TECHNISCHE MECHANIK, 32, 2-5, (2012), 546 – 553
submitted: October 17, 2011

Defects and Dissipation in a Four-Dimensional Material Manifold

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The material manifold is well established as the underlying space of material forces. This three-dimensional manifold can be augmented by a time-like dimension to a four-dimensional material space-time manifold. The motion of a body is then represented by a mapping of the material space-time to the physical space-time of relativity theory. Following Kijowski et al. (1990), the time-like coordinate of material space-time can be related to temperature. This can be expressed in an invariant manner by a time-like vector field. The geometric structure of the material space-time for an elastic solid is completed by three space-like vector fields which describe the stress-free length between adjacent particles. Defects are represented by a non-vanishing spatial part of the Cartan torsion defined by the four vector fields (tetrad field). Since temperature is also involved in the tetrad field, it is coupled to the defects in the differential geometric structure of the material space-time. In a Lagrangian formulation the derivative of the Lagrangian with respect to the tetrad yields a four-dimensional second-order tensor in which entropy density, entropy current, and the Eshelby tensor are coupled like energy density, energy current, and stress in the energy-momentum tensor of relativity.

1 Introduction

The description of defects by differential geometric quantities in a material manifold has a long tradition and goes back to Kondo (see e. g. Kondo (1955)) and Bilby et al. (1955). Moving defects lead to geometric quantities changing with time, so there are fields depending on four variables, the three material coordinates and time. This has motivated the introduction of a four-dimensional material manifold with a fourth time-like dimension. Four-dimensional formulations can be found in Günther (1967), Edelen and Lagoudas (1988), Kienzler and Herrmann (2003) and Epstein et al. (2006). In all these papers the material time is in the end identified with physical time by constraints which are put on the field quantities. Epstein et al. (2006) call this constraint “time consistency condition” and put into question if it is really needed. None of these works consider temperature, so they (tacitly) assume isothermal conditions.

A relativistic formulation of continuum mechanics, including non-equilibrium thermodynamics, can be found in Lianis (1974). A three-dimensional material manifold is used in this work. Kijowski et al. (1990) use a four-dimensional material space-time in relativistic hydrodynamics. The derivative of the material time is proportional to temperature if the material coordinates are suitably chosen. In this approach material time has a physical interpretation not only depending on physical time. In later works (e. g. Kijowski and Magli (1997)) also elastic materials are taken into account. But eigenstresses (and so implicitly defects) are considered only as prescribed quantities and not as dynamical variables.

In this paper the representation of temperature by a time-like material coordinate and the differential-geometric description of defects is combined. In this way the geometric structure of the four-dimensional material manifold is determined by defects and temperature. Most of the formulas in this article will be written in index notation. This seems appropriate since two spaces are involved and accordingly many two-point tensors show up. Three kinds of indices (physical, material, and numbering indices) are present. Indices with a tilde take values $1, 2, 3$ while ordinary indices run from $0 \dots 3$. A comma before an index means partial differentiation with respect to the respective coordinate.

2 Physical Space-Time

Material bodies are observed in physical space. The deformation history of a body is the configuration of the body in physical space evolving in time. Physical space coordinates and time are combined in a four-dimensional space in the theory of relativity. Since the formulation presented here will deal with a four-dimensional material manifold it is natural to take the space-time of relativity on the physical side. The history of a body can then be represented by a mapping between the two four-dimensional manifolds.

Coordinates in the physical space-time are denoted by x^i, $i = 0, 1, 2, 3$. Latin indices in the range i, j, k, l refer to physical space-time coordinates. In this paper only special relativity will be considered. So the physical space-time is flat. One can choose an inertial frame with Cartesian coordinates (x, y, z), one has $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ where t is the physical time and c is the velocity of light. In this case the metric of physical space-time has the form:

$$\eta_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

The history of a particle is described by a curve (world line) in the four-dimensional physical space-time. The tangent vector u^i represents the particle's velocity as shown in figure (1). It is a time-like vector normalized to the velocity of light: $\eta_{ij} u^i u^j = -c^2$ and called four-velocity. The scalar product of a time-like vector with itself is negative. This property distinguishes them from space-like vectors. Four-velocities can only appear in the sector of time-like vectors marked in Figure (1) because the speed of a particle cannot exceed the velocity of light. If a particle is (at the moment) at rest in the given frame (then called momentary rest frame), the space-like components of the four-velocity are zero, i. e. it points in x^0-direction.

