

Potential Method in the Linear Theories of Viscoelasticity and Thermoviscoelasticity for Kelvin-Voigt Materials

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In this paper the basic boundary value problems (BVPs) of steady vibrations in the linear theories of viscoelasticity and thermoviscoelasticity for Kelvin-Voigt materials are considered and some basic results of the classical theories of elasticity and thermoelasticity are generalized. The fundamental solutions of systems of equations of steady vibrations are constructed. The radiation conditions and basic properties of fundamental solutions are established. The properties of potentials of single-layer, double-layer and volume are given. The uniqueness theorems of the internal and external basic BVPs are established. Finally, the existence theorems for the internal and external basic BVPs are proved by means of the potential method and the theory of singular integral equations.

1 Introduction

Viscoelastic materials play an important role in many branches of engineering, technology and, in recent years, biomechanics (see Lakes, 2009). Viscoelastic materials, such as amorphous polymers, semicrystalline polymers, and biopolymers, can be modeled in order to determine their stress or strain interactions as well as their temporal dependencies. The study of viscoelastic behavior in bone is of interest in several contexts. Bone is hierarchical solid that contains structure at multiple length scales. Study of bone viscoelasticity is best placed in the context of strain levels and frequency components associated with normal activities and with applications of diagnostic tools (for details, see Lakes, 2009). The investigations of the solutions of viscoelastic wave equations, velocities of seismic wave propagating and the attenuation of seismic wave in the viscoelastic media are very important for geophysical prospecting technology.

The theories of viscoelasticity, which include the Maxwell model, the Kelvin-Voigt model, and the Standard Linear Solid model, are used to predict a material's response under different loading conditions. One of the simplest mathematical models constructed to describe the viscoelastic effects is the classical Kelvin-Voigt model (see Eringen, 1980). The basic idea concerning this model is that the stress is dependent on the deformation tensor and deformation-rate tensor. This model consists of a Newtonian damper and Hooke's elastic spring connected in parallel.

The modern theories of viscoelasticity and thermoviscoelasticity for materials with microstructure have become a subject of intensive study in recent years. The classical Kelvin-Voigt model by using a mixture consisting of a porous elastic solid and viscous fluid is generalized by Iesan (2004). The linear theory of porous thermoviscoelastic mixtures has been presented by Iesan and Quintanilla (2007). The linear theories of viscoelasticity and thermoviscoelasticity of binary mixtures where the individual components are modeled as Kelvin-Voigt viscoelastic materials are developed by Quintanilla (2005) and Iesan and Nappa (2008).

In this paper the basic BVPs of steady vibrations in the linear theories of viscoelasticity and thermoviscoelasticity for Kelvin-Voigt materials are considered and some basic results of the classical theories of elasticity and thermoelasticity (see Kupradze et al., 1979) are generalized. The fundamental solutions of systems of equations of steady vibrations are constructed. The radiation conditions and basic properties of fundamental solutions are established. The properties of potentials of single-layer, double-layer and volume are given. The uniqueness theorems of the internal and external basic BVPs are established. Finally, the existence theorems for the internal and external basic BVPs are proved by means of the potential method and the theory of singular integral equations.

The potential method makes it possible to investigate three-dimensional problems not only of classical theory of elasticity, but also problems of the theory of elasticity with conjugated fields. The main results obtained in this

area are presented in Kupradze et al. (1979). An extensive review of works on the potential method can be found in Gegelia and Jentsch (1994).

2 Basic Equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space R^3 and $\mathbf{D}_{\mathbf{x}} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$. We consider an isotropic homogeneous viscoelastic Kelvin-Voigt material that occupies the region Ω of R^3 . We assume that the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate and repeated indices are summed over the range (1,2,3).

The fundamental system of field equations in the linear theory of thermoviscoelasticity for Kelvin-Voigt material consists of the equations of steady vibrations (see Eringen, 1980)

$$t_{lj,j} = -\rho(\omega^2 u_l + F'_l), \quad (1)$$

the equation of balance of energy

$$-i\omega T_0 \rho \eta = q_{l,l} + \rho F_0, \quad (2)$$

the constitutive equations

$$\begin{aligned} t_{lj} &= \lambda_1 e_{rr} \delta_{lj} + 2\mu_1 e_{lj} - \gamma \theta \delta_{lj}, \\ \rho \eta &= \gamma e_{rr} + a \theta, \\ q_l &= k \theta_{,l}, \end{aligned} \quad (3)$$

