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Graphical Representations of the Regions of Rank-One-Convexity of some Strain Energies

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Isotropic elastic energies which are quadratic in the strain measures of the Seth family are known not to be rank-one-convex in the entire domain of invertible deformation gradients with positive determinant. Therefore, they are in principle capable of displaying a laminated microstructure. Nevertheless, they are commonly used for standard elastic solids. In general one does not observe a microstructure evolution due to the fact that the solution is not sought outside of the region of rank-one-convexity. Consequently, the question for the boundaries of the region of rank-one-convexity arises. We address this question by applying a set of necessary and sufficient conditions for rank-one-convexity to the mentioned elastic energies, and give graphical representations for the regions of rank-one-convexity.

1 Introduction

The modelling of elasticity of solids has reached a high level of sophistication, coming a long way from Hooke's observation that the force is proportional to the strain (1660) and Green's existence theorem (1839, Ferrers (1871)) of an elastic energy w. The latter is the starting point for most of the approaches to elasticity. However, several restrictions have to be imposed on w, both from purely mathematical and physical considerations. Starting from the theory of hyperelastic simple materials, i.e., w depends only on the deformation gradient $\boldsymbol{F}$, it is imposed that

- w has to be quasiconvex in $\boldsymbol{F}$ (Morrey, 1952),
- w has satisfy growth conditions, also known as coercivity,
- w should depend only through $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^2$ on $\boldsymbol{F}$, in order to make w independent of rotations (material objectivity),
- w should be positive for any deformation.

The first two conditions are necessary for the existence of a unique solution of the elastostatic boundary value problem. The third and fourth condition are imposed due to physical considerations. Unfortunately, it is very hard to change the order in which the constraints are imposed, since the linearity of $\boldsymbol{F}$ in the position vector $\boldsymbol{x}$ is an important property needed for the examination of the existence and uniqueness of a solution to the elastostatic boundary value problem.

Since it is difficult to verify the quasiconvexity of an elastic energy, other notions of convexity have been examined, e.g., by Ball (1977). Polyconvexity and rank-one-convexity proved to be more practical. Unlike quasiconvexity, a function $w(\boldsymbol{F})$ can be checked pointwise for specific $\boldsymbol{F}$ for poly- and rank-one-convexity. Moreover, both appear to be very close to quasiconvexity, where the implications

polyconvexity $\Rightarrow$ *quasiconvexity* $\Rightarrow$ *rank-one-convexity*

hold. Rank-one-convexity is interesting for two reasons:

- It is related to the kinematic compatibility condition. The deformation gradient can only undergo rank-one jumps, i.e.,

$$\boldsymbol{F}^+ - \boldsymbol{F}^- = \boldsymbol{a}\boldsymbol{b}, \qquad \boldsymbol{a}, \boldsymbol{b} \in R^3 \backslash \{\boldsymbol{0}\}. \tag{1}$$

This ensures that, at singular surfaces with the normal vector $\boldsymbol{b}$, the body does not open or overlap with itself. Poly- and quasiconvexity can also be linked to physical notions, namely to different stability criteria (Ball and Marsden, 1984). However, the kinematic compatibility condition is more elementary.

- It appears to be very close to quasiconvexity. There are examples of elastic energies which are rank-one-convex but not quasiconvex, firstly established by Sverak (1992), but these strain energies are not rotationally invariant. It is still unknown whether a rotational invariant rank-one-convex strain energy which is not quasiconvex exists (Šilhavý, 2002).

The failure of polyconvexity has been demonstrated for the isotropic St. Venant-Kirchhoff elastic energy (Raoult, 1986), and for isotropic linear stress strain relations based on the logarithmic strains by Bruhns et al. (2001). Bruhns et al. also gave lower bounds for the limits of the region of rank-one-convexity and state that, for modelling phenomena with discontinuous deformation gradients, it may be important to know the exact locations of these limits. In Bertram et al. (2007), all energy functions that are quadratic in Seth strain measures are shown to violate rank-one-convexity for one deformation state or another. However, since these states usually lie outside the domain within which a solution is sought, the functions are applied without problems in commercial finite element software. For example the FE system ABAQUS 6.7-1 employs an elastic law relating the Cauchy stresses linearly to the logarithmic strains in the large strain setting. (Although this information is not given in the documentation, it can be found easily in a uniaxial tension test.) Thus, before applying these laws to large deformations one should ask for the limit of the region of rank-one-convexity.

Notation. Vectors are symbolized by lowercase bold letters, second order tensors by uppercase bold letters. The tensor product $\boldsymbol{ab}$ is defined by $(\boldsymbol{ab}) \cdot \boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$. The dot represents a simple contraction, the number of dots corresponds to the number of contractions. Indices larger than 3 have to be taken *modulo* 3. The deformation gradient and its polar decomposition are $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$, $\boldsymbol{U} = \sqrt{\boldsymbol{C}}$, $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$. The eigenvalues of $\boldsymbol{U}$, resp. the singular values of $\boldsymbol{F}$, are denoted by $\lambda_{1,2,3}$. The strain energy is denoted by w, the indexing of w corresponds to the partial derivatives w.r.t. $\lambda_{1,2,3}$.

