{ij}^\pm \partial_j \theta(\varepsilon), \\ \sigma_{ij}^m(\varepsilon) = C_{ijk3}^m \partial_3 u_k(\varepsilon) + P_{3ij}^m \partial_3 \varphi(\varepsilon) + \varepsilon(C_{ijk\alpha}^m \partial_\alpha u_k(\varepsilon) + P_{\alpha ij}^m \partial_\alpha \varphi(\varepsilon) - X_{ij}^m \theta(\varepsilon)), \\ D_i^m(\varepsilon) = P_{ik3}^m \partial_3 u_k(\varepsilon) - H_{i3}^m \partial_3 \varphi(\varepsilon) + \varepsilon(P_{ik\alpha}^m \partial_\alpha u_k(\varepsilon) - H_{i\alpha}^m \partial_\alpha \varphi(\varepsilon) + p_i^m \theta(\varepsilon)), \\ q_i^m(\varepsilon) = -K_{i3}^m \partial_3 \theta(\varepsilon) - \varepsilon K_{i\alpha}^m \partial_\alpha \theta(\varepsilon). \end{cases}$$

By definition, we have that $\sigma_{ij}(\varepsilon), \; D_i(\varepsilon), \; q_i(\varepsilon) \in L^2(\Omega)$. Let us consider the following identity, in the sense of distributions,

$$\begin{aligned} &\int_\Omega \{\sigma_{ij}(\varepsilon) T_{ij} + D_i(\varepsilon) F_i + q_i(\varepsilon) G_i\}\, dx = \int_{\Omega^\pm} \left\{\sigma_{ij}^\pm(\varepsilon) T_{ij} + D_i^\pm(\varepsilon) F_i + q_i^\pm(\varepsilon) G_i\right\} dx + \\ &+ \int_{\Omega^m} \left\{(C_{ijk3}^m \partial_3 u_k(\varepsilon) + P_{3ij}^m \partial_3 \varphi(\varepsilon)) T_{ij} + (P_{ik3}^m \partial_3 u_k(\varepsilon) - H_{i3}^m \partial_3 \varphi(\varepsilon)) F_i - K_{i3}^m \partial_3 \theta(\varepsilon) G_i\right\} dx + \\ &+ \varepsilon \int_{\Omega^m} \left\{(C_{ijk\alpha}^m \partial_\alpha u_k(\varepsilon) + P_{\alpha ij}^m \partial_\alpha \varphi(\varepsilon) - X_{ij}^m \theta(\varepsilon)) T_{ij} + (P_{ik\alpha}^m \partial_\alpha u_k(\varepsilon) - H_{i\alpha}^m \partial_\alpha \varphi(\varepsilon) + p_i^m \theta(\varepsilon)) F_i - \right. \\ &\qquad \left. - K_{i\alpha}^m \partial_\alpha \theta(\varepsilon) G_i\right\} dx, \quad \text{for all } \; T_{ij} = T_{ji}, \; F_i, \; G_i \in \mathcal{D}(\Omega). \end{aligned} \tag{9}$$

The asymptotic expansions method allows to look for the stresses, electric displacements and heat flows as series of powers of ε, so that

$$\begin{cases} \sigma_{ij}(\varepsilon) = \sigma_{ij}^0 + \varepsilon \sigma_{ij}^1 + \varepsilon^2 \sigma_{ij}^2 + \dots \,, \\ D_i(\varepsilon) = D_i^0 + \varepsilon D_i^1 + \varepsilon^2 D_i^2 + \dots \,, \\ q_i(\varepsilon) = q_i^0 + \varepsilon q_i^1 + \varepsilon^2 q_i^2 + \dots \,. \end{cases} \tag{10}$$