and the geometrical equations

$$e_{lj} = \frac{1}{2}(u_{l,j} + u_{j,l}), \quad (4)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, θ is the deviation from a constant reference temperature $T_0 > 0$, t_{lj} and e_{lj} are the components of stress and strain tensors, respectively ($l, j = 1, 2, 3$); $\mathbf{F}' = (F'_1, F'_2, F'_3)$ is the body force, F_0 is the heat supply, η is the entropy per unit mass, q_l is the heat flux, ρ is the reference mass density ($\rho > 0$), ω is the oscillation frequency ($\omega > 0$),

$$\lambda_1 = \lambda - i\omega\lambda^*, \quad \mu_1 = \mu - i\omega\mu^*.$$

Here λ , μ and λ^* , μ^* are the Lamé and viscosity constants, respectively; a , k and γ are the constitutive coefficients.

Substituting equations (3) and (4) into (1) and (2), we obtain the following system of steady vibrations equations of the linear theory of thermoviscoelasticity for Kelvin-Voigt material expressed in terms of the displacement vector \mathbf{u} and the temperature θ

$$\begin{aligned} \mu_1 \Delta \mathbf{u} + (\lambda_1 + \mu_1) \text{grad div } \mathbf{u} - \gamma \text{grad } \theta + \rho \omega^2 \mathbf{u} &= -\rho \mathbf{F}', \\ (k\Delta + a_0) \theta + \gamma_0 \text{div } \mathbf{u} &= -\rho F_0, \end{aligned} \quad (5)$$

where Δ is the Laplacian, $a_0 = i\omega a$, $\gamma_0 = i\omega \gamma T_0$. Obviously, (5) is a system of PDEs with complex coefficients containing 10 real parameters.

We assume that

$$\mu^* > 0, \quad 3\lambda^* + 2\mu^* > 0, \quad k > 0, \quad a > 0. \quad (6)$$

In the isothermal case from (5) we obtain the system of equations of steady vibrations in the linear theory of viscoelasticity for Kelvin-Voigt material (see Eringen, 1980)

$$\mu_1 \Delta \mathbf{u} + (\lambda_1 + \mu_1) \text{grad div } \mathbf{u} + \rho \omega^2 \mathbf{u} = -\rho \mathbf{F}'. \quad (7)$$

We introduce the matrix differential operators

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}}) = (A_{pq}(\mathbf{D}_{\mathbf{x}}))_{4 \times 4}, \quad \mathbf{A}^{(1)}(\mathbf{D}_{\mathbf{x}}) = (A_{lj}^{(1)}(\mathbf{D}_{\mathbf{x}}))_{3 \times 3},$$

where

$$A_{lj}(\mathbf{D}_\mathbf{x}) = A_{lj}^{(1)}(\mathbf{D}_\mathbf{x}) = (\mu_1 \Delta + \rho \omega^2) \delta_{lj} + (\lambda_1 + \mu_1) \frac{\partial^2}{\partial x_l \partial x_j},$$

$$A_{l4}(\mathbf{D}_\mathbf{x}) = -\gamma \frac{\partial}{\partial x_l}, \quad A_{4l}(\mathbf{D}_\mathbf{x}) = \gamma_0 \frac{\partial}{\partial x_l}, \quad A_{44}(\mathbf{D}_\mathbf{x}) = k\Delta + a_0, \quad l, j = 1, 2, 3$$

and δ_{lj} is the Kronecker delta. The systems (5) and (7) can be written as

$$\mathbf{A}(\mathbf{D}_\mathbf{x}) \mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \quad (8)$$

and

$$\mathbf{A}^{(1)}(\mathbf{D}_\mathbf{x}) \mathbf{u}(\mathbf{x}) = -\rho \mathbf{F}'(\mathbf{x}),$$

respectively, where $\mathbf{U} = (\mathbf{u}, \theta)$, $\mathbf{F} = (-\rho \mathbf{F}', -\rho F_0)$ and $\mathbf{x} \in \Omega$.

3 Fundamental Solutions

In Kupradze and Burchuladze (1969), the fundamental solution of equations of steady vibrations in the classical theory of thermoelasticity has been presented. In this section the fundamental solution of the system of equations of steady vibrations in the linear theory of thermoviscoelasticity is constructed using a simple method like in the classical theory of thermoelasticity.