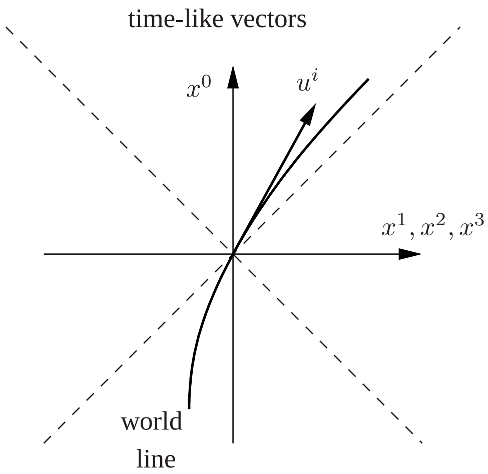

Figure 1: World line of a particle, two space dimensions are suppressed in the figure

3 Material Space-Time

The history of a body consists of the congruence of world lines of its particles. This congruence is often called a world tube. If the individual particles are indicated by space-like material coordinates $a^{\tilde{\alpha}}$, $\tilde{\alpha} = 1, 2, 3$ (Greek indices refer to material coordinates) and temperature is denoted by ϑ, the deformation and temperature history of a body can be described by the four functions

$$a^{\tilde{\alpha}}(x^i) \quad \vartheta(x^i) \tag{2}$$

An alternative description is given by Kijowski et al. (1990): An additional time-like material coordinate $a^0(x^i)$ is taken with

$$\beta \frac{\partial a^0}{\partial x^i} u^i = \vartheta(x^i) \tag{3}$$

where β is a constant. The temperature is proportional to the derivative of material time with respect to physical time in a rest frame ($u^i = (c, 0, 0, 0)$). There are now four material coordinates $a^\alpha = (a^0; a^{\tilde{\alpha}})$,$\alpha = 0, 1, 2, 3$ which can be taken as coordinates in a four-dimensional material manifold. The time-like coordinate serves as a potential function for the temperature.

The deformation and temperature history of the body can now be described by four invertible functions

$$a^\alpha(x^i) \quad \Leftrightarrow \quad x^i(a^\alpha) \quad . \tag{4}$$

These functions represent an invertible one-to-one mapping between the physical and the material space-time.

3.1 Time-like Material Vector

Equation (3) is not covariant since it is only valid in special coordinates, or more precisely, it defines these coordinates. A covariant formulation can be achieved by the introduction of a time-like material vector. To this end one takes the pull-back of four-velocity and temperature to the material manifold

$$u^\alpha = \frac{\partial a^\alpha}{\partial x^i} u^i \qquad \vartheta(a^\alpha) = \vartheta(x^i(a^\alpha)) \tag{5}$$

and sets

$$k_0^\alpha := \beta \frac{u^\alpha}{\vartheta} \quad . \tag{6}$$

In this equation only tensorial quantities are used and so it is valid in arbitrary coordinate systems. The lower index 0 is a number for the vector field. Three additional vector fields numbered $(1, 2, 3)$ will be introduced in the next subsection. It is easy to check that in coordinates with $k_0^\alpha = (1, 0, 0, 0)$ equation (3) is valid. It is also possible to transform to coordinates in which $x^0 = a^0$ is valid. In this case

$$k_0^\alpha = (\frac{\beta c}{\vartheta\sqrt{1-\frac{v^2}{c^2}}}; 0; 0; 0) \qquad x^0 = a^0 \quad , \tag{7}$$

where v is the absolute value of the particles three-velocity in the chosen frame. This vectorial representation of temperature is standard in relativistic thermodynamics.

The time-like material vector field not only serves to describe temperature, but is also necessary to identify Lagrangian coordinates in the material space-time. In the three-dimensional material space every coordinate system is Lagrangian by definition. In the four-dimensional material space-time transformations may also depend on the time-like coordinate and can lead to changing space-like coordinates for an individual particle. Coordinates are Lagrangian if and only if the spatial components of the time-like material vector field vanish. The presence of this field also indicates that there is no Lorentz symmetry in the material space-time. The Lorentz symmetry of the physical space-time means that there are no preferred time-like directions which in turn leads to the equivalence of all inertial frames. The time-like material vector defines a preferred direction (the rest frame of the particle) and so precludes Lorentz symmetry.

