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A Comparative Analysis of Linear and Nonlinear Kinematic Hardening Rules in Computational Elastoplasticity

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In this work a comparative analysis is presented between the linear and the nonlinear kinematic hardening assumptions for material modelling in the elastoplastic regime. For the simulation of typically adopted nonlinear kinematic hardening laws a solution procedure is considered which is able to preserve a quadratic rate of asymptotic convergence. Numerical computations and results are reported which allow to compare for different simulations the suitability of the assumptions of linear versus nonlinear kinematic hardening rules for elastoplastic materials. By considering different types of material properties the reported analysis gives indications on the conditions under which such assumptions can be considered efficacious.

1 Introduction

Over the last decades computational plasticity has achieved a deep mathematical and numerical understanding through the investigations of many researchers who have worked in the field. For a comprehensive account see, among others, Simo and Hughes (1998), Zienkiewicz and Taylor (2005) and Wriggers (2008). For rate-independent plasticity the assumption of a linear kinematic hardening behaviour (Prager (1949)) is frequently adopted in the literature. This assumption is quite advantageous since it provides numerical integration algorithms characterized by computational efficiency and a symmetric tangential stiffness matrix. However in recent years in the literature the use of nonlinear kinematic hardening rules has been proposed in order to properly and more accurately reproduce the experimental behaviour of materials, see, e.g., Armstrong and Frederick (1966), Dafalias and Popov (1975), Chaboche (1989), McDowell (1992), Lubliner et al. (1993), Auricchio and Taylor (1995). In particular this holds for the simulation of complex loading conditions, see for instance Chaboche and Cailletaud (1996) and Auricchio and Taylor (1995).

When a fast and robust numerical integration algorithm is required, the implementation of nonlinear kinematic hardening rules for plasticity models is not a trivial task to be accomplished. At present the research for fast and robust numerical integration algorithms for the simulation of nonlinear kinematic hardening models and complex loading conditions is an active topic of research in the literature. In this respect precursory proposals have been presented by, among others, Doghri (1993), Auricchio and Taylor (1995), Hopperstad and Remseth (1995), Chaboche and Cailletaud (1996). The current state of the art includes the contributions of many researchers who are actively working in this field and on the related topics. A physical approach to the formulation of constitutive laws based on nonlinear rheological models in finite thermoviscoplasticity has been proposed by Lion (2000). Lubarda and Benson (2002) proposed a numerical integration algorithm for the combined isotropic kinematic hardening plasticity with the Armstrong Frederick evolution equation for the back stress. A finite strain elastoplastic constitutive model capable to describe nonlinear kinematic hardening and nonlinear isotropic hardening has been presented by Wallin et al. (2003). Dettmer and Reese (2004) illustrated the theoretical and numerical modelling of Armstrong Frederick kinematic hardening in the finite strain regime. Tsakmakis and Willuweit (2004) discussed and compared the possible extensions of the Armstrong Frederick rule to finite deformations and implemented the models in a finite element code. A deformation gradient based kinematic hardening model has been introduced by Wallin and Ristinmaa (2005). Since the residual stresses are a manifestation of the distortion of the crystal lattice, a corresponding deformation gradient was introduced to represent this distortion. Menzel et al. (2005) proposed an algorithmic treatment of the phenomenological framework to capture anisotropic geometrically nonlinear inelasticity by considering the coupling of viscoplasticity with continuum damage. Hakansson et al. (2005) illustrated a comparative analysis of isotropic hardening and kinematic hardening in thermoplasticity. A computational modelling of inelastic large ratcheting strains has been described by Johansson et al. (2005). Svendsen et al. (2006) illustrated a thermodynamical approach to the modelling and simulation of induced elastic and inelastic material behavior and an application of the concept of evolving structure tensors to the modelling of initial and induced anisotropy at large deformation. Artioli et al. (2007) presented two second order numerical schemes for the von Mises plasticity with a combination of linear isotropic and nonlinear kinematic hardening. The derivation and the numerical implementation of a material model for nonlinear kinematic and isotropic hardening has been proposed by Vladimirov et al. (2008), where the kinematic hardening component represents a continuum extension of the rheological model of Armstrong Frederick kinematic hardening. Wang et al. (2008) illustrated the formulation and initial application of a phenomenological model for hardening effects in metals subject to non-proportional loading histories characterized by one or more loading-path changes. Barthel et al. (2008) presented a phenomenological model whose structure is motivated by polycrystalline modelling that takes into account the evolution of polarized dislocation structures on the grain level as the main cause of the induced flow anisotropy on the macroscopic level. The application of a model that combines both isotropic and kinematic hardening and utilizes a new algorithm based on the exponential map has been illustrated by Vladimirov et al. (2009). The derivation and numerical implementation of a material model for plastic anisotropy and nonlinear kinematic and isotropic hardening has been discussed by Vladimirov et al. (2010). Vladimirov et al. (2011) investigated the influence of kinematic hardening behaviour and plastic anisotropy in the context of continuum thermodynamics and a multiplicative formulation of elastoplasticity. Schwarze et al. (2011) illustrated a finite element formulation incorporating a material model for plastic anisotropy with Armstrong Frederick kinematic hardening and isotropic hardening. Pietryga et al. (2012) proposed a constitutive model for anisotropic metal plasticity that takes into account isotropic hardening, kinematic hardening and distortional hardening. For a comprehensive account and a review of some plasticity and viscoplasticity constitutive theories with linear and nonlinear kinematic hardening see, e.g., Chaboche (2008).

In this work attention is focused on a comparative analysis between the linear and the nonlinear kinematic hardening rules in elastoplasticity. Indications regarding the application of linear versus nonlinear kinematic hardening assumptions are presented for different material properties. A numerical algorithm which preserves a quadratic rate of convergence is used for the computational modelling of typically adopted nonlinear kinematic hardening laws. Numerical applications and computational results are reported in order to show the effectiveness of the algorithmic procedure. The adoption of the assumptions of linear and nonlinear kinematic hardening rules is made and a comparative analysis of the two models is illustrated for different types of material parameters. The reported analysis allows to have better indications on the conditions under which such assumptions can be considered as effective. In addition the performed analysis also provides useful indications on the degree of discrepancy that the two assumptions imply in the simulation and analysis of elastoplastic structural problems for different types of material properties.