By inserting (4) and (10) in (9), and by identifying the terms with identical power, we obtain

$$\begin{cases} \sigma_{ij}^{0,m} = C_{ijk3}^m \partial_3 u_k^0 + P_{3ij}^m \partial_3 \varphi^0 = \frac{1}{2h}(C_{ijk3}^m [\![u_k^0]\!] + P_{3ij}^m [\![\varphi^0]\!]), \\ D_i^{0,m} = P_{ik3}^m \partial_3 u_k^0 - H_{i3}^m \partial_3 \varphi^0 = \frac{1}{2h}(P_{ik3}^m [\![u_k^0]\!] - H_{i3}^m [\![\varphi^0]\!]), \\ q_i^{0,m} = -K_{i3}^m \partial_3 \theta^0 = -\frac{1}{2h} K_{i3}^m [\![\theta^0]\!], \end{cases}$$

whereas

$$\left\{\begin{array}{l} \sigma_{ij}^{0,\pm} = C_{ijk\ell}^{\pm} e_{k\ell}(\mathbf{u}^0) + P_{kij}^{\pm}\partial_k\varphi^0 - X_{ij}^{\pm}\theta^0, \\ D_i^{0,\pm} = P_{ijk}^{\pm} e_{jk}(\mathbf{u}^0) - H_{ij}^{\pm}\partial_j\varphi^0 + p_i^{\pm}\theta^0, \\ q_i^{0,\pm} = -K_{ij}^{\pm}\partial_j\theta^0. \end{array}\right.$$

5 The case of $p = -1$: the *strong* thermo-piezoelectric interface

In the sequel we identify the strong thermo-piezoelectric interface problem. By choosing $p = -1$, we obtain the following set of variational problems:

$$\begin{array}{ll} \mathcal{P}_{-2}^{-1}: & a^m(s^0, r) = 0, \\ \mathcal{P}_{-1}^{-1}: & a^m(s^1, r) + b^m(s^0, r) = 0, \\ \mathcal{P}_{0}^{-1}: & A^+(s^0, r) + A^-(s^0, r) + a^m(s^2, r) + b^m(s^1, r) + c^m(s^0, r) = L(r), \\ \mathcal{P}_{q}^{-1}: & A^+(s^q, r) + A^-(s^q, r) + a^m(s^{q+2}, r) + b^m(s^{q+1}, r) + c^m(s^q, r) = 0,\ q \geqslant 1. \end{array}$$

Let us consider problem $\mathcal{P}_{-2}^{-1}$. By choosing test functions $r \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD}) \times V(\Omega, \Gamma_{tD})$, one has, using the compact notation,

$$\int_{\Omega^m} \left\{(\mathbf{C}\partial_3\mathbf{u}^{0,m} + \mathbf{P}\partial_3\varphi^{0,m}) \cdot \partial_3\mathbf{v} + (H\partial_3\varphi^{0,m} - \mathbf{P}\cdot\partial_3\mathbf{u}^{0,m})\partial_3\psi + K\partial_3\theta^{0,m}\partial_3\eta\right\} dx = 0,$$

which is satisfied when $\mathbf{C}\partial_3\mathbf{u}^{0,m} + \mathbf{P}\partial_3\varphi^{0,m} = \mathbf{0}$, $H\partial_3\varphi^{0,m} - \mathbf{P}\cdot\partial_3\mathbf{u}^{0,m} = 0$ and $K\partial_3\theta^{0,m} = 0$. Hence, $\partial_3\mathbf{u}^{0,m} = \mathbf{0}$, $\partial_3\varphi^{0,m} = 0$ and $\partial_3\theta^{0,m} = 0$, and so, $\mathbf{u}^{0,m}$, $\varphi^{0,m}$ and $\theta^{0,m}$ are independent of x_3, i.e., $\mathbf{u}^{0,m} = \mathbf{u}^{0,m}(\tilde{x})$, $\varphi^{0,m} = \varphi^{0,m}(\tilde{x})$ and $\theta^{0,m} = \theta^{0,m}(\tilde{x})$.