Definition 1. The fundamental solution of the system (5) (the fundamental matrix of operator \mathbf{A}) is the matrix $\Gamma(\mathbf{x}) = (\Gamma_{pq}(\mathbf{x}))_{4 \times 4}$ satisfying (in the class of generalized functions) the following equation

$$\mathbf{A}(\mathbf{D}_\mathbf{x}) \Gamma(\mathbf{x}) = \delta(\mathbf{x}) \mathbf{J}, \quad (9)$$

where δ is the Dirac delta, $\mathbf{J} = (\delta_{pq})_{4 \times 4}$ is the unit matrix, and $\mathbf{x} \in R^3$.

In this section the matrix $\Gamma(\mathbf{x})$ is constructed in terms of elementary functions and some basic properties are established.

We consider the system of nonhomogeneous equations

$$\begin{aligned} \mu_1 \Delta \mathbf{u} + (\lambda_1 + \mu_1) \text{grad div } \mathbf{u} + \gamma_0 \text{grad } \theta + \rho \omega^2 \mathbf{u} &= \mathbf{G}', \\ (k\Delta + a_0) \theta - \gamma \text{div } \mathbf{u} &= G_0, \end{aligned} \quad (10)$$

where \mathbf{G}' is three-component vector function and G_0 is scalar function on R^3 . As one may easily verify, system (10) may be written in the form

$$\mathbf{A}^T(\mathbf{D}_\mathbf{x}) \mathbf{U}(\mathbf{x}) = \mathbf{G}(\mathbf{x}), \quad (11)$$

where \mathbf{A}^T is the transpose of matrix \mathbf{A} , $\mathbf{G} = (\mathbf{G}', G_0)$ is four-component vector function on R^3 and $\mathbf{x} \in R^3$. Applying the operator div to Eqs. (10)₁ from system (10) we obtain

$$\begin{aligned} \mu_0 \Delta \text{div } \mathbf{u} + \gamma_0 \Delta \theta &= \text{div } \mathbf{G}', \\ (k\Delta + a_0) \theta - \gamma \text{div } \mathbf{u} &= G_0, \end{aligned} \quad (12)$$

where $\mu_0 = \lambda_1 + 2\mu_1$. The system (12) implies

$$\Lambda_1(\Delta) \text{div } \mathbf{u} = \Phi_1, \quad \Lambda_1(\Delta) \theta = \Phi_2, \quad (13)$$

where $\Lambda_1(\Delta) = (\Delta + \tau_1^2)(\Delta + \tau_2^2)$; τ_1^2 and τ_2^2 are the roots of equation (with respect to τ)

$$(\mu_0 \tau - \rho \omega^2)(k\tau - a_0) - \gamma \gamma_0 \tau = 0$$

and

$$\begin{aligned} \Phi_1 &= \frac{1}{k\mu_0} [(k\Delta + a_0) \text{div } \mathbf{G}' - \gamma_0 \Delta G_0], \\ \Phi_2 &= \frac{1}{k\mu_0} [\gamma \text{div } \mathbf{G}' + (\mu_0 \Delta + \rho \omega^2) G_0]. \end{aligned} \quad (14)$$

Applying the operator $\Lambda_1(\Delta)$ to (10)₁ and taking into account (13), we obtain

$$\Lambda_2(\Delta) \mathbf{u} = \Phi, \quad (15)$$

where $\Lambda_2(\Delta) = \Lambda_1(\Delta)(\Delta + \tau_3^2)$, $\tau_3^2 = \frac{\rho\omega^2}{\mu_1}$ and

$$\Phi = \frac{1}{\mu_1} [\Lambda_1(\Delta) \mathbf{G}' - (\lambda_1 + \mu_1) \text{grad } \Phi_1 - \gamma_0 \text{grad } \Phi_2]. \quad (16)$$

On the basis of (13) and (15) we get

$$\Lambda(\Delta) \mathbf{U}(\mathbf{x}) = \tilde{\Phi}(\mathbf{x}), \quad (17)$$

where $\tilde{\Phi} = (\Phi, \Phi_2)$ is the four-component vector and $\Lambda(\Delta)$ is the following diagonal matrix

$$\begin{aligned} \Lambda(\Delta) &= (\Lambda_{pq}(\Delta))_{4 \times 4}, & \Lambda_{11}(\Delta) &= \Lambda_{22}(\Delta) = \Lambda_{33}(\Delta) = \Lambda_2(\Delta), \\ \Lambda_{44}(\Delta) &= \Lambda_1(\Delta), & \Lambda_{pq}(\Delta) &= 0, \quad p, q = 1, 2, 3, 4, \quad p \neq q. \end{aligned}$$