2 The Continuum Problem

Let $\Omega \subset \Re^n$, $1 \leq n \leq 3$, be the reference configuration of the body $\mathcal{B}$ with particles labelled by their position vector $\mathbf{x} \in \Omega$ relative to a Cartesian reference frame. Let $\mathcal{T} \subset \Re_+$ be the time interval of interest. We denote with $\boldsymbol{V}$ the space of displacements, $\boldsymbol{D}$ the strain space and $\boldsymbol{S}$ the dual stress space. We also denote by $\mathbf{u} : \Omega \times \mathcal{T} \to \boldsymbol{V}$ the displacement field and by $\boldsymbol{\sigma} : \Omega \times \mathcal{T} \to \boldsymbol{S}$ the stress field. The compatible strain field is defined as $\boldsymbol{\varepsilon}(\mathbf{u}) = \nabla^s(\mathbf{u}) : \Omega \times \mathcal{T} \to \boldsymbol{D}$, where ∇^s is the symmetric part of the gradient. The hypothesis of small strains is assumed to hold so that the total strain $\boldsymbol{\varepsilon} \in \boldsymbol{D}$ is additively split into an elastic part $\boldsymbol{\varepsilon}^e$ and a plastic part $\boldsymbol{\varepsilon}^p$, resulting $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$. The constitutive elastic relation between the elastic strain $\boldsymbol{\varepsilon}^e \in \boldsymbol{D}$ and the stress $\boldsymbol{\sigma} \in \boldsymbol{S}$ is governed by an elastic operator $\mathcal{E}$, and its inverse $\mathcal{E}^{-1}$, so that $\boldsymbol{\sigma} = \mathcal{E}(\boldsymbol{\varepsilon}^e)$, and $\boldsymbol{\varepsilon}^e = \mathcal{E}^{-1}(\boldsymbol{\sigma})$. Assuming linear elasticity the elastic operator $\mathcal{E}$ and its inverse $\mathcal{E}^{-1}$ are linear and will be denoted by $\mathbf{E}$ and $\mathbf{E}^{-1}$. Thus the constitutive elastic relations specialize to $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\varepsilon}^e = \mathbf{E}^{-1}\boldsymbol{\sigma}$. The elastic energy $\mathcal{W} : \boldsymbol{D} \to \Re$ and the complementary elastic energy $\mathcal{W}^* : \boldsymbol{S} \to \Re$ in case of linear elasticity are expressed in the quadratic forms $\mathcal{W}(\boldsymbol{\varepsilon}^e) = \frac{1}{2}\langle \mathbf{E}\boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^e \rangle$ and $\mathcal{W}^*(\boldsymbol{\sigma}) = \frac{1}{2}\langle \boldsymbol{\sigma}, \mathbf{E}^{-1}\boldsymbol{\sigma} \rangle$, where $\mathbf{E}$ is the elastic stiffness and the symbol $\langle \cdot , \cdot \rangle$ denotes a non-degenerate bilinear form on dual spaces. We define the kinematic internal variable $\boldsymbol{\alpha} \in \boldsymbol{X} \times \Re$ and the dual static internal variable $\boldsymbol{\chi} \in \boldsymbol{X}' \times \Re$ as

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}_{kin} \\ \alpha_{iso} \end{bmatrix}, \qquad \boldsymbol{\chi} = \begin{bmatrix} \boldsymbol{\chi}_{kin} \\ \chi_{iso} \end{bmatrix}, \tag{1}$$

where $\boldsymbol{\alpha}_{kin} \in \boldsymbol{X}$ and $\boldsymbol{\chi}_{kin} \in \boldsymbol{X}'$ are introduced to model kinematic hardening, $\alpha_{iso} \in \Re$ and $\chi_{iso} \in \Re$ model isotropic hardening and $\boldsymbol{X}$ and $\boldsymbol{X}'$ are dual spaces. Accordingly, the hardening matrix $\mathbf{H} = \text{diag}[\mathbf{H}_{kin}, H_{iso}]$ is introduced, where $\mathbf{H}_{kin}$ and H_{iso} denote the kinematic and isotropic hardening moduli, so that the static and kinematic internal variables are linked by the relation $\boldsymbol{\chi} = \mathbf{H}\boldsymbol{\alpha}$. A yield function $f(\boldsymbol{\sigma}, \boldsymbol{\chi}_{kin}, \chi_{iso})$ defines the convex elastic domain as $\mathcal{C} = \{(\boldsymbol{\sigma}, \boldsymbol{\chi}_{kin}, \chi_{iso}) \in \boldsymbol{S} \times \boldsymbol{X}' \times \Re : f(\boldsymbol{\sigma}, \boldsymbol{\chi}_{kin}, \chi_{iso}) \leq 0\}$. An important class of plastic hardening materials arises when the yield function is expressed as $f(\boldsymbol{\sigma}, \boldsymbol{\chi}_{kin}, \chi_{iso}) = F(\boldsymbol{\sigma} - \boldsymbol{\chi}_{kin}) - \chi_{iso} - y_o$, where y_o is a material parameter. Furtherly, we consider a formulation within the framework provided by the generalized standard material model (Halphen and Nguyen (1975)). Accordingly, a suitable definition of the internal variables and of the generalized yielding function is considered for modelling elastoplastic phenomena. Generalized strains and stresses are introduced as

$$\tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \mathbf{o} \end{bmatrix}, \qquad \tilde{\boldsymbol{\varepsilon}}^e = \begin{bmatrix} \boldsymbol{\varepsilon}^e \\ \boldsymbol{\alpha} \end{bmatrix}, \qquad \tilde{\boldsymbol{\varepsilon}}^p = \begin{bmatrix} \boldsymbol{\varepsilon}^p \\ -\boldsymbol{\alpha} \end{bmatrix}, \qquad \tilde{\boldsymbol{\sigma}} = \begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\chi} \end{bmatrix}, \tag{2}$$