3.2 Space-like Material Vectors

To describe the structure (including defects) of the material a triad of three additional space-like material vectors is introduced. This approach is now standard in the continuum theory of dislocations and goes back to Kondo (1955) and Bilby et al. (1955). The spatial components (in Lagrangian coordinates) can be denoted by

$$k_{\tilde{r}}^{\tilde{\alpha}}(a^\lambda) \quad \tilde{r} = 1, 2, 3 \tag{8}$$

where $\tilde{r}$ is a numbering index. Indices in the range p, q, r, s, t are numbering indices for the material vectors. The components can be arranged in an invertible 3x3 matrix:

$$k_{\tilde{r}}^{\tilde{\alpha}} \quad h_{\tilde{\alpha}}^{\tilde{r}} \quad k_{\tilde{r}}^{\tilde{\alpha}} h_{\tilde{\beta}}^{\tilde{r}} = \delta_{\tilde{\beta}}^{\tilde{\alpha}} \quad h_{\tilde{\beta}}^{\tilde{r}} k_{\tilde{s}}^{\tilde{\beta}} = \delta_{\tilde{s}}^{\tilde{r}} \tag{9}$$

The mutual scalar products are considered as given by a positive definite and symmetric 3x3 matrix

$$\mathbf{k}_{\tilde{r}} \cdot \mathbf{k}_{\tilde{s}} = \overset{o}{C}_{\tilde{r}\tilde{s}} \quad . \tag{10}$$

These products define length and mutual angles in the stress-free state of the material at a fixed reference temperature and so include a first constitutive assumption by the existence of this state. In a materially uniform body the matrix $\overset{o}{C}_{\tilde{r}\tilde{s}}$ is constant. Defects (dislocations) are represented by the Cartan torsion of the material triad

$$S_{\tilde{\alpha}\tilde{\beta}}^{\tilde{r}} = h_{\tilde{\alpha},\tilde{\beta}}^{\tilde{r}} - h_{\tilde{\beta},\tilde{\alpha}}^{\tilde{r}} \quad . \tag{11}$$

The torsion vanishes if no dislocations are present.

3.3 Material Tetrad

Together with the time-like material vector the space-like triad constitutes a tetrad field in the material space-time. The components of the tetrad can be arranged in a 4x4 matrix and its inverse

$$k_r^\alpha \quad h_\alpha^r \quad k_r^\alpha h_\beta^r = \delta_\beta^\alpha \quad h_\beta^r k_s^\beta = \delta_s^r \quad . \tag{12}$$

In Lagrangian coordinates with $x^0 = a^0$ the components can be represented as

$$h_\alpha^r = \begin{pmatrix} V & V A_{\tilde{\alpha}} \\ 0^{\tilde{r}} & h_{\tilde{\alpha}}^{\tilde{r}} \end{pmatrix} \quad k_r^\alpha = \begin{pmatrix} 1/V & -A_{\tilde{\beta}} k_{\tilde{r}}^{\tilde{\beta}} \\ 0^{\tilde{\alpha}} & k_{\tilde{r}}^{\tilde{\alpha}} \end{pmatrix} \quad . \tag{13}$$

The zero components $0^{\tilde{r}}$ and $0^{\tilde{\alpha}}$ indicate the Lagrangian coordinates. The component $h_0^0 = V$ is related to temperature as shown in equation (7). The components involving $A_{\tilde{\alpha}}$ are new and will be discussed later.

The mutual scalar products are now extended to

$$\mathbf{k}_r \cdot \mathbf{k}_s = g_{rs} = \begin{pmatrix} -1 & 0 \\ 0 & \overset{o}{C}_{\tilde{r}\tilde{s}} \end{pmatrix} \quad . \tag{14}$$

This definition implies that the time-like material vector is considered orthogonal to the space-like vectors. The product of the time-like vector with itself is negative. This yields the distinction between time-like and space-like vectors similar to relativity. But again it is stressed that the form of the metric coefficients in equation (14) does not imply Lorentz symmetry since the geometry of the material manifold is governed by the tetrad field an not by the metric properties.