2 Necessary and Sufficient Conditions for Rank-One-Convexity

Unfortunately, the full set of necessary and sufficient conditions for rank-one-convexity is known only for isotropic (Rosakis, 1990; Šilhavý, 1999; Dacorogna, 2001) and incompressible isotropic (Zee and Sternberg, 1983) strain energies. The starting point for the derivation is

$$0 \leq (\boldsymbol{ab}) \cdot\cdot \frac{\partial^2 w}{\partial \boldsymbol{F}^2} \cdot\cdot (\boldsymbol{ab}) \qquad \text{for all} \quad \boldsymbol{a} \in \mathbb{R}^3, \boldsymbol{b} \in \mathbb{R}^3. \tag{2}$$

Further, in the case of incompressibility, $\boldsymbol{a}$ and $\boldsymbol{b}$ must obey

$$\boldsymbol{b} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{a} = 0 \tag{3}$$

due to the fact that the jump of $\boldsymbol{F}$ at a singular surface must be volume preserving. Limiting the domain of $\boldsymbol{a}$ or $\boldsymbol{b}$ by the latter inequality, the restrictions for rank-one-convexity are less strict in the case of incompressibility. Due to the isotropy, w depends only on the singular values of $\boldsymbol{F}$.

Eliminating the quantifiers in eq. (2) is a laborious work. One way of doing this is

- normalize $\boldsymbol{a}$ and $\boldsymbol{b}$
- determine $\partial^2 w / \partial \boldsymbol{F}^2$ at $\boldsymbol{F} = \boldsymbol{U}$ (the substitution $\boldsymbol{a} = \boldsymbol{R} \cdot \tilde{\boldsymbol{a}}$ is always possible)
- carry out the contractions with $\boldsymbol{b}$, which gives the acoustic tensor $\boldsymbol{A}$
- assure positive definiteness of $\boldsymbol{A}$

The last step can be done by assuring the positivity of the principle minors. In doing so, one arrives at the following set of inequalities:

- The first principle minor gives the separate convexity conditions

$$0 < w_{ii}, \qquad \text{(no sum)} \tag{4}$$

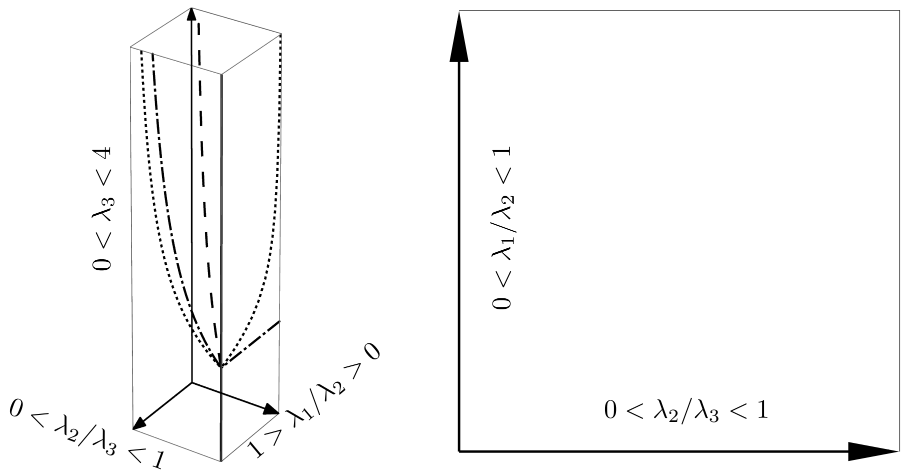


Figure 1: Scheme for graphical representation. For later reference, some characteristic strain-driven tests are depicted, namely a simple shear test (dashed), isochoric extension/compression (dotted), uniaxial extension/compression without lateral straining (dash-dot), pure volume change (solid line).

and the Baker-Ericksen-inequalities

$$0 < \frac{w_i \lambda_i - w_j \lambda_j}{\lambda_i - \lambda_j}, \qquad i \neq j. \tag{5}$$

- The second principle minor gives inequalities involving mixed derivatives

$$0 < m_{ij}^{\pm} + \sqrt{w_{ii} w_{jj}}, \quad i \neq j \tag{6}$$

$$m_{ij}^{\pm} = \pm\, w_{ij} + \frac{w_i - \pm w_j}{\lambda_i - \pm \lambda_j}. \tag{7}$$

- The third principle minor gives inequalities involving mixed derivatives

$$0 < m_{12}^{\pm}\sqrt{w_{33}} + m_{13}^{\pm}\sqrt{w_{22}} + m_{23}^{\pm}\sqrt{w_{11}} + \sqrt{w_{11} w_{22} w_{33}}, \tag{8}$$

  which must be evaluated for the combinations $\{m_{12}^{+}, m_{13}^{+}, m_{23}^{+}\}$, $\{m_{12}^{-}, m_{13}^{+}, m_{23}^{-}\}$, $\{m_{12}^{+}, m_{13}^{-}, m_{23}^{-}\}$, and $\{m_{12}^{-}, m_{13}^{-}, m_{23}^{+}\}$.

These results can be found in Dacorogna (2001). For the incompressible case, a similar procedure is possible. However, one has to take the derivative with respect to $\boldsymbol{F}$ on the subspace $\det(\boldsymbol{F}) = 1$, and incorporate eq. (3).

