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A Process for Stochastic Material Analysis based on Empirical Data

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Material properties are often dominated by imperfections and geometrical variations in micro-scale. The manufacturing process of complex parts as stringers and their assembly creates specific microscopic imperfections whose influence to phenomena like delamination growth can not be understood with a deterministic homogenised material model. This paper describes a general approach to develop a stochastic model of anisotropic micro-structure on the basis of high-resolution image data. This approach uses a surrogate model for approximating material properties of meso-scale material blocks. The empirical material properties provided by the surrogate model are analysed for their marginal distribution and spatial covariance.

1 Motivation

An uncertain material parameter may be discretised by a random field — a function defined on the tensor product of the underlying geometrical domain and the specified stochastic space. The most important work in modelling tensor-valued random fields is done in Soize (2006). The identification procedure proposed by Soize requires experiments and image data characterising the inner state of considered material under load. The objective of our work is to model the random field of fluctuations in material properties induced by microscopic fluctuation in the geometry of phases in a forward approach.

The difference to our approach is mainly visible in the direction of the analysis. While Soize is using a backward identification process, our approach uses a forward surrogate model. We describe abstractly a procedure to extract stochastic information of random fluctuations from high resolution cross-section images using a forward approach.

2 Stochastic Analysis Based on Random Fields

The goal of our approach is to determine random fields matching real-world circumstances. The number of parameters to describe the stochastic fluctuation in the material is therefore deduced from the analysis of given empirical material samples.

In our approach, we approximate the set of admissible material geometries inside a material block in a low-dimensional vector space representation and use a surrogate model to evaluate given geometry inside a block to its local fourth-order anisotropic stiffness tensor $c \in \mathbb{C}$, where $\mathbb{C}$ is an admissible subset of the set of real symmetric positive definite 6×6-matrices $\mathbb{M}_6^+(\mathbb{R})$, thus $\mathbb{C} \subseteq \mathbb{M}_6^+(\mathbb{R})$. A stochastic model for the spatial fluctuations of the material property $\mathbb{C}$ is then available and fluctuations can be derived from a given set of sample data without putting additional artificial assumptions. The resulting model for fluctuations is done in an appropriate low-dimensional subspace of the original data.

The approach uses an empirically determined non-linear mapping from normal distributed independent variables to the proper material. This provides a generator for the random material.

3 Abstract Description of the Proposed Procedure

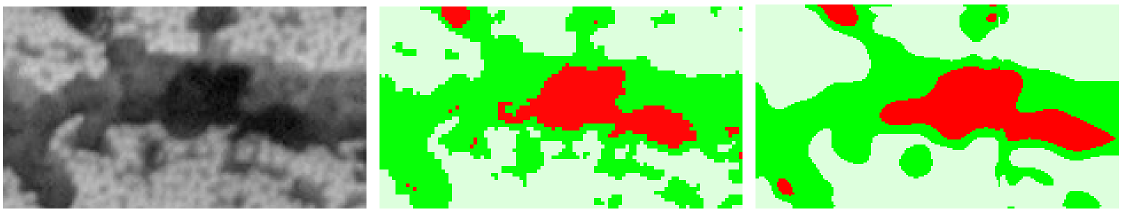
Generally speaking the approach takes as input high-quality image data, determines blockwise local material properties and analyses its stochastic moments, namely marginal distributions and spatial covariances. This information is used to obtain a generator for the material defined onto standard Gaussian variables. Therefor a sampling of the marginal distributions and an analysis of the covariances is required. The generator itself is already a complete tool, which may be applied e.g. in the Monte Carlo method or stochastic collocation methods described in Ganapathysubramanian and Zabaras (2007); Nobile et al. (2008); Blatman and Sudret (2008); Foo and Karniadakis (2010) to sample the material. In our work this generator is used to obtain a Karhunen-Loève expansion (KLE) — a spectral decomposition — of the material. As therefor a further sampling is required, the mentioned process identifies a bootstrapping. The KLE of the material supports a comfortable embedding in the spectral stochastic finite element method described in Ghanem and Spanos (2003); Bieri and Schwab (2009); Doostan et al. (2007); Matthies and Zander (2009); Nouy (2008); Krosche and Niekamp (2010) to approximate the solution of a stochastic partial differential equation involving the mentioned material.

4 Marginal Stochastic Model for Material Blocks

A marginal stochastic model is developed to describe the stochastic material properties of each material block. The development of this marginal material model is summarised in the following sections.