4 Metric Tensors

Two metric tensors can now be established in the material space-time. The first one is the pull-back of the physical metric to the material manifold

$$\eta_{\alpha\beta} = \eta_{ij} \frac{\partial x^i}{\partial a^\alpha} \frac{\partial x^j}{\partial a^\beta} \tag{15}$$

and the second one is defined by the material scalar products

$$g_{\alpha\beta} = g_{rs} h_\alpha^r h_\beta^s \quad . \tag{16}$$

The spatial length of material space-time coordinate differentials is calculated by taking the spatial part of the metric tensors. The unstressed length at reference temperature $\mathrm{d}\overset{o}{s}$ is determined by the material metric

$$(\mathrm{d}\overset{o}{s})^2 = \overset{o}{C}_{\alpha\beta} \mathrm{d}a^\alpha \mathrm{d}a^\beta \quad \text{with} \quad \overset{o}{C}_{\alpha\beta} = g_{\alpha\beta} + \frac{g_{\alpha\lambda} k_0^\lambda g_{\beta\mu} k_0^\mu}{-g_{\lambda\mu} k_0^\lambda k_0^\mu} \tag{17}$$

while the actual length is of course determined by the physical metric

$$(ds)^2 = C_{\alpha\beta} da^\alpha da^\beta \quad \text{with} \quad C_{\alpha\beta} = \eta_{\alpha\beta} + \frac{\eta_{\alpha\lambda} k_0^\lambda \eta_{\beta\mu} k_0^\mu}{-\eta_{\lambda\mu} k_0^\lambda k_0^\mu} . \tag{18}$$

$C_{\alpha\beta}$ is the well known relativistic equivalent of the Cauchy-Green tensor in relativistic elasticity (see e. g. Kijowski and Magli (1997)). The spatial parts of the metrics are determined by eliminating the contribution of the component in direction of the four-velocity (or equivalently the time-like material vector) of the squared vector.

5 Lagrangian Formulation for a Hyper-Elastic Solid

In the variational formulation of relativistic elasticity the Lagrangian function consists of the energy equivalent of mass density and the free energy density in the rest frame. If the material coordinates are taken as independent variables and the densities are referred to unstressed volume at reference temperature the Lagrangian can be expressed as

$$L_0(x^i; x^i_{,\alpha}; h_\alpha^r) = L_0(\eta_{ij} x^i_{,\alpha} x^j_{,\beta}; h_\alpha^r) = \sqrt{-g} \frac{\sqrt{-\eta_0}}{\sqrt{-g_0}} (\overset{o}{\rho} c^2 + \Psi) \tag{19}$$

where the abbreviations

$$g = \det(g_{\alpha\beta}) \quad g_0 = g_{\alpha\beta} k_0^\alpha k_0^\beta = -1 \quad \eta_0 = \eta_{\alpha\beta} k_0^\alpha k_0^\beta \tag{20}$$

are used. $\overset{o}{\rho}$ is the mass and Ψ the free energy per unit undeformed volume. The free energy depends on the strain relative to the unstressed state, so it depends on $h_{\tilde{\alpha}}^{\tilde{r}}$ and $C_{\alpha\beta}$. The dependence on temperature should involve the time-like components of the material vectors. To find a reasonable ansatz the structure of the material tetrad has to be further investigated. To simplify the following considerations components are written in Lagrangian coordinates and in a momentary rest frame with

$$\eta_{\tilde{\alpha}0} = 0 \quad \eta_{00} = -1 \quad \sqrt{-g}\frac{\sqrt{-\eta_0}}{\sqrt{-g_0}} = \sqrt{\det(\overset{o}{C}_{\tilde{\alpha}\tilde{\beta}})} = \sqrt{\overset{o}{C}} \tag{21}$$

The component representation of the tetrad is written down in equation (13). The component $h_0^0 = V = \vartheta/(\beta c)$ represents the temperature (cf. equation (7), the square root term does not appear because of the rest frame). The physical meaning of the components involving $A_{\tilde{\alpha}}$ is still open. In older works on four-dimensional material space-time (e. g. Epstein et al. (2006)) these components are constrained to zero. V is assumed constant which agrees with isothermal conditions. The following scalar invariants can be derived from the time-like components and the physical metric. Note that $\overset{-1}{\eta}{}^{\alpha\beta}\eta_{\beta\gamma} = \delta_\gamma^\alpha$.