3 Graphical Representations of the Regions of Rank-One-Convexity

Using the inequalities (4, 5, 6, 8), we are able to determine the regions of rank-one-convexity. $w(\lambda_1, \lambda_2, \lambda_3)$ is symmetric w.r.t. index permutations. Therefore, the plotting range can be restricted by the ordering $\lambda_1 \leq \lambda_2 \leq \lambda_3$, which is achieved by introducing the parameters $q_1 = \lambda_1/\lambda_2$ and $q_2 = \lambda_2/\lambda_3$. The graphical representations are given in terms of $0 \leq q_1 \leq 1, 0 \leq q_2 \leq 1, 0 \leq \lambda_3 \leq 4$. They have been constructed with the help of a computer algebra system. The figure layout is the same for all graphics. On the left, a 3D-representation is given, where the non-region of rank-one-convexity is filled. The lower back corner corresponds to the origin of the coordinates q_1, q_2, λ_3, and the upper front corner to $q_1 = 1$, $q_2 = 1$, $\lambda_3 = 4$, see Fig. 1. Additionally, the region of relatively small strains $\sum_i (1 - \lambda_i)^2 < 0.2^2$ is represented. On the right, a contour plot of λ_3 over λ_2/λ_3 (the horizontal direction) and λ_1/λ_2 (the vertical direction) is given, which is more suitable for extracting data. It corresponds to a projection of the rank-one-convex boundary into the q_1–q_2–plane.

3.1 Elastic Energies which are Quadratic in Seth Strain Measures

In the case of a linear and isotropic stress strain relation, the elastic law and hence the elastic energy are completely determined by the choice of the strain measure and two elasticity constants. The elastic strain energy is then given by

$$w = \left(\mu + \frac{\lambda}{2}\right)(E_1^2 + E_2^2 + E_3^2) + \lambda(E_1E_2 + E_2E_3 + E_1E_3) \tag{9}$$

or normalized

$$w = E_1^2 + E_2^2 + E_3^2 + \beta(E_1E_2 + E_2E_3 + E_1E_3) \qquad \beta := \frac{\lambda}{\mu + \frac{\lambda}{2}} \tag{10}$$

with Lamé's constants λ, μ and the eigenvalues E_i of the strain tensor. Lamé's constants are related to the bulk- and shear modulus K and G by

$$\mu = G \qquad \lambda = K - \frac{2}{3}G. \tag{11}$$

The Legendre-Hadamard condition is ensured by $K > 0$, $G > 0$, which allows to determine the range for valid β,

$$-1 < \beta < 2. \tag{12}$$

One can write β as a function of Poisson's ratio ν

$$\beta = \frac{2\nu}{1-\nu} \tag{13}$$

where $-1 < \nu < \frac{1}{2}$ is admissible. Usually, the strains are defined as isotropic functions of the material stretching

$$\boldsymbol{E} = \sum_i E_m(\lambda_i)\boldsymbol{u}_i\boldsymbol{u}_i. \tag{14}$$

The function $E(\lambda)$ defines the strain measure. Typically one uses a Seth strain, for which

$$E_m(\lambda) := \frac{1}{m}(\lambda^m - 1), \qquad m \neq 0, \tag{15}$$

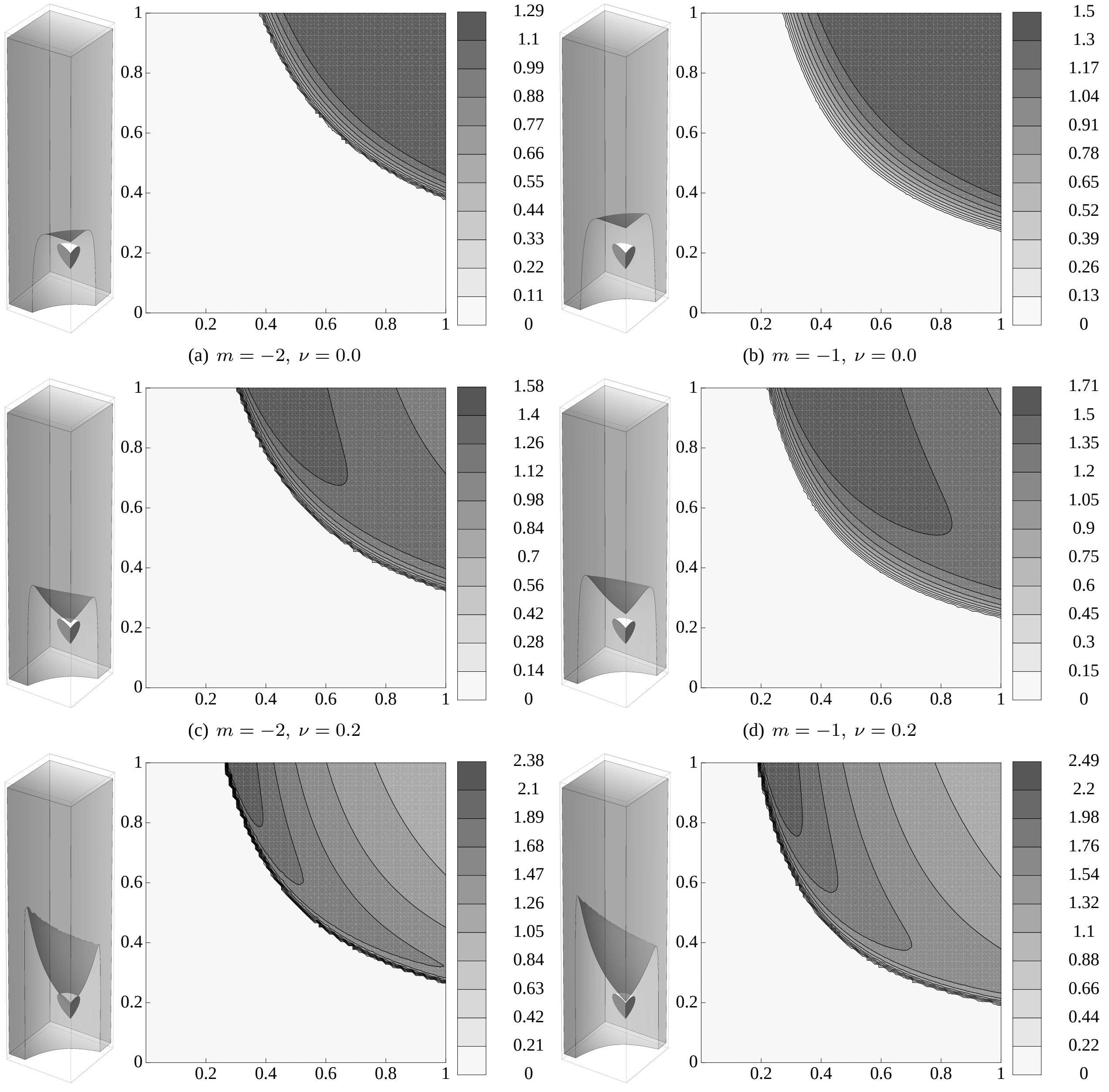
$$E_0(\lambda) := \ln(\lambda), \tag{16}$$