In view of (14) and (16) the vector $\tilde{\Phi}$ we can written in the form

$$\tilde{\Phi}(\mathbf{x}) = \mathbf{L}^T(\mathbf{D}_\mathbf{x}) \mathbf{G}(\mathbf{x}), \quad (18)$$

where

$$\begin{aligned} \mathbf{L} &= (L_{pq})_{4 \times 4}, & L_{lj}(\mathbf{D}_\mathbf{x}) &= \frac{1}{k\mu_0\mu_1} \left\{ k\mu_0 \Lambda_1(\Delta) \delta_{lj} - [(\lambda_1 + \mu_1)(k\Delta + a_0) + \gamma\gamma_0] \frac{\partial^2}{\partial x_l \partial x_j} \right\}, \\ L_{l4}(\mathbf{D}_\mathbf{x}) &= \frac{\gamma}{k\mu_0} \frac{\partial}{\partial x_l}, & L_{4l}(\mathbf{D}_\mathbf{x}) &= -\frac{\gamma_0}{k\mu_0\mu_1} (\mu_1\Delta + \rho\omega^2) \frac{\partial}{\partial x_l}, \\ L_{44}(\mathbf{D}_\mathbf{x}) &= \frac{1}{k\mu_0} (\mu_0\Delta + \rho\omega^2), & l, j &= 1, 2, 3. \end{aligned} \quad (19)$$

By virtue of (11) and (18) from (17) it follows that $\Lambda \mathbf{U} = \mathbf{L}^T \mathbf{A}^T \mathbf{U}$. It is obvious that $\mathbf{L}^T \mathbf{A}^T = \Lambda$ and, hence,

$$\mathbf{A}(\mathbf{D}_\mathbf{x}) \mathbf{L}(\mathbf{D}_\mathbf{x}) = \Lambda(\Delta). \quad (20)$$

We assume that $\tau_1^2 \neq \tau_2^2 \neq \tau_3^2 \neq \tau_1^2$. Let

$$\mathbf{Y}(\mathbf{x}) = (Y_{pq}(\mathbf{x}))_{4 \times 4}, \quad Y_{11}(\mathbf{x}) = Y_{22}(\mathbf{x}) = Y_{33}(\mathbf{x}) = \sum_{j=1}^3 \eta_j h_j(\mathbf{x}), \quad (21)$$

$$Y_{44}(\mathbf{x}) = \frac{1}{\tau_2^2 - \tau_1^2} [h_1(\mathbf{x}) - h_2(\mathbf{x})], \quad Y_{pq}(\mathbf{x}) = 0, \quad p, q = 1, 2, 3, 4, \quad p \neq q,$$

where

$$h_j(\mathbf{x}) = -\frac{1}{4\pi |\mathbf{x}|} e^{i\tau_j |\mathbf{x}|}, \quad \eta_j = \prod_{l=1, l \neq j}^3 (\tau_l^2 - \tau_j^2)^{-1}, \quad j = 1, 2, 3.$$

It is easily to see that \mathbf{Y} is the fundamental matrix of operator $\Lambda(\Delta)$, that is

$$\Lambda(\Delta) \mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x}) \mathbf{J}. \quad (22)$$

We introduce the matrix

$$\Gamma(\mathbf{x}) = \mathbf{L}(\mathbf{D}_\mathbf{x}) \mathbf{Y}(\mathbf{x}). \quad (23)$$

Using identity (20) from (22) and (23) we obtain

$$\mathbf{A}(\mathbf{D}_\mathbf{x}) \Gamma(\mathbf{x}) = \mathbf{A}(\mathbf{D}_\mathbf{x}) \mathbf{L}(\mathbf{D}_\mathbf{x}) \mathbf{Y}(\mathbf{x}) = \Lambda(\Delta) \mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x}) \mathbf{J}.$$

Hence, the matrix $\Gamma(\mathbf{x})$ is a solution to (9) and is constructed by elementary functions h_1, h_2 and h_3 . We have thereby proved the following theorem.

Theorem 1. The matrix $\Gamma(\mathbf{x})$ defined by (23) is the fundamental solution of system (5), where $\mathbf{L}(\mathbf{D}_{\mathbf{x}})$ and $\mathbf{Y}(\mathbf{x})$ are given by (19) and (21), respectively.