$$\eta_0 = \eta_{\alpha\beta} k_0^\alpha k_0^\beta = -\frac{1}{V^2} = -\frac{\beta^2 c^2}{\vartheta^2} \tag{22}$$

and

$$\eta^0 = \overset{-1}{\eta}{}^{\alpha\beta} h_\alpha^0 h_\beta^0 = -V^2(1 + \overset{-1}{\eta}{}^{\tilde{\alpha}\tilde{\beta}} A_{\tilde{\alpha}} A_{\tilde{\beta}}) \quad . \tag{23}$$

The simplest choice would be to assume that the free energy depends on η_0, i. e. on temperature. A dependence on η^0 would be nearly the same if the assumption

$$\overset{-1}{\eta}{}^{\tilde{\alpha}\tilde{\beta}} A_{\tilde{\alpha}} A_{\tilde{\beta}} \ll 1 \tag{24}$$

is valid. It will turn out that the ansatz

$$\Psi = \Psi(h_{\tilde{\alpha}}^{\tilde{r}}; C_{\alpha\beta}; \eta^0) \tag{25}$$

leads to a reasonable description including heat flux. The assumption leads to

$$\frac{\partial \Psi}{\partial \eta^0} \approx -\frac{1}{2}\frac{\partial \Psi}{\partial \vartheta}\frac{\beta^2 c^2}{\vartheta} \quad . \tag{26}$$

A central quantity in relativity is the energy-momentum tensor $T^{\alpha\beta}$. It unifies energy density, energy flux (proportional to momentum density) and momentum flux (negative stress) in a four-dimensional second order tensor. In a Lagrangian formulation it shows up as the derivative of the Lagrangian with respect to the physical metric

$$T^{\alpha\beta} = -\frac{2}{\sqrt{\overset{o}{C}}}\frac{\partial L_0}{\partial \eta_{\alpha\beta}} \tag{27}$$

The purely time-like component T^{00} is the energy equivalent of mass plus the internal energy density. The Lagrangian used here yields

$$T^{00} \approx (\overset{o}{\rho} c^2 + \Psi - \frac{\partial \Psi}{\partial \vartheta}\vartheta) \quad . \tag{28}$$

Since the entropy density is $s = -\frac{\partial \Psi}{\partial \vartheta}$ the internal energy can be easily recognized. The purely space-like components yield the negative stress tensor:

$$T^{\tilde{\alpha}\tilde{\beta}} = -2\frac{\partial \Psi}{\partial C_{\tilde{\alpha}\tilde{\beta}}} \tag{29}$$

The mixed space- and time-like components of the energy-momentum tensor represent the energy current $q^{\tilde{\alpha}}$ divided by c. The only energy current occuring here is heat flux, since the components refer to Lagrangian coordinates and a rest frame.

$$T^{\tilde{\alpha}0} = \frac{q^{\tilde{\alpha}}}{c} \approx \frac{\partial \Psi}{\partial \vartheta}\vartheta \eta^{\tilde{\alpha}\tilde{\beta}} A_{\tilde{\beta}} \tag{30}$$

From these equations one can conclude that the quantities $A_{\tilde{\beta}}$ are related to heat flux and are of order $1/c$. This in turn yields that the left hand side of the assumption (24) is of order $1/c^2$ and so the assumption is justified. The physical interpretation of the quantities $A_{\tilde{\beta}}$ is now clear.

The components of the material vectors also appear as variables in the Lagrangian function and so the derivatives with respect to them should be investigated. First the derivatives with respect to the time-like material vector h^0_α are considered: The purely time-like component yields

$$\frac{\partial L_0}{\partial h^0_0} \approx \beta c\sqrt{-g}\frac{\sqrt{-\eta_0}}{\sqrt{-g_0}}\frac{\partial \Psi}{\partial \vartheta} \approx \beta c\sqrt{-g}\frac{\sqrt{-\eta_0}}{\sqrt{-g_0}}(-s) \tag{31}$$

and the mixed components are

$$\frac{\partial L_0}{\partial h^0_{\tilde\alpha}} \approx -\beta c\sqrt{-g}\frac{\sqrt{-\eta_0}}{\sqrt{-g_0}}\frac{\partial \Psi}{\partial \vartheta}\overset{-1}{\eta}{}^{\tilde\alpha\tilde\beta}A_{\tilde\beta} \approx -\beta\sqrt{-g}\frac{\sqrt{-\eta_0}}{\sqrt{-g_0}}\frac{q^{\tilde\alpha}}{\vartheta} \quad . \tag{32}$$