Considering problem $\mathcal{P}_{-1}^{-1}$ with test functions $r \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD}) \times V(\Omega, \Gamma_{tD})$, we get

$$\begin{aligned} \int_{\Omega^m} \Big\{ & (C_{i3j3}^m\partial_3 u_j^{1,m} + P_{3i3}^m\partial_3\varphi^{1,m} + C_{i3j\alpha}^m\partial_\alpha u_j^{0,m} + P_{\alpha i3}^m\partial_\alpha\varphi^{0,m} - X_{i3}^m\theta^{0,m})\partial_3 v_i + (K_{33}^m\partial_3\theta^{1,m} + \\ & + K_{\alpha 3}^m\partial_\alpha\theta^{0,m})\partial_3\eta + (H_{33}^m\partial_3\varphi^{1,m} - P_{3i3}^m\partial_3 u_i^{1,m} - P_{3i\alpha}^m\partial_\alpha u_i^{0,m} + H_{\alpha 3}^m\partial_\alpha\varphi^{0,m} - p_3^m\theta^{0,m})\partial_3\psi \Big\} dx = 0. \end{aligned}$$

The previous variational problem is verified when

$$\left\{\begin{array}{l} C_{i3j3}^m\partial_3 u_j^{1,m} + P_{3i3}^m\partial_3\varphi^{1,m} = -C_{i3j\alpha}^m\partial_\alpha u_j^{0,m} - P_{\alpha i3}^m\partial_\alpha\varphi^{0,m} + X_{i3}^m\theta^{0,m}, \\ H_{33}^m\partial_3\varphi^{1,m} - P_{3i3}^m\partial_3 u_i^{1,m} = P_{3i\alpha}^m\partial_\alpha u_i^{0,m} - H_{\alpha 3}^m\partial_\alpha\varphi^{0,m} + p_3^m\theta^{0,m}, \\ \partial_3\theta^{1,m} = -\frac{K_{\alpha 3}^m}{K_{33}^m}\partial_\alpha\theta^{0,m}. \end{array}\right. \tag{11}$$

Now we can easily compute $\partial_3 u_i^{1,m}$, $\partial_3\varphi^{1,m}$ and $\partial_3\theta^{1,m}$ in terms of $\partial_\alpha u_i^{0,m}$, $\partial_\alpha\varphi^{0,m}$ and $\theta^{0,m}$. Let $(d_{ij}) := (C_{i3j3}^m)^{-1}$, we obtain

$$\begin{aligned} &\partial_3 u_i^{1,m} = -d_{ij}\left\{\left(C_{j3k\alpha}^m + k' P_{3j3}^m P'_{3k\alpha}\right)\partial_\alpha u_k^{0,m} + \left(P_{\alpha j3}^m - k' P_{3j3}^m H'_{\alpha 3}\right)\partial_\alpha\varphi^{0,m} - \left(X_{j3}^m - k' P_{3j3}^m p'_3\right)\theta^{0,m}\right\}, \\ &\partial_3\varphi^{1,m} = k'\left\{P'_{3i\alpha}\partial_\alpha u_i^{0,m} - H'_{\alpha 3}\partial_\alpha\varphi^{0,m} + p'_3\theta^{0,m}\right\}, \end{aligned} \tag{12}$$

with $P'_{3i\alpha} := P_{3i\alpha}^m - P_{3k3}^m d_{kj} C_{j3i\alpha}^m$, $H'_{k3} := H_{k3}^m + P_{3i3}^m d_{ij} P_{kj3}^m$, $p'_3 := p_3^m + P_{3i3}^m d_{ij} X_{j3}^m$ and $k' := \frac{1}{H'_{33}}$.

We are now in position to characterize the limit problem. Let us consider problem $\mathcal{P}_0^{-1}$ and let us choose test function $r \in \mathbf{Y}(\Omega, \Gamma_{mD}) \times Y(\Omega, \Gamma_{eD}) \times Y(\Omega, \Gamma_{tD})$, where

$$\begin{aligned} &Y(\Omega, \Sigma) := \{v \in L^2(\Omega);\ v^\pm \in H^1(\Omega^\pm),\ L^2(\Omega^m) \ni \partial_3 v^m = 0, v^m \in H^1(\Omega^m),\ v = 0 \text{ on } \Sigma,\ v^\pm = v^m \text{ on } S^\pm\}, \\ &\mathbf{Y}(\Omega, \Sigma) := [Y(\Omega, \Sigma)]^3. \end{aligned}$$