and take into account actual strains and stresses and kinematic and static internal variables. The generalized variables are defined in product spaces, respectively $\tilde{\boldsymbol{D}} = \boldsymbol{D} \times \boldsymbol{X} \times \Re$ and $\tilde{\boldsymbol{S}} = \boldsymbol{S} \times \boldsymbol{X}' \times \Re$ and for their rapresentation is equivalently used the notation $\tilde{\boldsymbol{\varepsilon}} = (\boldsymbol{\varepsilon}, \mathbf{o})$ and $\tilde{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$. A generalized convex elastic domain $\tilde{\boldsymbol{C}} \subseteq \tilde{\boldsymbol{S}}$ is introduced to define the admissibility condition on the generalized stress $\tilde{\boldsymbol{\sigma}}$ as $\tilde{\boldsymbol{C}} = \{\tilde{\boldsymbol{\sigma}} \in \tilde{\boldsymbol{S}} : \tilde{f}(\tilde{\boldsymbol{\sigma}}) \leq 0\}$, where $\tilde{f} : \tilde{\boldsymbol{S}} \rightarrow \Re$ is a convex generalized yield function. Consequently, the duality products between generalized variables are introduced as $\langle \tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\varepsilon}} \rangle = \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \rangle$, $\langle \tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\varepsilon}}^e \rangle = \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}^e \rangle + \langle \boldsymbol{\chi}, \boldsymbol{\alpha} \rangle$, $\langle \tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\varepsilon}}^p \rangle = \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p \rangle - \langle \boldsymbol{\chi}, \boldsymbol{\alpha} \rangle$, and they are defined by the duality products between the corresponding elements of $\boldsymbol{S}$ and $\boldsymbol{D}$ and between the corresponding elements of $\boldsymbol{X}' \times \Re$ and $\boldsymbol{X} \times \Re$. For a comprehensive treatment see, among others, Lemaitre and Chaboche (1990).

3 Constitutive Model in Plasticity

The constitutive formulations of plasticity problems may be efficiently considered within the context of the theory of continuum thermodynamics with internal variables, see for instance Halphen and Nguyen (1975). For a survey account we refer, e.g., to Germain et al. (1983), and Lemaitre and Chaboche (1990). In an internal variable formulation of plasticity an important role is played by the form of the evolutive equations. A generalized way to consider this problem has been proposed by assuming that evolution equations are expressed in terms of a potential. This formulation has been illustrated in detail by Halphen and Nguyen (1975) and Germain et al. (1983). The thermodynamic potential provides the description of the thermodynamic forces conjugate to the state variables and allows the formulation of the evolutive laws of the state variables which represent the irreversible process.

The maximum plastic dissipation principle supplies the expression of the dissipation function

$$\mathcal{D}(\dot{\tilde{\boldsymbol{\varepsilon}}}^p) = \sup_{\tilde{\boldsymbol{\tau}} \in \tilde{\boldsymbol{C}}} \{\langle \tilde{\boldsymbol{\tau}}, \dot{\tilde{\boldsymbol{\varepsilon}}}^p \rangle\} = \sup_{(\boldsymbol{\tau}, \mathbf{q}) \in \tilde{\boldsymbol{C}}} \{\langle \boldsymbol{\tau}, \dot{\boldsymbol{\varepsilon}}^p \rangle - \langle \mathbf{q}, \dot{\boldsymbol{\alpha}} \rangle\}, \tag{3}$$

where $\tilde{\boldsymbol{\tau}} = (\boldsymbol{\tau}, \mathbf{q})$ has been used to denote the generic generalized stress state, while $\tilde{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$ has been used to indicate the value at solution. For the given generalized plastic strain rate $\dot{\tilde{\boldsymbol{\varepsilon}}}^p$, the Lagrangian of the plastic constitutive problem with hardening is introduced as

$$\tilde{\mathcal{L}}^p(\tilde{\boldsymbol{\tau}}, \dot{\delta}) = -\langle \tilde{\boldsymbol{\tau}}, \dot{\tilde{\boldsymbol{\varepsilon}}}^p \rangle + \dot{\delta}\tilde{f}(\tilde{\boldsymbol{\tau}}) - \sqcup_{\Re^+}(\dot{\delta}) = -\langle \boldsymbol{\tau}, \dot{\boldsymbol{\varepsilon}}^p \rangle + \langle \mathbf{q}, \dot{\boldsymbol{\alpha}} \rangle + \dot{\delta}\tilde{f}(\boldsymbol{\tau}, \mathbf{q}) - \sqcup_{\Re^+}(\dot{\delta}), \tag{4}$$

where $\sqcup_{\Re^+}(\dot{\delta})$ denotes the convex indicator function (Hiriart-Urruty and Lemaréchal (1993)) of the set of non-negative real numbers $\Re^+$

$$\sqcup_{\Re^+}(\dot{\delta}) \stackrel{def}{=} \begin{cases} 0 & \text{if} \quad \dot{\delta} \geq 0 \\ +\infty & \text{if} \quad \dot{\delta} < 0. \end{cases} \tag{5}$$

Here $\dot{\delta}$ indicates the generic Lagrange multiplier, while $\dot{\gamma}$ denotes the value at the solution point, with the meaning of a plastic multiplier. The solution of the maximum plastic dissipation problem (3) is given by the point $(\tilde{\boldsymbol{\sigma}}, \dot{\gamma}) \in \tilde{\boldsymbol{S}} \times \Re^+$ which fulfills the Kuhn-Tucker optimality conditions

$$\begin{aligned} 0 &\in \left[\partial_{\tilde{\boldsymbol{\tau}}} \tilde{\mathcal{L}}^p(\tilde{\boldsymbol{\tau}}, \dot{\delta})\right]_{(\tilde{\boldsymbol{\sigma}}, \dot{\gamma})} &&\Leftrightarrow &&\dot{\tilde{\boldsymbol{\varepsilon}}}^p \in \dot{\gamma}\partial\tilde{f}(\tilde{\boldsymbol{\sigma}}) \\ 0 &\in \left[\partial_{\dot{\delta}} \tilde{\mathcal{L}}^p(\tilde{\boldsymbol{\tau}}, \dot{\delta})\right]_{(\tilde{\boldsymbol{\sigma}}, \dot{\gamma})} &&\Leftrightarrow &&\tilde{f}(\tilde{\boldsymbol{\sigma}}) \in \partial \sqcup_{\Re^+}(\dot{\delta}), \end{aligned} \tag{6}$$

and by making explicit the terms related to the generalized variables the solution is given by

$$\begin{aligned} 0 &\in \left[\partial_{\boldsymbol{\tau}} \tilde{\mathcal{L}}^p(\boldsymbol{\tau}, \mathbf{q}, \dot{\delta})\right]_{(\boldsymbol{\sigma}, \boldsymbol{\chi}, \dot{\gamma})} &&\Leftrightarrow &&\dot{\boldsymbol{\varepsilon}}^p \in \dot{\gamma}\partial_{\boldsymbol{\sigma}}\tilde{f}(\boldsymbol{\sigma}, \boldsymbol{\chi}) \\ 0 &\in \left[\partial_{\mathbf{q}} \tilde{\mathcal{L}}^p(\boldsymbol{\tau}, \mathbf{q}, \dot{\delta})\right]_{(\boldsymbol{\sigma}, \boldsymbol{\chi}, \dot{\gamma})} &&\Leftrightarrow &&-\dot{\boldsymbol{\alpha}} \in \dot{\gamma}\partial_{\boldsymbol{\chi}}\tilde{f}(\boldsymbol{\sigma}, \boldsymbol{\chi}) \\ 0 &\in \left[\partial_{\dot{\delta}} \tilde{\mathcal{L}}^p(\boldsymbol{\tau}, \mathbf{q}, \dot{\delta})\right]_{(\boldsymbol{\sigma}, \boldsymbol{\chi}, \dot{\gamma})} &&\Leftrightarrow &&\tilde{f}(\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \partial \sqcup_{\Re^+}(\dot{\delta}). \end{aligned} \tag{7}$$