Together these form the components of a four-vector density. It is a density and not a regular vector because of the factor $\sqrt{-g}$. Besides the constant factor $-\beta$ the time-like component is the entropy density multiplied by c and the space-like components are the entropy current. The partial divergence of a vector density is a scalar density (see e. g. Lovelock and Rund (1975)) and so a tensorial quantity. Here it reads (notice that $a^0 = x^0 = ct$)

$$\partial_\alpha\left(\frac{\partial L_0}{\partial h^0_\alpha}\right) = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial h^0_0}\right) + \partial_{\tilde\alpha}\left(\frac{\partial L_0}{\partial h^0_{\tilde\alpha}}\right) \quad . \tag{33}$$

and represents the (negative) entropy production. The derivative of the Lagrangian with respect to the time-like material vector has thus a clear physical meaning as the four-current of entropy.

Next the derivatives with respect to the space-like components are calculated. First it could be stated that

$$\frac{\partial L_0}{\partial h^{\tilde s}_0} = k^\mu_{\tilde s}\eta_{\lambda\mu}\frac{\partial L_0}{\partial h^0_\lambda}h^0_0\overset{-1}{\eta}{}^{00} \quad , \tag{34}$$

so no essentially new information could be expected. The remaining derivatives yield

$$\frac{\partial L_0}{\partial h^{\tilde r}_{\tilde\alpha}} = \sqrt{-g}\frac{\sqrt{-\eta_0}}{\sqrt{-g_0}}\left[\overset{o}{\rho}c^2k^{\tilde\alpha}_{\tilde r} + \Psi k^{\tilde\alpha}_{\tilde r} - 2\frac{\partial \Psi}{\partial C_{\tilde\alpha\tilde\beta}}\eta_{\tilde\beta\tilde\gamma}k^{\tilde\gamma}_{\tilde r}\right] \quad . \tag{35}$$

The last two terms in the square brackets can be recognized as the Eshelby tensor. The first term is due to the energy equivalent of mass in relativity. The factor $\overset{o}{\rho}c^2$ is constant in a materially uniform body.

The derivatives contain one coordinate index and one numbering index, so they constitute four fields of vector densities. A second order tensor density can be established by

$$\frac{\partial L_0}{\partial h^r_\alpha}h^r_\beta \quad . \tag{36}$$

In this tensor entropy, entropy current, and the Eshelby tensor are coupled, similarly as energy, energy current and stress are coupled in the energy-momentum tensor.

6 Invariance Identities

The Lagrangian function $L_0 = L_0(\eta_{\alpha\beta}; h^r_\alpha)$ is a scalar density depending on tensorial variables. As described in Lovelock and Rund (1975), the transformation laws of the Lagrangian and its arguments lead to invariance identities. Following the procedure given in Lovelock and Rund (1975), the identities

$$L_0\delta^\alpha_\beta = \frac{\partial L_0}{\partial h^r_\alpha}h^r_\beta + 2\frac{\partial L_0}{\partial \eta_{\alpha\gamma}}\eta_{\gamma\beta} \tag{37}$$

$$-\left[\partial_\alpha\left(\frac{\partial L_0}{\partial x^i_{,\alpha}}\right) - \frac{\partial L_0}{\partial x^i}\right]x^i_{,\gamma} = -2\overset{\eta}{\nabla}_\alpha\left(\frac{\partial L_0}{\partial \eta_{\alpha\beta}}\eta_{\beta\gamma}\right) \tag{38}$$

$$-2\overset{\eta}{\nabla}_\alpha\left(\frac{\partial L_0}{\partial \eta_{\alpha\beta}}\eta_{\beta\gamma}\right) = \partial_\alpha\left(\frac{\partial L_0}{\partial h^r_\alpha}\right)h^r_\gamma + \frac{\partial L_0}{\partial h^r_\alpha}(h^r_{\gamma,\alpha} - h^r_{\alpha,\gamma}) \tag{39}$$

can be derived. The symbol $\overset{\eta}{\nabla}_\alpha$ denotes the covariant derivative with respect to the physical metric while ∂_α is the partial derivative.