Thus, $\mathcal{P}_0^{-1}$ takes the following simplified form

$$\begin{aligned} A^\pm(s^0, r) + \int_{\Omega^m} \{ & C_{i\beta j\alpha}^m\partial_\beta u_i^{0,m} + P_{\beta\alpha j}^m\partial_\beta\varphi^{0,m} + C_{i3j\alpha}^m\partial_3 u_i^{1,m} + P_{3\alpha j}^m\partial_3\varphi^{1,m} - X_{j\alpha}^m\theta^{0,m})\partial_\alpha v_j + \\ +(K_{\alpha 3}^m\partial_3\theta^{1,m} + K_{\beta\alpha}^m\partial_\beta\theta^{0,m})\partial_\alpha\eta & + (H_{\alpha\beta}^m\partial_\beta\varphi^{0,m} - P_{\alpha\beta i}^m\partial_\beta u_i^{0,m} - P_{\alpha i3}^m\partial_3 u_i^{1,m} + H_{\alpha 3}^m\partial_3\varphi^{1,m} - p_\alpha^m\theta^{0,m})\partial_\alpha\psi\} dx = 0. \end{aligned} \tag{13}$$

By substituting expression (12) in problem (13) we obtain, as customary, the limit problem:

$$\begin{cases} \text{Find } s^0 \in \mathbf{Y}(\Omega,\Gamma_{mD}) \times Y(\Omega,\Gamma_{eD}) \times Y(\Omega,\Gamma_{tD}) \text{ such that} \\ A^-(s^0,r) + A^+(s^0,r) + \mathcal{A}^m(s^0,r) = L(r), \end{cases} \tag{14}$$

for all $r \in \mathbf{Y}(\Omega,\Gamma_{mD}) \times Y(\Omega,\Gamma_{eD}) \times Y(\Omega,\Gamma_{tD})$, where

$$\begin{aligned} \mathcal{A}^m(s^0,r) := \int_{\Omega^m} \Big\{ & \left(\tilde{C}^m_{i\beta j\alpha} \partial_\beta u^0_i + \tilde{P}^m_{\beta\alpha j} \partial_\beta \varphi^0 - \tilde{X}^m_{j\alpha} \theta^0 \right) \partial_\alpha v_j + \tilde{K}^m_{\alpha\beta} \partial_\beta \theta^0 \partial_\alpha \eta + \\ & + \left(\tilde{H}^m_{\beta\alpha} \partial_\beta \varphi^0 - \tilde{P}^m_{\beta\alpha i} \partial_\beta u^0_i - \tilde{p}^m_\alpha \theta^0 \right) \partial_\alpha \psi \Big\} \, dx. \end{aligned}$$

The reduced coefficients $\tilde{C}^m_{i\beta j\alpha}$, $\tilde{P}^m_{\beta\alpha j}$, $\tilde{H}^m_{\beta\alpha}$, $\tilde{X}^m_{j\alpha}$, $\tilde{p}^m_\alpha$ and $\tilde{K}^m_{\alpha\beta}$ are defined as follows

$$\begin{aligned} &\tilde{C}^m_{i\beta j\alpha} := C^m_{i\beta j\alpha} - C^m_{p3j\alpha} d_{pq} \left(C^m_{q3i\beta} + k' P^m_{3q3} P'_{3i\beta} \right) + k' P^m_{3j\alpha} P'_{3i\beta}, \\ &\tilde{P}^m_{\beta\alpha j} := P^m_{\beta\alpha j} - C^m_{p3j\alpha} d_{pq} \left(P^m_{\beta q3} - k' P^m_{3q3} H'_{\beta 3} \right) - k' P^m_{3j\alpha} H'_{\beta 3}, \\ &\tilde{H}^m_{\beta\alpha} := H^m_{\beta\alpha} - P^m_{\alpha p3} d_{pq} \left(P^m_{\beta q3} - k' P^m_{3q3} H'_{\beta 3} \right) - k' H^m_{\alpha 3} H'_{\beta 3}, \\ &\tilde{X}^m_{j\alpha} := X^m_{j\alpha} - C^m_{p3q\alpha} d_{pq} \left(X^m_{q3} - k' P^m_{3q3} p'_3 \right) - k' P^m_{3j\alpha} p'_3, \\ &\tilde{p}^m_\alpha := p^m_\alpha + P^m_{\alpha p3} d_{pq} \left(X^m_{q3} - k' P^m_{3q3} p'_3 \right) - k' H^m_{\alpha 3} p'_3, \\ &\tilde{K}^m_{\alpha\beta} := K^m_{\alpha\beta} - \tfrac{K^m_{\alpha 3} K^m_{\beta 3}}{K^m_{33}}. \end{aligned}$$