4.1 Deriving Vector-Space Representation of Image-Data

It is assumed that the geometry of the phase transition in multiphase material can be approximated by iso-surfaces of a polynomial expansion in $L^2([-1,1]^2)$. Each image is thereby used as a piecewise (pixel-wise) constant function I on a finite interval of size $\delta_x * n_x \times \delta_y * n_y$, where $\delta_{\{x,y\}}$ is the physical resolution and $n_{\{x,y\}}$ is the number of pixels of the given image. The resulting coefficient vector $g \in \mathbb{R}^{(n+1)*(m+1)} =: \mathbb{G}$ is now a spectral



(a) original image (b) discrete phases (c) L^2 approximation

Figure 1: Original image of a cross-section 1(a), its segmentation to discrete phases 1(b) and its L^2 approximation 1(c)

representation of I. Its quality is controlled by the polynomial order. With these steps a compression of the image-data to a small set of coefficients is obtained. Figure 1 shows a result of the application of the L^2 map for a given cross-section image. In this way it is possible to create a 3D model from a coherent sequence of consecutive cross-section images by interpolating the vector space representation of each image between two images.

4.2 A Surrogate Model for Mechanical Properties of Material Blocks

The basic idea of our surrogate model is to select representative realisations from the set of admissible realisations. We take the ansatz

$$H_I(\tilde{c}) = \prod_{i=1}^{k_{\mathbb{G}}} H_{I_i}(\tilde{c}_i),$$

where I is a $k_{\mathbb{G}}$-dimensional index-set adapted to the variance in each dimension, and H_{I_i} is the one-dimensional Hermite polynomial of order I_i, and $\tilde{c} \in \mathbb{G}_k$ is the centred geometrical coefficient vector $\tilde{c} := c - E\{c\}$, and c_i is the ith entry of the vector. $E\{c\}$ denotes the expected value of c.

Representative realisations of block-geometries are selected by using sample-points of Gauss-Hermite quadrature for the projection described above. This quadrature leads to a finite set of evaluations of the FE-Model. An example for the fluctuation of first Young's modulus determined with the surrogate model is given in figure 2.

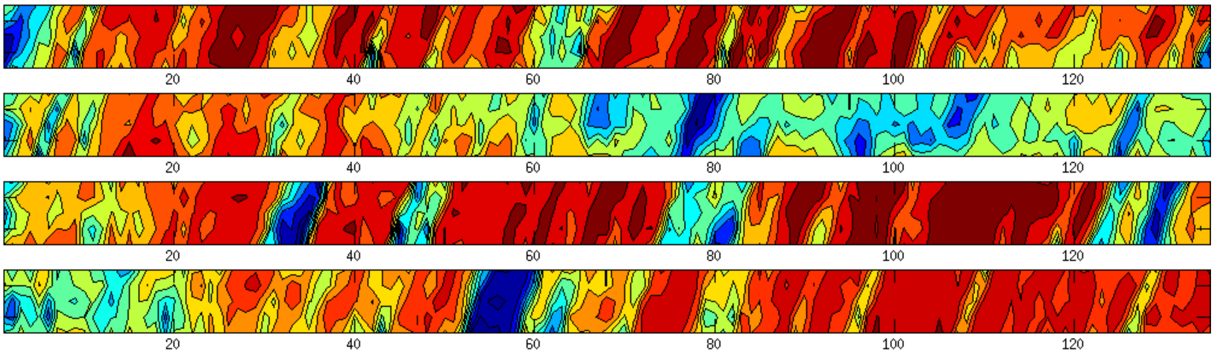


Figure 2: The figure shows four empirical material fields of a laminate layer determined using our forward surrogate model.

5 Developing the Model in Identical Independent Random Variables

The surrogate model of the previous section is now used to determine the normalised matrix logarithm of the material tensor for a given 3D image sample. The resulting data can now be analysed statistically and location-dependent statistical moments can be evaluated. To minimise the number of marginal variables, a small rank basis for the marginal variables is determined. In order to obtain statistically independent variables, the well-known Rosenblatt-Transformation described in Rosenblatt (1952) is applied.

As result it is now possible to model the samples of random material in only two marginal variables

$$x \mapsto rosenblatt(rank_2(\mathbb{C}^{log}(x))) \in \mathbb{R}^2,$$

which are identical independent normal Gaussian distributed. The mapping into this variables is completely invertible, so that fields generated in these independent variables can be mapped back to fields of the positive definite material tensor.

6 Conclusion and Outlook

The procedure described in this paper provides an approach to stochastic analysis of random fields based on empirical data. It can be used to directly model the stochastic nature of real material in a forward approach. Generally even non-homogeneous settings can be analysed with this method. This opens a large set of application in analysing the influence of voids in more complex specimens like stringers.

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