In the follows we assume that $\text{Im } \tau_j > 0$, $j = 1, 2, 3, 4$. The theorem 1 leads to the following results.

Theorem 2. Each column of the matrix $\Gamma(\mathbf{x})$ is a solution of system (5) at every point $\mathbf{x} \in R^3$ except the origin.

Theorem 3. The relations

$$\Gamma_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad \frac{\partial^m}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \Gamma_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1-m})$$

hold in a neighborhood of the origin, where $l, j = 1, 2, 3, 4$, $m = m_1 + m_2 + m_3$, $m \geq 1$.

Theorem 4. The relations

$$\Gamma_{lj}(\mathbf{x}) = e^{-\tau_0|\mathbf{x}|} O(|\mathbf{x}|^{-1})$$

hold in a neighborhood of the infinity ($|\mathbf{x}| \gg 1$), where $\tau_0 = \min \{\text{Im } \tau_j, j = 1, 2, 3, 4\} > 0$, $l, j = 1, 2, 3, 4$.

Quite similarly we can construct the fundamental solution of system (7) (fundamental matrix of operator $\mathbf{A}^{(1)}$) and establish basic properties. We have the following results.

Theorem 5. The matrix $\Gamma^{(1)}(\mathbf{x})$ defined by

$$\Gamma^{(1)}(\mathbf{x}) = \left(\Gamma_{lj}^{(1)}(\mathbf{x}) \right)_{3 \times 3},$$

$$\Gamma_{lj}^{(1)}(\mathbf{x}) = \frac{1}{\mu_1(\tau_4^2 - \tau_3^2)} \left[(\Delta + \tau_4^2) \delta_{lj} - \frac{\lambda_1 + \mu_1}{\mu_0} \frac{\partial^2}{\partial x_1 \partial x_j} \right] [h_3(\mathbf{x}) - h_4(\mathbf{x})]$$

is the fundamental solution of system (7), where

$$h_4(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|} e^{i\tau_4|\mathbf{x}|}, \quad \tau_4^2 = \frac{\rho\omega^2}{\mu_0}.$$

Theorem 6. Each column of the matrix $\Gamma^{(1)}(\mathbf{x})$ is a solution of system (7) at every point $\mathbf{x} \in R^3$ except the origin.

Theorem 7. The relations

$$\Gamma_{lj}^{(1)}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad \frac{\partial^m}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \Gamma_{lj}^{(1)}(\mathbf{x}) = O(|\mathbf{x}|^{-1-m})$$

hold in a neighborhood of the origin, where $l, j = 1, 2, 3$, $m = m_1 + m_2 + m_3$, $m \geq 1$.

Theorem 8. The relations

$$\Gamma_{lj}^{(1)}(\mathbf{x}) = e^{-\tau_0^{(1)}|\mathbf{x}|} O(|\mathbf{x}|^{-1})$$

hold in a neighborhood of the infinity ($|\mathbf{x}| \gg 1$), where $\text{Im } \tau_4 > 0$, $\tau_0^{(1)} = \min \{\text{Im } \tau_3, \text{Im } \tau_4\} > 0$, $l, j = 1, 2, 3, 4$.

4 Boundary Value Problems. Uniqueness Theorems

Let S be the closed surface surrounding the finite domain Ω^+ in R^3 , $S \in C^{2,\nu}$, $0 < \nu \leq 1$, $\bar{\Omega}^+ = \Omega^+ \cup S$, $\Omega^- = R^3 \setminus \bar{\Omega}^+$.

Definition 2. A vector function $\mathbf{U} = (U_1, U_2, U_3, U_4)$ is called regular in Ω^- (or Ω^+) if

1)

$$U_l \in C^2(\Omega^-) \cap C^1(\bar{\Omega}^-) \quad (\text{or } U_l \in C^2(\Omega^+) \cap C^1(\bar{\Omega}^+)),$$

2)

$$\mathbf{U} = \sum_{j=1}^3 \mathbf{U}^{(j)}, \quad \mathbf{U}^{(j)} = (U_1^{(j)}, U_2^{(j)}, U_2^{(j)}, U_4^{(j)}),$$

$$U_l^{(j)} \in C^2(\Omega^-) \cap C^1(\bar{\Omega}^-),$$

3)

$$(\Delta + \tau_j^2)U_l^{(j)}(\mathbf{x}) = 0$$

and

$$\left(\frac{\partial}{\partial|\mathbf{x}|} - i\tau_j\right)U_l^{(j)}(\mathbf{x}) = e^{i\tau_j|\mathbf{x}|}o(|\mathbf{x}|^{-1}) \quad \text{for} \quad |\mathbf{x}| \gg 1, \quad (24)$$

where $U_4^{(3)} = 0$ and $j = 1, 2, 3$, $l = 1, 2, 3, 4$.