We notice that $Y(\Omega,\Sigma)$ is isomorphic to $\tilde{Y}(\tilde{\Omega},\Sigma) := \{v \in H^1(\tilde{\Omega}),\ v|_\omega \in H^1(\omega),\ v = 0 \text{ on } \Sigma\}$, $\tilde{\Omega} := \Omega^+ \cup \omega \cup \Omega^-$. We can integrate $\mathcal{A}^m(\cdot,\cdot)$ along x_3 and obtain the reduced form of the limit problem:

$$\begin{cases} \text{Find } s^0 \in \tilde{\mathbf{Y}}(\tilde{\Omega},\Gamma_{mD}) \times \tilde{Y}(\tilde{\Omega},\Gamma_{eD}) \times \tilde{Y}(\tilde{\Omega},\Gamma_{tD}) \text{ such that} \\ A^-(s^0,r) + A^+(s^0,r) + \tilde{\mathcal{A}}^m(s^0,r) = L(r), \end{cases}$$

for all $r \in \tilde{\mathbf{Y}}(\tilde{\Omega},\Gamma_{mD}) \times \tilde{Y}(\tilde{\Omega},\Gamma_{eD}) \times \tilde{Y}(\tilde{\Omega},\Gamma_{tD})$, with

$$\begin{aligned} \tilde{\mathcal{A}}^m(s^0,r) := 2h \int_{\omega} \Big\{ & \left(\tilde{C}^m_{i\beta j\alpha} \partial_\beta u^0_i + \tilde{P}^m_{\beta\alpha j} \partial_\beta \varphi^0 - \tilde{X}^m_{j\alpha} \theta^0 \right) \partial_\alpha v_j + \tilde{K}^m_{\alpha\beta} \partial_\beta \theta^0 \partial_\alpha \eta + \\ & + \left(\tilde{H}^m_{\beta\alpha} \partial_\beta \varphi^0 - \tilde{P}^m_{\beta\alpha i} \partial_\beta u^0_i - \tilde{p}^m_\alpha \theta^0 \right) \partial_\alpha \psi \Big\} \, d\tilde{x}. \end{aligned}$$

Remark 3. The variational limit problem results into a non classical transmission problem between Ω^+ and Ω^- with ad hoc transmission conditions at the interface ω. This problem represents a generalization, taking into account the influence of temperature, of the Ventcel-type transmission conditions obtained for strong piezoelectric interfaces in Serpilli (2015). After an integration by parts we can rewrite problem (14) in its differential form, so that

Thermal problems in $\Omega^\pm$

$$\begin{cases} \partial_i q^\pm_i = j & \text{in } \Omega^\pm, \\ q^\pm_i n_i = w & \text{on } \Gamma_{tN}, \\ \theta^0 = 0 & \text{on } \Gamma_{tD}, \\ \tilde{q}_\alpha \nu_\alpha = 0 & \text{on } \gamma_{tN}, \end{cases}$$

Electrostatic problems in $\Omega^\pm$

$$\begin{cases} \partial_i D^\pm_i = \rho_e & \text{in } \Omega^\pm, \\ D^\pm_i n_i = d & \text{on } \Gamma_{eN}, \\ \varphi^0 = 0 & \text{on } \Gamma_{eD}, \\ \tilde{D}_\alpha \nu_\alpha = 0 & \text{on } \gamma_{eN}, \end{cases}$$