hold. For example, $m = -2$ corresponds to the Piola-Almansi strain. In the 1-dimensional homogeneous case, $m = -1$ corresponds to $\varepsilon = \Delta l/l$, i.e., one relates the length change to the actual (true) length. The case $m = 0$ corresponds to the logarithmic (Hencky) strain which is also referred to as the true strain. In the 1-dimensional homogeneous case, $m = 1$ corresponds to $\varepsilon = \Delta l/l_0$, i.e., one relates the length change to the reference length. Thus, it is referred to as the nominal strain, engineering strain or Biot strain. The case $m = 2$ leads to the the Green, Lagrange or Finger strain.

In Figs. 2 to 4, the regions of rank-one-convexity for $m = -2, -1, 0, 1, 2$ and $\nu = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ are plotted. It is found in general that for $\nu \to 0.5$, the limit of the region of rank-one-convexity approaches the unstretched state $\lambda_1 = \lambda_2 = \lambda_3 = 1$, independent of m. Further, for $m \leq 0$, one can impose arbitrarily large compressive strains without violating rank-one-convexity, while for $0 < m$, one can impose an arbitrarily large volume increase without violating rank-one-convexity.

The cases $m = 0$, $m = 1$ and $m = 2$ are most interesting, and examined in detail. It is found that

- for $m = 0$, eq. 5 imposes no restrictions (the right hand side is positive for all strains and approaches 0 as $\nu \to 0.5$). eq. 8 is globally weaker than eq. 4 and eq. 6 together.
- for $m = 1$, eq. 4 imposes no restrictions (the right hand side is just 2). Eq. eq. 5 and eq. 6 restrict the very same domain, which is given by $0 < \beta(\lambda_i - 2) + 2(\lambda_{i+1} + \lambda_{i+2} - 1)$. The latter approaches eq. 8 as $\nu \to 0.5$. Eq. 8 is globally weaker than eq. 5 (resp. eq. 6).
- for $m = 2$, none of eq. 4, eq. 5 or eq. 6 includes the other one globally. However, they include together eq. 6.