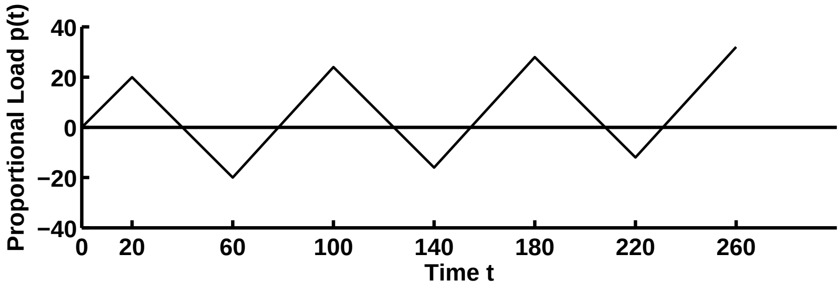

Figure 1: Loading program in tension-compression with increasing mean stress.

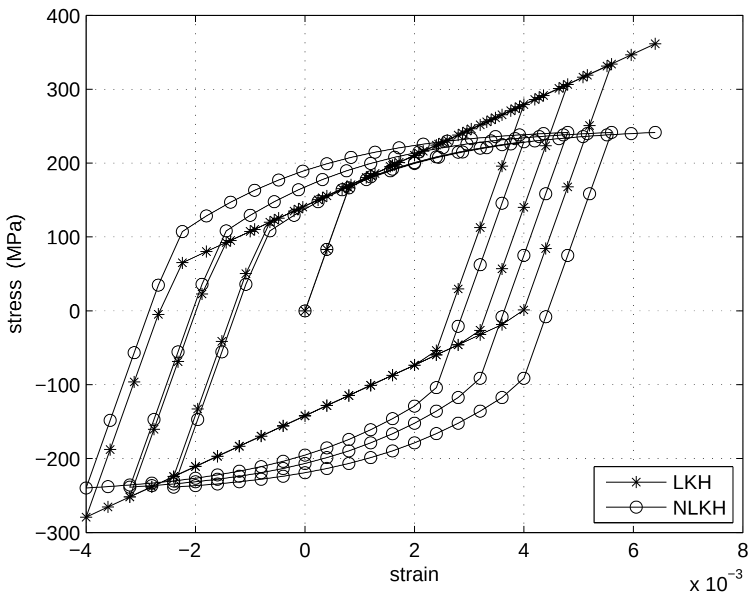

Figure 2: Comparative analysis between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity for chromium-nickel stainless steel X5CrNi 18·9. Loading conditions in tension-compression with loading program reported in Fig.1.

Relation $(6)_1$ is the normality law of the generalized plastic flow for the plasticity problem with hardening and by expliciting the generalized variables it supplies the flow law $(7)_1$ for the plastic strain and the evolutive law $(7)_2$ for the internal variables. Furtherly, since it results $\mathcal{N}_{\Re^+}(\dot{\delta}) = \partial \sqcup_{\Re^+}(\dot{\delta})$ (see, e.g., Hiriart-Urruty and Lemaréchal (1993)), equations $(6)_2$ and $(7)_3$ can be expressed as

$$\tilde{f}(\tilde{\boldsymbol{\sigma}}) = \tilde{f}(\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \mathcal{N}_{\Re^+}(\dot{\delta}), \tag{8}$$

which is an equivalent expression for the loading/unloading conditions in the complementarity form

$$\tilde{f}(\tilde{\boldsymbol{\sigma}}) = \tilde{f}(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leq 0, \qquad \dot{\gamma} \geq 0, \qquad \dot{\gamma}\tilde{f}(\tilde{\boldsymbol{\sigma}}) = \dot{\gamma}\tilde{f}(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0. \tag{9}$$

For hardening plasticity the principle of maximum plastic dissipation supplies

$$\langle (\tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\sigma}}), \dot{\tilde{\boldsymbol{\varepsilon}}}^p \rangle \leq 0, \qquad \forall\, \tilde{\boldsymbol{\tau}} \in \tilde{\boldsymbol{C}}, \tag{10}$$

and, given the definition of a normal cone to the convex set $\tilde{\boldsymbol{C}}$ (Hiriart-Urruty and Lemaréchal (1993)), it follows

$$\dot{\tilde{\boldsymbol{\varepsilon}}}^p \in \mathcal{N}_{\tilde{\boldsymbol{C}}}(\tilde{\boldsymbol{\sigma}}), \tag{11}$$

which represents the normality law in plasticity with hardening, with the physical meaning for the generalized plastic strain rate to belong to the normal cone to the generalized convex elastic domain $\tilde{\boldsymbol{C}}$ at $\tilde{\boldsymbol{\sigma}}$. We now observe

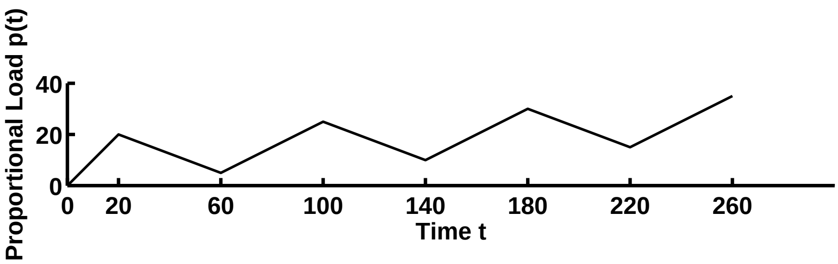

Figure 3: Loading program with increasing levels of loading.