7 Conservation and Balance Laws

The conservation of energy and momentum in relativity is expressed by the vanishing divergence of the energy-momentum tensor:

$$\overset{\eta}{\nabla}_\alpha \left(\frac{\partial L_0}{\partial \eta_{\alpha\beta}} \eta_{\beta\gamma} \right) = 0 \quad . \tag{40}$$

The invariance identity (38) shows, that this is equivalent to the Euler equations for the variation of $x^i(a^\alpha)$ for an action integral based on the Lagrangian L_0:

$$\partial_\alpha \left(\frac{\partial L_0}{\partial x^i_{,\alpha}} \right) - \frac{\partial L_0}{\partial x^i} = 0 \tag{41}$$

The invariance identity (39) yields as equivalent to the conservation of energy and momentum:

$$\partial_\alpha \left(\frac{\partial L_0}{\partial h^r_\alpha} \right) h^r_\gamma + \frac{\partial L_0}{\partial h^r_\alpha} (h^r_{\gamma,\alpha} - h^r_{\alpha,\gamma}) = 0 \quad . \tag{42}$$

After multiplying by k^γ_s this can be split up in

$$-\partial_\alpha \left(\frac{\partial L_0}{\partial h^0_\alpha} \right) = \frac{\partial L_0}{\partial h^r_\alpha} (h^r_{\gamma,\alpha} - h^r_{\alpha,\gamma}) k^\gamma_0 \tag{43}$$

and

$$\partial_\alpha \left(\frac{\partial L_0}{\partial h^{\tilde{s}}_\alpha} \right) = -\frac{\partial L_0}{\partial h^r_\alpha} (h^r_{\gamma,\alpha} - h^r_{\alpha,\gamma}) k^\gamma_{\tilde{s}} \quad . \tag{44}$$

Equation (43) gives an expression for the entropy production, so the second law of thermodynamics demands that the it must be non-negative. Equation (44) states a source term for the (modified) Eshelby tensor (see equation (34) and (35)). It is easily recognized that the divergence vanishes if no temperature gradient, no heat flux, and no defects are present.

8 Lagrangian for Defects and Heat Conduction

So far only the Lagrangian function $L_0 = L_0(\eta_{\alpha\beta}; h^r_\alpha)$ has been considered. If the material tetrad is taken as an additional dependent variable, the complete Lagrangian must depend on (at least first) derivatives of these quantities. One ansatz could be

$$L = L_0 + L_1 \quad L_1 = L_1(h^r_\alpha; h^r_{\alpha,\beta}) \quad \frac{\partial L_1}{\partial \eta_{\alpha\beta}} = 0 \tag{45}$$

In this case the conservation of energy and momentum will remain unchanged since the additional Lagrangian does not depend on $\eta_{\alpha\beta}$ (and so not on x^i and $x^i_{,\alpha}$). It can be derived from invariance identities that the Euler equations for the variation of x^i are the integrability conditions for the Euler equations arrising from the variation of h^r_α. This is similar to the situation in general relativity, where the divergence of the Einstein tensor vanishes identically.

The ansatz (45) has a serious drawback, since it admits no solutions with h^r_α =const. which would describe a isothermal state with no defects and homogeneous stress/stain. A possible remedy could be the introduction of an additional scalar potential and/or the use of Null Lagrangians as in Edelen and Lagoudas (1988). The development of a reasonable Lagrangian is a matter of future work.

9 Conclusions

In this paper the well established space-like material vector triad in material space is augmented by a fourth time-like vector in a four-dimensional material space-time. Together, the four vectors constitute a tetrad field which has of course more components than the original triad. A part of the new components serves to identify Lagrangian coordinates in the material space-time, the rest is related to temperature and heat flux. In this way a fundamental coupling between defects and thermodynamic quantities is established because temperature and heat flux also become part of the geometry of material space-time. Since the motion of defects in a material is a dissipative process this coupling seems very plausible.

The derivatives of the Lagrangian of an elastic solid with respect to the material tetrad can be arranged in a second order tensor density in which entropy, entropy current, and the Eshelby tensor are coupled, similarly as energy, energy current and stress are coupled in the energy-momentum tensor. Invariance identities lead to expressions for entropy production and for source terms for the Eshelby tensor. These expressions are valid if and only if the conservation of energy and momentum is fulfilled.

For a complete theory an additional Lagrangian describing the defects is necessary. A fully satisfactory ansatz has not been found yet, but looking for it seems to be a worthy task. It could lead to a theory in which the dissipative character of defect motion is a priori incorporated.

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