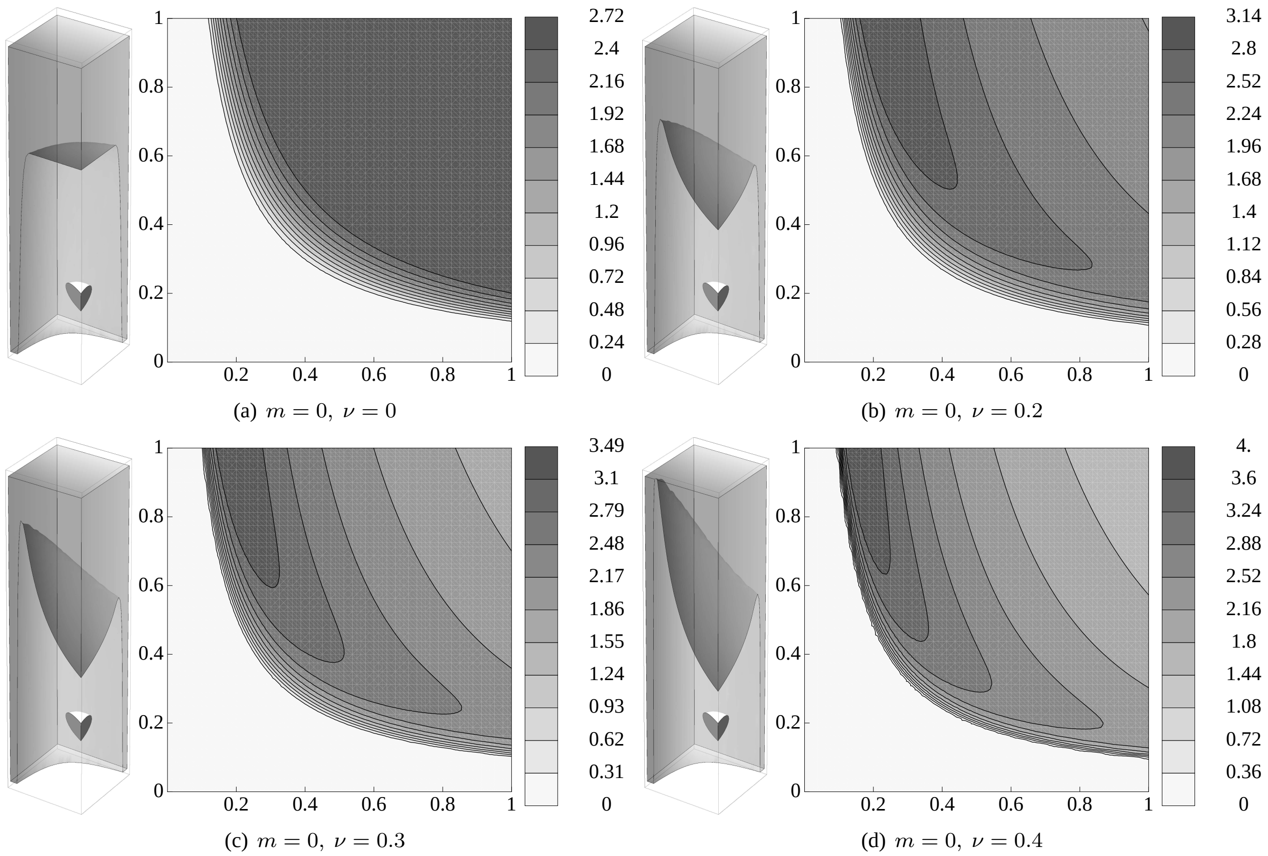


(a) $m = -2,\ \nu = 0.0$ (b) $m = -1,\ \nu = 0.0$

(c) $m = -2,\ \nu = 0.2$ (d) $m = -1,\ \nu = 0.2$

(e) $m = -2,\ \nu = 0.4$ (f) $m = -1,\ \nu = 0.4$

Figure 2: Regions of rank-one-convexity for $m = -2$, $m = -1$ and $\nu = 0.0$, $\nu = 0.2$, $\nu = 0.4$.



(a) $m = 0,\ \nu = 0$ (b) $m = 0,\ \nu = 0.2$ (c) $m = 0,\ \nu = 0.3$ (d) $m = 0,\ \nu = 0.4$

Figure 3: Regions of rank-one-convexity for $m = 0$ and $\nu = 0.0$, $\nu = 0.2$, $\nu = 0.3$, $\nu = 0.4$.

3.2 Ciarlet-Geymonat Strain Energy

The Ciarlet-Geymonat strain energy (Ciarlet and Geymonat, 1982) is conveniently denoted in terms of principal invariants of $\boldsymbol{B}$,

$$w_{\rm CG} = \frac{\lambda}{4}(III_{\boldsymbol{B}} - \ln III_{\boldsymbol{B}} - 1) + \frac{\mu}{2}(I_{\boldsymbol{B}} - \ln III_{\boldsymbol{B}} - 3) \tag{17}$$

with $III_{\boldsymbol{B}} = (\lambda_1\lambda_2\lambda_3)^2$ and $I_{\boldsymbol{B}} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ and Lamé's constants (Simo and Hughes, 1998; **?**). The original function contains six material parameters (Ciarlet (1988), Exercise 4.23), which are reduced by approaching a linearly elastic isotropic material with λ and μ as $\boldsymbol{F} \to \boldsymbol{I}$. It is derived from the family of Ogden materials, but with the restriction to be polyconvex for positive values of λ and μ (Ciarlet (1988), page 185). Thus, $w_{\rm CG}$ is rank-one-convex in the entire domain of positive $\lambda_1, \lambda_2, \lambda_3$ if $\nu \geq 0$. We can normalize $w_{\rm CG}$ by multiplying with $2/\mu$, where we are able to replace

$$\frac{\lambda}{2\mu} = \frac{\nu}{1 - 2\nu}. \tag{18}$$

This allows, similar to the preceding section, to parametrize the range of valid material parameters by ν on the normalized strain energy. For negative ν, the region of rank-one-convexity is bounded. It is smallest for $\nu = -1$, for which a graphical representation is given in Fig. 5(a).

3.3 Blatz-Ko Strain Energy

The strain energy of Blatz and Ko is of the form

$$w_{\rm Blatz-Ko} = \frac{\mu f}{2}\left[I_{\boldsymbol{B}} - 3 + \frac{2\mu}{\lambda}(III_{\boldsymbol{B}}^{-\frac{\lambda}{\mu}} - 1)\right] + \frac{\mu(1-f)}{2}\left[I_{\boldsymbol{B}^{-1}} - 3 + \frac{2\mu}{\lambda}(III_{\boldsymbol{B}}^{\frac{\lambda}{\mu}} - 1)\right], \tag{19}$$