Elasticity problems in $\Omega^\pm$

$$\begin{cases} -\partial_j \sigma^\pm_{ij} = f_i & \text{in } \Omega^\pm, \\ \sigma^\pm_{ij} n_j = g_i & \text{on } \Gamma_{mN}, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_{mD}, \\ \tilde{\sigma}_{\alpha i} \nu_\alpha = 0 & \text{on } \gamma_{mN}, \end{cases}$$

Transmission conditions on ω

$$\begin{cases} [\![\mathbf{u}^0]\!] = \mathbf{0},\ [\![\varphi^0]\!] = 0,\ [\![\theta^0]\!] = 0 & \text{on } \omega, \\ [\![\sigma_{i3}]\!] = -2h \partial_\alpha \tilde{\sigma}_{\alpha i} & \text{on } \omega, \\ [\![D_3]\!] = 2h \partial_\alpha \tilde{D}_\alpha & \text{on } \omega, \\ [\![q_3]\!] = 2h \partial_\alpha \tilde{q}_\alpha & \text{on } \omega, \end{cases}$$

where $\tilde{\sigma}_{\alpha i} := \tilde{C}^m_{j\beta i\alpha} \partial_\beta u^0_j + \tilde{P}^m_{\beta\alpha i} \partial_\beta \varphi^0 - \tilde{X}^m_{i\alpha} \theta^0$, $\tilde{D}_\alpha := \tilde{P}^m_{\beta\alpha i} \partial_\beta u^0_i - \tilde{H}^m_{\beta\alpha} \partial_\beta \varphi^0 + \tilde{p}^m_\alpha \theta^0$ and $\tilde{q}_\alpha := -\tilde{K}^m_{\alpha\beta} \partial_\beta \theta^0$ represent, respectively, the reduced two-dimensional interface stress tensor, the reduced interface electric displacement and the reduced interface heat flow defined over ω and (ν_α) denotes the unit normal vector to the uncharged thermo-electromechanical boundaries $\gamma_{eN},\ \gamma_{mN},\ \gamma_{tN} \subset \partial\omega$.

Remark 4. Let us estimate the stresses, electric displacements and heat flows in Ω^m. Using the same procedure adopted in Remark 2, by applying the rescaling method, one has

$$\begin{cases} \sigma_{ij}^m(\varepsilon) = \frac{1}{\varepsilon^2}(C_{ijk3}^m \partial_3 u_k(\varepsilon) + P_{3ij}^m \partial_3 \varphi(\varepsilon)) + \frac{1}{\varepsilon}(C_{ijk\alpha}^m \partial_\alpha u_k(\varepsilon) + P_{\alpha ij}^m \partial_\alpha \varphi(\varepsilon) - X_{ij}^m \theta(\varepsilon)), \\ D_i^m(\varepsilon) = \frac{1}{\varepsilon^2}\left(P_{ik3}^m \partial_3 u_k(\varepsilon) - H_{i3}^m \partial_3 \varphi(\varepsilon)\right) + \frac{1}{\varepsilon}\left(P_{ik\alpha}^m \partial_\alpha u_k(\varepsilon) - H_{i\alpha}^m \partial_\alpha \varphi(\varepsilon) + p_i^m \theta(\varepsilon)\right), \\ q_i^m(\varepsilon) = -\frac{1}{\varepsilon^2} K_{i3}^m \partial_3 \theta(\varepsilon) - \frac{1}{\varepsilon} K_{i\alpha}^m \partial_\alpha \theta(\varepsilon). \end{cases}$$