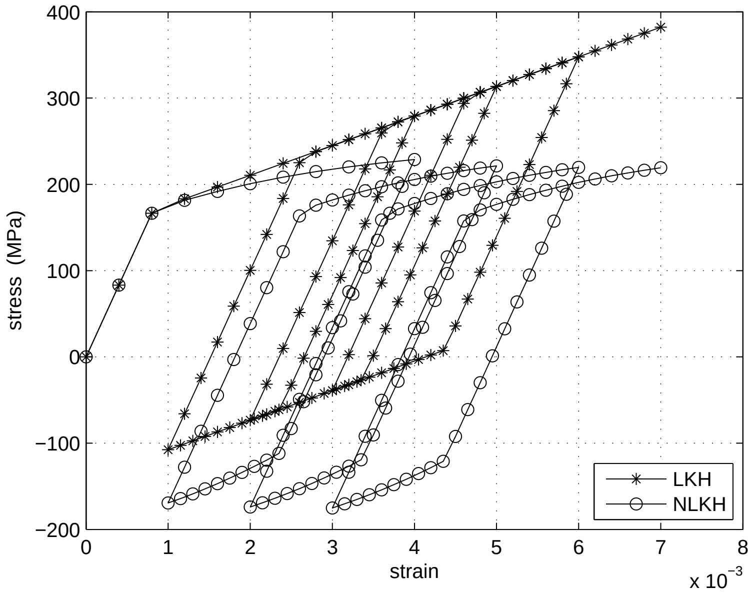

Figure 4: Comparative analysis between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity for chromium-nickel stainless steel X5CrNi 18·9. Loading conditions with increasing levels of loading and with loading program reported in Fig.3.

that $\mathcal{N}_{\tilde{\boldsymbol{C}}}(\tilde{\boldsymbol{\sigma}}) = \partial \sqcup_{\tilde{\boldsymbol{C}}}(\tilde{\boldsymbol{\sigma}})$, where the indicator function $\sqcup_{\tilde{\boldsymbol{C}}}(\tilde{\boldsymbol{\sigma}})$ (Hiriart-Urruty and Lemaréchal (1993)) of the convex generalized elastic domain $\tilde{\boldsymbol{C}}$ is defined as

$$\sqcup_{\tilde{\boldsymbol{C}}}(\tilde{\boldsymbol{\sigma}}) \stackrel{def}{=} \begin{cases} 0 & \text{if} \quad \tilde{\boldsymbol{\sigma}} \in \tilde{\boldsymbol{C}} \\ +\infty & \text{if} \quad \tilde{\boldsymbol{\sigma}} \notin \tilde{\boldsymbol{C}}. \end{cases} \tag{12}$$

Accordingly, the normality law in plasticity with hardening can be formulated as

$$\dot{\tilde{\boldsymbol{\varepsilon}}}^p \in \partial \sqcup_{\tilde{\boldsymbol{C}}}(\tilde{\boldsymbol{\sigma}}), \tag{13}$$

which is an equivalent form of the flow rule in terms of generalized variables. The advocated internal variable treatment proves to be ideally suited to derive a complete variational formulation of the structural model in plasticity and viscoplasticity, see, e.g., DeAngelis (2000).

4 Computational Formulation of the Evolutive Laws in Elastoplasticity with Hardening

From a computational point of view it is convenient to consider an additive decomposition of the stress tensor into the deviatoric and spherical parts $\boldsymbol{\sigma} = \mathbf{s} + p\,\mathbf{1}$, where the pressure is $p = \frac{1}{3}tr(\boldsymbol{\sigma})$, the spherical part is $p\,\mathbf{1}$, the rank two identity tensor is denoted by $\mathbf{1}$ and the stress deviator is represented by $\mathbf{s} = \text{dev}\boldsymbol{\sigma} = \boldsymbol{\sigma} - p\,\mathbf{1}$. Similarly, the strain tensor is decomposed into the deviatoric and volumetric parts $\boldsymbol{\varepsilon} = \mathbf{e} + \frac{1}{3}\,\theta\,\mathbf{1}$, where θ is the change in volume and $\mathbf{e} = \text{dev}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} - \frac{1}{3}\,\theta\,\mathbf{1}$ represents the strain deviator. The linear elastic relation between the volumetric part of the stress and the volumetric part of the strain is formulated as $p = K\,\theta$ where K denotes the bulk modulus. The elastic relation between the deviatoric stress and the elastic deviatoric strain is $\mathbf{s} = 2G\,\mathbf{e}^e = 2G[\mathbf{e} - \mathbf{e}^p]$, where G denotes the shear modulus and, by assuming small strain elastoplasticity, the deviatoric part of the total strain is additively decomposed into an elastic and a plastic part $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p$. The relative stress is denoted by $\boldsymbol{\Sigma} = \mathbf{s} - \boldsymbol{\beta}$, where $\boldsymbol{\beta}$ represents the deviatoric back stress. A J_2 material model behaviour is considered in the sequel and, accordingly, a von Mises yield criterion is adopted in the form

$$f(\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa) = \|\mathrm{dev}\boldsymbol{\sigma} - \boldsymbol{\beta}\| - \kappa(\chi_{iso}) = \|\mathbf{s} - \boldsymbol{\beta}\| - \sqrt{\frac{2}{3}}(\sigma_{yo} + \chi_{iso}) \leq 0, \tag{14}$$

where $\kappa(\chi_{iso}) = \sqrt{\frac{2}{3}}\,\sigma_y = \sqrt{\frac{2}{3}}(\sigma_{yo} + \chi_{iso})$ is the current radius of the yield surface in the deviatoric plane and σ_{yo} is the uniaxial yield stress of the virgin material. The static internal variable related to isotropic hardening is expressed as $\chi_{iso} = H_{iso}\bar{e}^p$, and the dual kinematic internal variable $\bar{e}^p$ denotes the equivalent (accumulated) plastic strain $\bar{e}^p = \int_0^t \sqrt{\frac{2}{3}}\|\dot{\mathbf{e}}^p\|\mathrm{d}\,t$. The deviatoric plastic strain rate supply the evolutive flow law for the constitutive equation in associative plasticity

$$\dot{\mathbf{e}}^p = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\Sigma}} = \dot{\gamma}\,\mathbf{n}, \tag{15}$$

in which $\dot{\gamma}$ is the plastic multiplier and the second rank tensor $\mathbf{n}$ is expressed as $\mathbf{n} = \frac{\boldsymbol{\Sigma}}{\|\boldsymbol{\Sigma}\|}$, and it is characterized by unit norm. The accumulated equivalent plastic strain rate is expressed as $\dot{\bar{e}}^p = \sqrt{\frac{2}{3}}\,\dot{\gamma}$. By assuming linear kinematic hardening behaviour the back stress rate is expressed in the form originally proposed by Prager (1949)