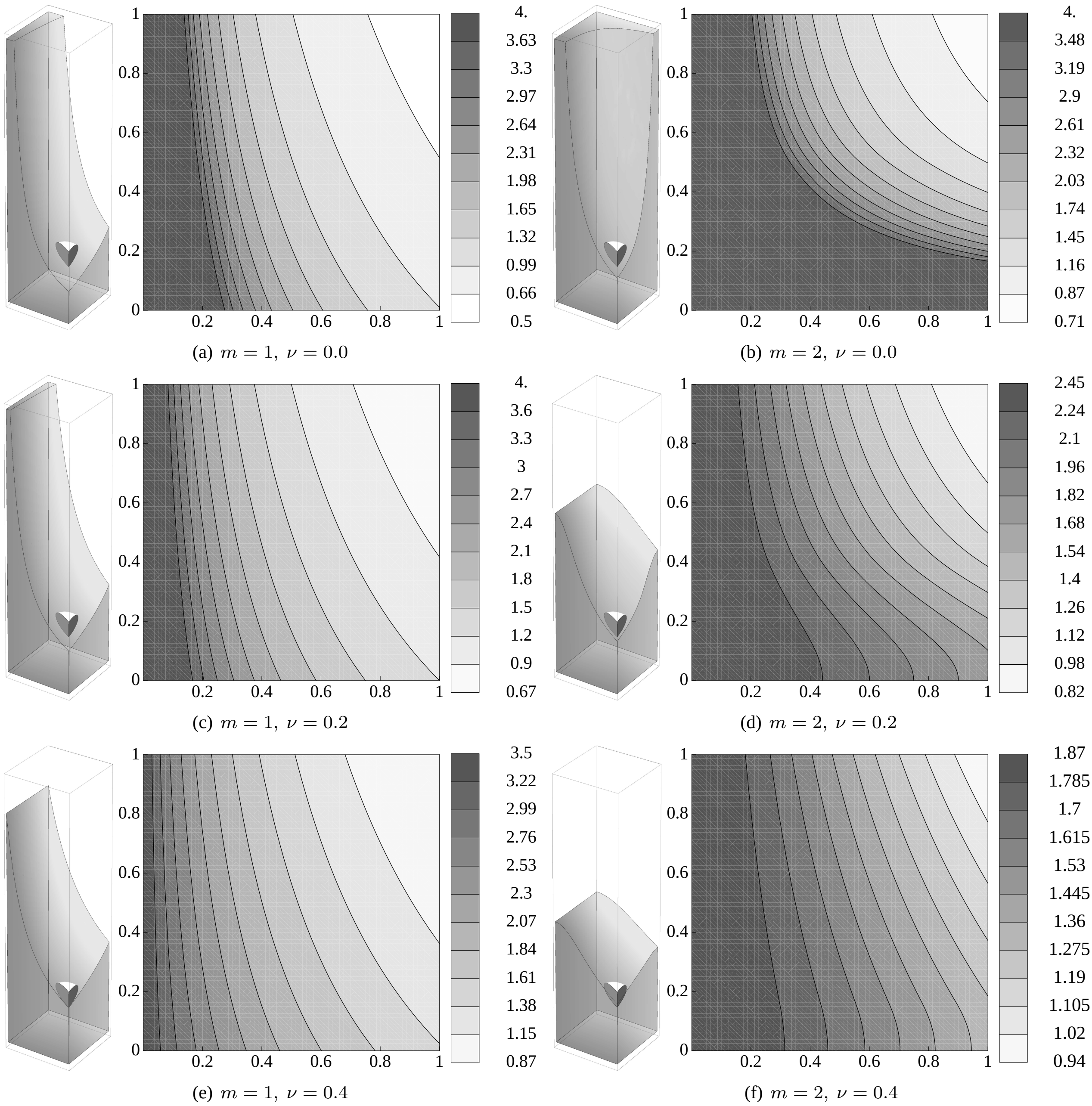
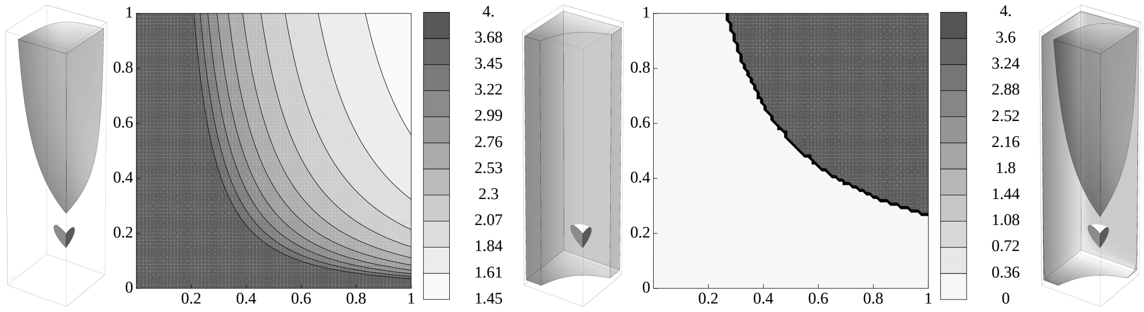


Figure 4: Regions of rank-one-convexity for $m = 1$, $m = 2$ and $\nu = 0.0$, $\nu = 0.2$, $\nu = 0.4$.