The asymptotic expansions method allows to look for the stresses and electric displacements as series of powers of ε, so that

$$\begin{cases} \sigma_{ij}^m(\varepsilon) = \frac{1}{\varepsilon^2}\sigma_{ij}^{-2,m} + \frac{1}{\varepsilon}\sigma_{ij}^{-1,m} + \sigma_{ij}^{0,m} + \varepsilon\sigma_{ij}^{1,m} + \dots\,, \\ D_i^m(\varepsilon) = \frac{1}{\varepsilon^2}D_i^{-2,m} + \frac{1}{\varepsilon}D_i^{-1,m} + D_i^{0,m} + \varepsilon D_i^{1,m} + \dots\,, \\ q_i^m(\varepsilon) = \frac{1}{\varepsilon^2}q_i^{-2,m} + \frac{1}{\varepsilon}q_i^{-1,m} + q_i^{0,m} + \varepsilon q_i^{1,m} + \dots\,. \end{cases} \tag{15}$$

By using (4) and (15), and relations (12), by identifying the terms with identical power, we obtain

$$\begin{cases} \sigma_{ij}^{-2,m} = C_{ijk3}^m \partial_3 u_k^0 + P_{3ij}^m \partial_3 \varphi^0 = 0, \\ D_i^{-2,m} = P_{ik3}^m \partial_3 u_k^0 - H_{i3}^m \partial_3 \varphi^0 = 0, \\ q_i^{-2,m} = -K_{i3}^m \partial_3 \theta^0 = 0, \\ \sigma_{ij}^{-1,m} = C_{ijk3}^m \partial_3 u_k^1 + P_{3ij}^m \partial_3 \varphi^1 + C_{ijk\alpha}^m \partial_\alpha u_k^0 + P_{\alpha ij}^m \partial_\alpha \varphi^0 - X_{ij}^m \theta^0 = \mathbb{C}_{ijk\alpha}^m \partial_\alpha u_k^0 + \mathbb{P}_{\alpha ij}^m \partial_\alpha \varphi^0 - \mathbb{X}_{ij}^m \theta^0, \\ D_i^{-1,m} = P_{ik3}^m \partial_3 u_k^1 - H_{i3}^m \partial_3 \varphi^1 + P_{ik\alpha}^m \partial_\alpha u_k^0 - H_{i\alpha}^m \partial_\alpha \varphi^0 + p_i^m \theta^0 = \mathbb{P}_{ik\alpha}^m \partial_\alpha u_k^0 - \mathbb{H}_{i\alpha}^m \partial_\alpha \varphi^0 + \mathbb{M}_i^m \theta^0, \\ q_i^{-1,m} = -K_{i3}^m \partial_3 \theta^1 - K_{i\alpha}^m \partial_\alpha \theta^0 = -\mathbb{K}_{i\alpha}^m \partial_\alpha \theta^0. \end{cases}$$

Expressions above are thought as a first approximation of the stress field, the electric displacement field and the heat flow in Ω^m: in order to have a better estimation of both stresses and electric displacements, we need to characterize the successive terms of the asymptotic expansions for the displacement field and electric potential field, such as $\mathbf{u}^2$, φ^2 and θ^2. For what concerns with the stresses, electric displacements and heat flows in $\Omega^\pm$ we obtain, as customary,

$$\begin{cases} \sigma_{ij}^{0,\pm} = C_{ijk\ell}^\pm e_{k\ell}(\mathbf{u}^0) + P_{kij}^\pm \partial_k \varphi^0 - X_{ij}^\pm \theta^0, \\ D_i^{0,\pm} = P_{ijk}^\pm e_{jk}(\mathbf{u}^0) - H_{ij}^\pm \partial_j \varphi^0 + p_i^\pm \theta^0, \\ q_i^{0,\pm} = -K_{ij}^\pm \partial_j \theta^0. \end{cases}$$

6 CONCLUDING REMARKS

In the present work we derive two limit interface models corresponding to a generic piezoelectric assembly with a piezoelectric interphase, taking into account the thermal effect, through the asymptotic expansions method. We analyze two particular cases: the first case, for $p = 1$, corresponding from a mechanical point of view to a soft weakly conducting thermo-piezoelectric interphase, leads to the *weak* thermo-piezoelectric interface model; the latter, for $p = -1$, corresponding to a rigid highly conducting interphase between two thermo-piezoelectric media, leads to the *strong* thermo-piezoelectric interface model. For what concerns with the first case, the interphase disappears from a geometrical point view and is described only by means of a surface energy, namely $\tilde{a}^m(s^0, s^0)$, depending on the discontinuities of the displacement field, electric potential and temperature at the interface. In the second case, the interphase is substituted by a material surface which behaves as a thermo-piezoeletric membrane.