$$\dot{\boldsymbol{\beta}} = \frac{2}{3}\,H_{kin}\,\dot{\mathbf{e}}^p, \tag{16}$$

where it has been assumed $\mathbf{H}_{kin} = \frac{2}{3}\,H_{kin}\mathbf{I}$. In the assumption of a nonlinear kinematic hardening behaviour in the literature it is often adopted the model originally proposed by Armstrong and Frederick (1966) (see e.g. Chaboche (1989)), which can be expressed as

$$\dot{\boldsymbol{\beta}} = \frac{2}{3}\,H_{kin}\,\dot{\mathbf{e}}^p - H_{nl}\,\dot{\bar{e}}^p\boldsymbol{\beta}, \tag{17}$$

where H_{nl} is a non-dimensional material dependent parameter characterizing nonlinear kinematic hardening behaviour. In equation (17) the second term represents a recall term and for $H_{nl} = 0$ the linear kinematic hardening behaviour is reproduced. A better approximation consists in adding several models such as (17) with different recall constants (Chaboche (1989), Chaboche (2008))

$$\boldsymbol{\beta} = \sum_{i=1}^{M}\boldsymbol{\beta}_i, \qquad \dot{\boldsymbol{\beta}}_i = \frac{2}{3}\,H_{kin,i}\,\dot{\mathbf{e}}^p - H_{nl,i}\,\dot{\bar{e}}^p\boldsymbol{\beta}_i. \tag{18}$$

For numerical integration algorithms in computational plasticity a comprehensive treatment of the subject is provided by, among others, Simo and Hughes (1998) and Zienkiewicz and Taylor (2005). A general solution procedure holding for general yield criteria is provided by, e.g., Alfano et al. (2001).

5 Comparative Analysis of Linear and Nonlinear Kinematic Hardening Modelling

In this section a comparative analysis between the linear and the nonlinear kinematic hardening assumptions is illustrated and discussed. The search for fast and robust integration methods relative to nonlinear kinematic hardening models is nowadays an active topic of investigation between researchers. In this paper a solution procedure which preserves the quadratic rate of asymptotic convergence is used. The aim of the present paper is focused on the investigation of the comparative analysis of linear and nonlinear kinematic hardening rules for different material properties. A more detailed description of the adopted algorithmic solution procedure and the development of a consistent tangent operator for nonlinear kinematic hardening plasticity is described thoroughly in DeAngelis (2012). For the numerical tests we use a three-dimensional finite element, based on a mixed approach and implemented into the Finite Element Analysis Program FEAP (Zienkiewicz and Taylor (2005)). In the simulations the load is enforced on a cubic specimen of side length equal to 5, by prescribing a uniform displacement on the top boundary of the specimen and with the appropriate boundary conditions. The sample is modelled with only one element. In simulating the material constitutive behaviour the computational tests are performed for different types of material parameters so that in each case the effectiveness of the adoption of linear versus nonlinear kinematic hardening can be properly evaluated.

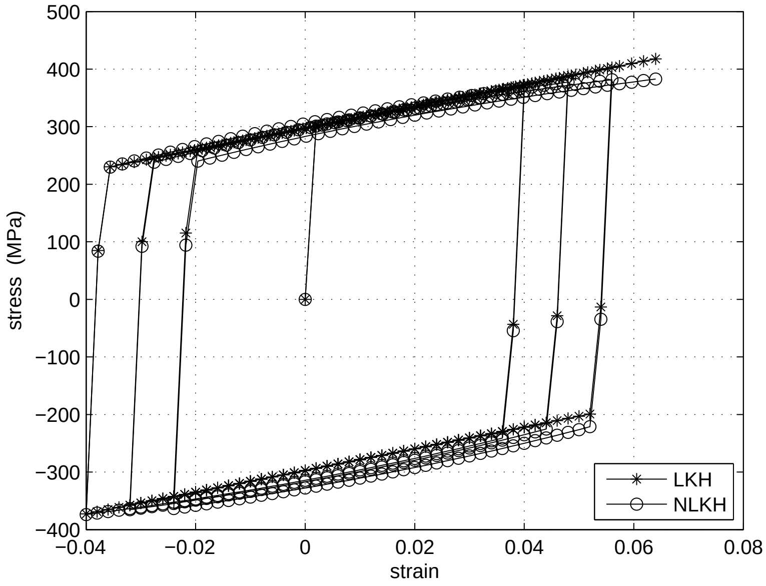

Figure 5: Comparative analysis between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity for carbon steel Ck 15. Loading conditions in tension-compression with loading program reported in Fig.1.

5.1 Computational Results: Chromium-Nickel Stainless Steel X5CrNi 18·9

In the first simulation test we consider the material properties which identify the chromium-nickel stainless steel X5CrNi 18·9 (see, e.g., Hartmann et al. (1997)). The material properties are: elastic modulus $E = 208000$ MPa, Poisson's ratio $\nu = 0.3$, yield limit $\sigma_{yo} = 170$ MPa, kinematic hardening modulus $H_{kin} = 41080$ MPa, nonlinear kinematic hardening parameter $H_{nl} = 525$, isotropic hardening modulus $H_{iso} = 0$ MPa. On the top boundary of the specimen a vertical displacement is prescribed and set equal to u_o = 0.001. The proportional load coefficient $p(t)$ describes the evolution with time, by amplifying the prescribed displacement. Accordingly, the loading history is given by the relation $u(t) = p(t)u_o$. A simulation of the material behaviour is performed by assigning a loading program in tension-compression with increasing mean stress and with loading history illustrated in Figure 1. The related stress-strain curves are reported in Figure 2, where a comparative analysis of the linear kinematic hardening behaviour (LKH) and the nonlinear kinematic hardening behaviour (NLKH) is illustrated. Another test for the simulation of the material behaviour is performed by assigning a loading program with increasing levels of loading and with loading history illustrated in Figure 3. The stress-strain curves for this loading condition are illustrated in Figure 4, where it is possible to evaluate the different assumptions of linear and nonlinear hardening models on the material constitutive behaviour. A comparative analysis of the different assumptions of a linear kinematic hardening rule (LKH) and a nonlinear kinematic hardening rule (NLKH) in elastoplasticity is shown in Figure 4, where the differences between the two assumptions on the kinematic hardening rule are clearly illustrated.