(a) Ciarlet-Geymonat strain energy at $\nu = -1$ (b) Reduced Blatz-Ko-strain energy (c)

Figure 5: Region of rank-one-convexity of the Ciarlet-Geymonat strain energy at $\nu = -1$ (a), the reduced Blatz-Ko strain energy (b) and Attard's strain energy (c) for the material parameters of eq. (22).

with $I_{\boldsymbol{B}^{-1}} = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}$. Compared to its original form (Blatz and Ko (1962), eq. 50) the expression $(1-2\nu)/\nu$ is replaced by $2\mu/\lambda$ (eq. 18) and J_3 is replaced by $\sqrt{III_{\boldsymbol{B}}}$. By fixing $f = 0$ and $\lambda/\mu = 1$ ($\nu = 0.25$) it is reduced to

$$w^*_{\mathrm{Blatz-Ko}} = \frac{\mu}{2}(I_{\boldsymbol{B}^{-1}} + 2\sqrt{III_{\boldsymbol{B}}} - 5) = \frac{\mu}{2}(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} + 2\lambda_1\lambda_2\lambda_3 - 5) \tag{20}$$

see Blatz and Ko (1962) eq. (67). The strain energy $w^*_{\mathrm{Blatz-Ko}}$ is used to model polymeric foams (Horgan, 1996). However, Horgan noted that strong ellipticity is lost in case of $0 \leq f < 1$, i.e., $w^*_{\mathrm{Blatz-Ko}}$ displays loss of rank-one-convexity. In $w^*_{\mathrm{Blatz-Ko}}$, there is no material parameter which affects the region of rank-one-convexity, i.e., a single graph is sufficient (Fig.5(b)). It is found that $w^*_{\mathrm{Blatz-Ko}}$ does not fail to be rank-one-convex under purely volumetric deformations. To be more specific, rank-one-convexity is lost only if $\lambda_i/\lambda_j < 0.2688$.

3.4 Attard's Strain Energy

Attard (2003) proposed a strain energy of the form

$$w_{\mathrm{Attard}} = \sum_{n=1}^{N}\left(\frac{A_n}{2n}(I_{\boldsymbol{C}^n} - 3) + \frac{B_n}{2n}(I_{\boldsymbol{C}^{-n}} - 3)\right) + \sum_{m=1}^{M}\frac{C_m}{2m}(\ln J)^{2m} - \ln J\sum_{n=1}^{N}(A_n - B_n). \tag{21}$$

The first sum represents a shape-change energy contribution, the second and third sum incorporate the dependence of w_{Attard} on volume changes. Usually, nonlinear strain energies employ more than two material parameters, i.e., it is a tedious task to make a parameter-study of the regions of rank-one-convexity. Therefore, we examine the rank-one-convexity for only one of the resultant strain energies, obtained by adopting Attard's strain energy to a rubber material. The nonzero material parameters (Attard (2003) eq. 69) are

$$A_1 = 0.361\mathrm{MPa} \qquad B_1 = 0.22\mathrm{MPa} \qquad A_2 = 0.1\mathrm{MPa} \qquad C_1 = 2000\mathrm{MPa}. \tag{22}$$

The region of rank-one-convexity is, unlike the other examples presented here, not representable by a single projection of the rank-one-convex boundary into the λ_2/λ_3–λ_1/λ_2–plane (Fig. 5(c)). Characteristic points for the loss of rank-one-convexity are eigenvalues very close to zero and $\lambda_{1,2,3} \approx 1.4$.

4 Failure of Rank-One-Convexity in Characteristic Tests

For completeness, we have examined five characteristic tests, namely

- simple shear: $\lambda_1 = \sqrt{1 + \gamma^2/2 - \gamma\sqrt{4+\gamma^2}/2}$, $\lambda_2 = 1$, $\lambda_3 = \sqrt{1 + \gamma^2/2 + \gamma\sqrt{4+\gamma^2}/2}$,
- uniaxial stress state: $\lambda_1 = u$, λ_2 and λ_3 from $w_2 = w_3 = 0$,
- isochoric tension/compression: $\lambda_1 = u,\ \lambda_2 = \lambda_3 = 1/\sqrt{u}$,
- simple tension/compression: $\lambda_1 = u$, $\lambda_2 = \lambda_3 = 1$,
- dilatation: $\lambda_1 = \lambda_2 = \lambda_3 = u$,