As future developments, we would like to study more complex interface problems taking into account thermo-electromagnetoelastic couplings and time-dependent phenomena. Moreover we are numerically implementing the model by adapting the domain decomposition algorithm, presented in Geymonat et al. (1998, 2014), to the present case.

References

Benveniste, Y.: An $o(h^n)$ interface model for a three-dimensional curved interphase in conduction phenomena. *Proc. Royal Soc. A*, 462, (2006), 1619 – 1627.

Benveniste, Y.: Two models of three-dimensional thin interphases with variable conductivity and their fulfillment of the reciprocal theorem. *J. Mech. Phys. Solids*, 60(10), (2009), 1740 –1752.

Bessoud, A. L.; Krasucki, F.; Michaille, G.: Multi-materials with strong interface: variational modelings. *Asympt. Anal.*, 1, (2009), 1 – 19.

Bessoud, A. L.; Krasucki, F.; Serpilli, M.: Plate-like and shell-like inclusions with high rigidity. *C. R. Aca. Sci. Paris Ser. I*, 346, (2008), 697 – 702.

Bessoud, A. L.; Krasucki, F.; Serpilli, M.: Asymptotic analysis of shell-like inclusions with high rigidity. *J. Elasticity*, 103, (2011), 153 – 172.

Blanchard, D.; Francfort, G.: Asymptotic thermoelastic behavior of flat plates. *Quart. Appl. Math.*, 45, (1987), 645 – 667.

Ciarlet, P. G.: *Mathematical Elasticity, vol. II: Theory of Plates*. Elsevier, North-Holland, Amsterdam (1997).

Figueiredo, I.; Franco Leal, C.: A piezoelectric anisotropic plate model. *Asympt. Anal.*, 44, (2005), 327 – 346.

Geymonat, G.; Hendili, S.; Krasucki, F.; Serpilli, M.; Vidrascu, M.: Asymptotic expansions and domain decomposition. In: *Lecture notes in Computational Science and Engineering*, Springer (2014).

Geymonat, G.; Krasucki, F.; Lenci, S.: Mathematical analysis of a bonded joint with a soft thin adhesive. *Math. Mech. Solids*, 4, (1999), 201 – 225.

Geymonat, G.; Krasucki, F.; Marini, D.; Vidrascu, M.: A domain decomposition method for a bonded structure. *Math. Model. Meth. Appl. Sci.*, 8, (1998), 1387 – 1402.

Krasucki, F.; Münch, A.; Ousset, Y.: Mathematical analysis of nonlinear bonded joint models. *Math. Models Meth. Appl. Sci.*, 14, (2004), 1 – 22.

Lebon, F.; Rizzoni, R.: Asymptotic analysis of a thin interface: the case involving similar rigidity. *Int. J. Eng. Sci.*, 48, (2010), 473 – 486.

Lebon, F.; Rizzoni, R.: Asymptotic behavior of a hard thin linear interphase: An energy approach. *Int. J. Solids Struct.*, 48, (2011), 441 – 449.

Miara, B.; Suarez, J.: Asymptotic pyroelectricity and pyroelasticity in thermopiezo-electric plate. *Asymptot. Anal.*, 81, (2013), 211 – 250.

Serpilli, M.: Mathematical modeling of weak and strong piezoelectric interfaces. *J. Elasticity*, (published online), (2015), DOI: 10.1007/s10659–015–9526–5.

Weller, T.; Licht, C.: Asymptotic modeling of thin piezoelectric plates. *Ann. Solid. Struct. Mech.*, 1, (2010), 173 – 188.

Address: Department of Civil and Building Engineering, and Architecture,
Polytechnic University of Marche,
via Brecce Bianche 60131, Ancona, Italy
email: `m.serpilli@univpm.it`