5.2 Computational Results: Carbon Steel Ck15

In this section we consider a simulation test in which the material properties identify the plain carbon steel Ck15 (see, e.g., Dettmer and Reese (2004) and Luhrs et al. (1997)). The material properties are: elastic modulus $E = 208000$ MPa, Poisson's ratio $\nu = 0.3$, yield limit $\sigma_{yo} = 300$ MPa, kinematic hardening modulus $H_{kin} = 1900$ MPa, nonlinear kinematic hardening parameter $H_{nl} = 8.5$, isotropic hardening modulus $H_{iso} = 0$ MPa. On the top boundary of the test specimen a prescribed vertical displacement is assigned and set equal to u_o = 0.01. A proportional load coefficient $p(t)$ describes the evolution with time of the prescribed displacement and defines the loading history according to the relation $u(t) = p(t)u_o$. The simulation of the material behaviour is performed by assigning a loading program in tension-compression with loading history illustrated in Figure 1. The related stress-strain curves are reported in Figure 5. A comparative analysis of the different assumptions of a linear kinematic hardening rule (LKH) and a nonlinear kinematic hardening rule (NLKH) for this plain carbon steel can be made from the observation of Figure 5. A second test for the simulation of the material behaviour is

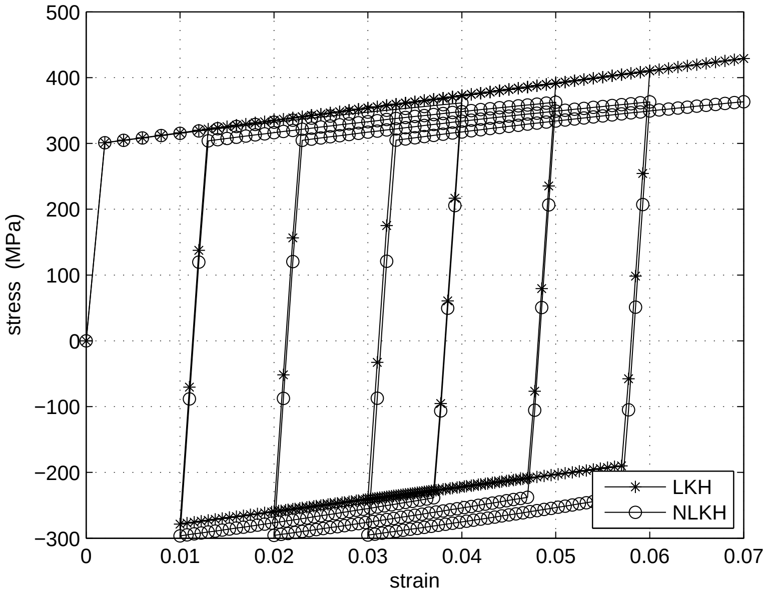

Figure 6: Comparative analysis between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity for carbon steel Ck 15. Loading conditions with increasing levels of loading and with loading program reported in Fig.3.

performed by assigning a loading history illustrated in Figure 3 with increasing levels of loading. For this type of loading condition the stress-strain curves are reported in Figure 6, where the discrepancies between the different hardening rule assumptions on the material constitutive behaviour can be evaluated. The comparative analysis and the differences between the assumptions of a linear kinematic hardening rule (LKH) and a nonlinear kinematic hardening rule (NLKH) in elastoplasticity are clearly illustrated in Figure 6.

6 Conclusions

In this work typical linear and nonlinear hardening laws for elastoplasticity have been investigated. A comparative analysis between the assumptions of linear versus nonlinear kinematic hardening behaviour has been presented with illustrative numerical simulations. In fact in finite element applications of large scale elastoplastic structural analysis the linear kinematic hardening rule is usually adopted since this assumption ensures a symmetric tangent stiffness matrix and computationally efficient solution procedures. However, in the literature it has been discussed the opportunity of adopting nonlinear kinematic hardening rules in order to properly simulate experiments on real materials. The computational implementation and research for fast and effective numerical procedures for nonlinear kinematic hardening rules is not trivial especially for large structural simulations and complex loading conditions which involve large computing times.

In the present article a comparative analysis of the linear and the nonlinear kinematic hardening rules has been performed for different material properties in order to have a better comprehension of the suitability of applying the different assumptions in the appropriate modelling of the material behaviour. Numerical tests and computational results have been performed for different types of material parameters in order to understand in which case the adoption of more complex kinematic hardening rules is considered to be advisable or necessary. Another purpose of the present analysis is to be able to evaluate in a broad sense the type of disagreement or lack of conformity of the two different assumptions on the kinematic hardening rule for different material properties.

Consequently, a comparative analysis between the adoption of linear versus nonlinear kinematic hardening rules has been illustrated for different material properties and for different loading conditions. The performed comparative analysis allows to have a better understanding on the conditions under which the different assumptions on the hardening rules are considered adequate and favorable. In addition the performed analysis also gives useful insights on the degree of discrepancy that the two different hardening rule assumptions imply in the simulation of elastoplastic structures subject to complex loading conditions.

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References

Alfano, G.; DeAngelis, F.; Rosati, L.: General solution procedure in elasto/viscoplasticity. *Computer Methods in Applied Mechanics and Engineering*, 190, (2001), 5123–5147.

Armstrong, P.; Frederick, C.: A mathematical representation of the multiaxial baushinger effect. Tech. rep., Central Electricity Generating Board, Berkeley, UK (1966), report RD/B/N731.

Artioli, E.; Auricchio, F.; da Veiga, L. B.: Second-order accurate integration algorithms for the von-mises plasticity with a nonlinear kinematic hardening mechanism. *Computer Methods in Applied Mechanics and Engineering*, 196, (2007), 1827–1846.

Auricchio, F.; Taylor, R.: Two material models for cyclic plasticity: non-linear kinematic hardening and generalized plasticity. *International Journal of Plasticity*, 11, (1995), 65–98.

Barthel, C.; Levkovitch, V.; Svendsen, B.: Modeling of sheet metal forming processes taking into account distorsional hardening. *International Journal of Material Forming*, 1 (1), (2008), 105–108.

Chaboche, J.: Constitutive equations for cyclic plasticity and cyclic visco-plasticity. *International Journal of Plasticity*, 5, (1989), 247–302.

Chaboche, J.: A review of some plasticity and viscoplasticity constitutive theories. *International Journal of Plasticity*, 24, (2008), 1642–1693.

Chaboche, J.; Cailletaud, G.: Integration methods for complex plastic constitutive equations. *Computer Methods in Applied Mechanics and Engineering*, 133, (1996), 125–155.