m	ν	Simple Shear	Uniax. Stress	Isochor. T/C	Simple T/C	Dilatation
-2	0.0	$\gamma < 0.496$	$0.436 < u < 1.29$	$0.657 < u < 1.289$	$0.436 < u < 1.29$	$0 < u < 1.29$
	0.1	$\gamma < 0.55$	$0.513 < u < 1.296$	$0.626 < u < 1.33$	$0.394 < u < 1.29$	$0 < u < 1.243$
	0.2	$\gamma < 0.641$	$0.569 < u < 1.31$	$0.578 < u < 1.403$	$0.351 < u < 1.286$	$0 < u < 1.201$
	0.3	$\gamma < 0.861$	$0.612 < u < 1.332$	$0.506 < u < 1.675$	$0.31 < u < 1.265$	$0 < u < 1.143$
	0.4	$\gamma < 1.29$	$0.646 < u < 1.364$	$0.432 < u < 2.337$	$0.271 < u < 1.185$	$0 < u < 1.069$
-1	0.0	$\gamma < 0.768$	$0.33 < u < 1.5$	$0.548 < u < 1.494$	$0.33 < u < 1.5$	$0 < u < 1.5$
	0.1	$\gamma < 0.827$	$0.39 < u < 1.511$	$0.529 < u < 1.559$	$0.302 < u < 1.5$	$0 < u < 1.409$
	0.2	$\gamma < 0.909$	$0.439 < u < 1.544$	$0.503 < u < 1.655$	$0.272 < u < 1.498$	$0 < u < 1.333$
	0.3	$\gamma < 1.043$	$0.48 < u < 1.595$	$0.462 < u < 1.837$	$0.239 < u < 1.478$	$0 < u < 1.269$
	0.4	$\gamma < 1.366$	$0.514 < u < 1.67$	$0.391 < u < 2.434$	$0.203 < u < 1.368$	$0 < u < 1.142$
0	0.0	$\gamma < 1.849$	$0.141 < u < 2.718$	$0.303 < u < 2.638$	$0.141 < u < 2.718$	$0 < u < 2.718$
	0.1	$\gamma < 1.941$	$0.165 < u < 2.78$	$0.294 < u < 2.829$	$0.133 < u < 2.718$	$0 < u < 2.266$
	0.2	$\gamma < 2.061$	$0.187 < u < 3.031$	$0.283 < u < 3.074$	$0.124 < u < 2.718$	$0 < u < 1.947$
	0.3	$\gamma < 2.226$	$0.204 < u < 3.469$	$0.268 < u < 3.411$	$0.114 < u < 2.718$	$0 < u < 1.713$
	0.4	$\gamma < 2.485$	$0.215 < u < 4.081$	$0.245 < u < 3.932$	$0.103 < u < 2.718$	$0 < u < 1.535$
1	0.0	$\gamma < \infty$	$0 < u < \infty$	$0 < u < 4.0$	$0 < u < \infty$	$0.501 < u < \infty$
	0.1	$\gamma < \infty$	$0 < u < 11.0$	$0 < u < \infty$	$0.112 < u < \infty$	$0.579 < u < \infty$
	0.2	$\gamma < \infty$	$0 < u < 5.999$	$0 < u < \infty$	$0.251 < u < \infty$	$0.667 < u < \infty$
	0.3	$\gamma < \infty$	$0 < u < 4.333$	$0 < u < \infty$	$0.429 < u < \infty$	$0.765 < u < \infty$
	0.4	$\gamma < \infty$	$0 < u < 3.5$	$0 < u < \infty$	$0.667 < u < \infty$	$0.876 < u < \infty$
2	0.0	$\gamma < 1.121$	$0.586 < u < \infty$	$0.578 < u < 2.0$	$0.586 < u < \infty$	$0.708 < u < \infty$
	0.1	$\gamma < \infty$	$0.592 < u < \infty$	$0.513 < u < \infty$	$0.603 < u < \infty$	$0.761 < u < \infty$
	0.2	$\gamma < \infty$	$0.594 < u < \infty$	$0 < u < \infty$	$0.639 < u < \infty$	$0.817 < u < \infty$
	0.3	$\gamma < \infty$	$0.593 < u < \infty$	$0 < u < \infty$	$0.706 < u < \infty$	$0.875 < u < \infty$
	0.4	$\gamma < \infty$	$0.588 < u < \infty$	$0 < u < \infty$	$0.825 < u < \infty$	$0.936 < u < \infty$
Blatz-Ko		$\gamma < \sqrt{2}$	$0.349 < u < 2.867$	$0.416 < u < 2.406$	$0.268 < u < 3.732$	$0 < u < \infty$

Table 1: Rank-one-convex ranges of characteristic tests for the Hookean strain energies and the reduced Blatz-Ko strain energy.

where u and γ are process parameters. Note that the eigenvalues in the list above are not ordered. Table 1 contains the rank-one-convex ranges for the characteristic tests for the Hookean strain energies and Blatz-Ko's reduced strain energy.

5 Summary

We examined the strain energies which are quadratic in the generalized Seth strain measures. It has been found that

- in all cases, in the incompressible limit, the rank-one-convex boundary approaches the undeformed state,
- for Seth strains with negative exponent and the logarithmic strains, a compressive volume change does not lead to the loss of rank-one-convexity,
- for Seth strains with positive exponent, a volume expansion does not lead to the loss of rank-one-convexity.

From the Hookean strain energies which have been examined, the one which is quadratic in the nominal strain ($m = 1$) displayed the largest rank-one-convex ranges in characteristic tests.

We also examined three compressible non-Hookean strain energies, namely Ciarlet-Geymonat's, Blatz-Ko's and Attard's strain energy.

We confirmed that the two-parameter Ciarlet-Geymonat strain energy has no rank-one-convex limit for positive Poisson's ratios. It is worth noting that Ciarlet-Geymonat's strain energy and the corresponding Cauchy stresses are given by $\boldsymbol{\sigma} = (\mu(\boldsymbol{B} - \boldsymbol{I}) + \lambda(J^2 - 1)/2\boldsymbol{I})/J$, i.e. no explicit calculation of $\boldsymbol{U}$ or $\boldsymbol{V}$ is required.

The reduced Blatz-Ko strain energy (with $\nu = 0.25$) fails to be rank-one-convex if and only if $\lambda_i/\lambda_j < 0.2688$, i.e., no rank-one-convex failure under purely dilatorical deformations is observed. In the characteristic tests, the rank-one-convex range of the Blatz-Ko strain energy turns out to be similar to those Hookean strain energies.

No specific conclusions could be drawn for Attard's strain energy, except for that for more complicated strain energies the regions of rank-one-convexity may have a complex shape in the space of the singular values of $\boldsymbol{F}$.

We observed that, out of the set of necessary and sufficient conditions that we used here, the eq. (8) are always weaker than the restrictions eq. (4, 5, 6) together, and thus had no impact for the strain energies under consideration. At most, this restriction coincided with one of the others. However, this is due to the specific strain energies under consideration, and does not hold in general.

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