Dafalias, Y.; Popov, E.: A model of nonlinearly hardening materials for complex loading. *Acta Mechanica*, 21, (1975), 173–192.

DeAngelis, F.: An internal variable variational formulation of viscoplasticity. *Computer Methods in Applied Mechanics and Engineering*, 190, (2000), 35–54.

DeAngelis, F.: Return mapping algorithms and numerical analysis for elastoplasticity with nonlinear kinematic hardening. (in preparation) (2012).

Dettmer, W.; Reese, S.: On the theoretical and numerical modelling of armstrong-frederick kinematic hardening in the finite strain regime. *Computer Methods in Applied Mechanics and Engineering*, 193, (2004), 87–116.

Doghri, I.: Fully implicit integration and consistent tangent modulus in elasto-plasticity. *International Journal for Numerical Methods in Engineering*, 36, (1993), 3915–3932.

Germain, P.; Nguyen, Q.; Suquet, P.: Continuum thermodynamics. *Jour. Appl. Mech.*, 50, (1983), 1010–1020.

Hakansson, P.; Wallin, M.; Ristinmaa, M.: Comparison of isotropic hardening and kinematic hardening in thermoplasticity. *International Journal of Plasticity*, 21, (2005), 1435–1460.

Halphen, B.; Nguyen, Q.: Sur les matériaux standards généralisés. *Jour. Mech.*, 14, (1975), 39–63.

Hartmann, S.; Luhrs, G.; Haupt, P.: An efficient stress algorithm with applications in viscoplasticity and plasticity. *International Journal for Numerical Methods in Engineering*, 40, (1997), 991–1013.

Hiriart-Urruty, J.; Lemaréchal, C.: *Convex Analysis and Minimization Algorithms*. Springer-Verlag, Berlin (1993).

Hopperstad, O.; Remseth, S.: A return mapping algorithm for a class of cyclic plasticity models. *International Journal for Numerical Methods in Engineering*, 38, (1995), 549–564.

Johansson, G.; Ekh, M.; Runesson, K.: Computational modeling of inelastic large ratcheting strains. *International Journal of Plasticity*, 21, (2005), 955–980.

Lemaitre, J.; Chaboche, J.: *Mechanics of solid materials*. Cambridge University Press (1990).

Lion, A.: Constitutive modelling in finite thermoviscoplasticity: a physical approach based on nonlinear rheological models. *International Journal of Plasticity*, 16, (2000), 469–494.

Lubarda, V.; Benson, D.: On the numerical algorithm for isotropic-kinematic hardening with the armstrong-frederick evolution of the back stress. *Computer Methods in Applied Mechanics and Engineering*, 191, (2002), 3583–3596.

Lubliner, J.; Taylor, R.; Auricchio, F.: A new model of generalized plasticity. *International Journal of Solids and Structures*, 30, (1993), 3171–3184.

Luhrs, G.; Hartmann, S.; Haupt, P.: On the numerical treatment of finite deformations in elastoviscoplasticity. *Computer Methods in Applied Mechanics and Engineering*, 144, (1997), 1–21.

McDowell, D.: A non-linear kinematic hardening theory for cyclic thermo-plasticity and thermo-visco-plasticity. *International Journal of Plasticity*, 8, (1992), 695–728.

Menzel, A.; Ekh, M.; Runesson, K.; Steinmann, P.: A framework for multiplicative elastoplasticity with kinematic hardening coupled to anisotropic damage. *International Journal of Plasticity*, 21, (2005), 397–434.

Pietryga, M.; Vladimirov, I.; Reese, S.: A finite deformation model for evolving flow anisotropy with distortional hardening including experimental validation. *Mechanics of Materials*, 44, (2012), 163–173.

Prager, W.: Recent developments in the mathematical theory of plasticity. *J. Appl. Phys.*, 20, (3), (1949), 235–241.

Schwarze, M.; Vladimirov, I.; Reese, S.: Sheet metal forming and springback simulation by means of a new reduced integration solid-shell finite element technology. *Computer Methods in Applied Mechanics and Engineering*, 200, (2011), 454–476.

Simo, J.; Hughes, T.: *Computational inelasticity*. Springer-Verlag, New York (1998).

Svendsen, B.; Levkovitch, V.; Wang, J.; Reusch, F.; Reese, S.: Application of the concept of evolving structure tensors to the modeling of initial and induced anisotropy at large deformation. *Computers and Structures*, 84, (2006), 1077–1085.

Tsakmakis, C.; Willuweit, A.: A comparative study of kinematic hardening rules at finite deformations. *International Journal of Non-Linear Mechanics*, 39, (2004), 539–554.

Vladimirov, I.; Pietryga, M.; Reese, S.: On the modelling of non-linear kinematic hardening at finite strains with application to springback - comparison of time integration algorithms. *International Journal for Numerical Methods in Engineering*, 75, (2008), 1–28.

Vladimirov, I.; Pietryga, M.; Reese, S.: Prediction of springback in sheet forming by a new finite strain model with nonlinear kinematic and isotropic hardening. *Journal of Materials Processing Technology*, 209, (2009), 4062–4075.

Vladimirov, I.; Pietryga, M.; Reese, S.: Anisotropic finite elastoplasticity with nonlinear kinematic and isotropic hardening and application to sheet metal forming. *International Journal of Plasticity*, 26, (2010), 659–687.

Vladimirov, I.; Pietryga, M.; Reese, S.: On the influence of kinematic hardening on plastic anisotropy in the context of finite strain plasticity. *International Journal of Material Forming*, 4, (2011), 255–267.

Wallin, M.; Ristinmaa, M.: Deformation gradient based kinematic hardening model. *International Journal of Plasticity*, 21, (2005), 2025–2050.

Wallin, M.; Ristinmaa, M.; Ottosen, N. S.: Kinematic hardening in large strain plasticity. *European Journal of Mechanics - A:Solids*, 22, (2003), 341–356.

Wang, J.; Levkovitch, V.; Reusch, F.; Svendsen, B.; Huetink, J.; van Riel, M.: On the modeling of hardening in metals during non-proportional loading. *International Journal of Plasticity*, 24, (2008), 1039–1070.

Wriggers, P.: *Nonlinear Finite Element Methods*. Springer Verlag, Berlin (2008).

Zienkiewicz, O.; Taylor, R.: *The Finite Element Method for Solid and Structural Mechanics*. Elsevier, Oxford, 6th edn. (2005).

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