Equalities in (24) are Sommerfeld-Kupradze type radiation conditions in the linear theory of thermoviscoelasticity for Kelvin-Voigt materials.

In the sequel we use the matrix differential operators

1)

$$\mathbf{P}^{(1)}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = (P_{lj}^{(1)}(\mathbf{D}_\mathbf{x}, \mathbf{n}))_{3 \times 3}, \quad P_{lj}^{(1)}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \mu_1 \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \mu_1 n_j \frac{\partial}{\partial x_l} + \lambda_1 n_l \frac{\partial}{\partial x_j},$$

2)

$$\mathbf{P}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = (P_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}))_{4 \times 4}, \quad P_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = P_{lj}^{(1)}(\mathbf{D}_\mathbf{x}, \mathbf{n}),$$

$$P_{l4}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -\gamma n_l, \quad P_{4l}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = 0, \quad P_{44}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = k \frac{\partial}{\partial \mathbf{n}},$$

3)

$$\tilde{\mathbf{P}}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = (\tilde{P}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}))_{4 \times 4}, \quad \tilde{P}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = P_{lj}(D_x, n), \quad \tilde{P}_{l4}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -\gamma_0 n_l, \quad (25)$$

$$\tilde{P}_{4l}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = 0, \quad \tilde{P}_{44}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = P_{44}(\mathbf{D}_\mathbf{x}, \mathbf{n}),$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the unit vector, $\frac{\partial}{\partial \mathbf{n}}$ is the derivative along the vector \mathbf{n} and $l, j = 1, 2, 3$.

Remark 1. The matrix differential operator $\mathbf{P}^{(1)}(\mathbf{D}_\mathbf{x}, \mathbf{n})$ is the stress operator in the linear theory of viscoelasticity for Kelvin-Voigt materials (see Eringen, 1980).

The basic internal and external BVPs of steady vibration in the linear theory of thermoviscoelasticity for Kelvin-Voigt materials are formulated as follows.

Find a regular (classical) solution to system (8) for $\mathbf{x} \in \Omega^+$ satisfying one of the following boundary conditions

$$\lim_{\Omega^+ \ni \mathbf{x} \rightarrow \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the Problem $(I)_{\mathbf{F}, \mathbf{f}}^+$, and

$$\{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the Problem $(II)_{\mathbf{F}, \mathbf{f}}^+$.

Find a regular (classical) solution to system (8) for $\mathbf{x} \in \Omega^-$ satisfying one of the following boundary conditions

$$\lim_{\Omega^- \ni \mathbf{x} \rightarrow \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the Problem $(I)_{\mathbf{F}, \mathbf{f}}^-$, and

$$\{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the Problem $(II)_{\mathbf{F}, \mathbf{f}}^-$, where \mathbf{F} and \mathbf{f} are the four-component known vector functions, and $\text{supp } \mathbf{F}$ is a finite domain in Ω^- .

We have the following results (uniqueness theorems).

Theorem 9. If condition (6) is satisfied, then the internal BVP $(K)_{\mathbf{F}, \mathbf{f}}^+$ admits at most one regular solution, where $K = I, II$.

Theorem 10. If condition (6) is satisfied, then the external BVP $(K)_{\mathbf{F}, \mathbf{f}}^-$ admits at most one regular solution, where $K = I, II$.

5 Basic Properties of Potentials

In this section we present the basic properties of thermoviscoelastopotentials. We introduce the potential of single-layer

$$\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) = \int_S \Gamma(\mathbf{x} - \mathbf{y}) \mathbf{g}(\mathbf{y}) d_{\mathbf{y}} S,$$

the potential of double-layer

$$\mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) = \int_S [\tilde{\mathbf{P}}(\mathbf{D}_{\mathbf{y}}, \mathbf{n}(\mathbf{y})) \Gamma^{\mathbf{T}}(\mathbf{x} - \mathbf{y})]^{\mathbf{T}} \mathbf{g}(\mathbf{y}) d_{\mathbf{y}} S,$$

and the potential of volume

$$\mathbf{Z}^{(3)}(\mathbf{x}, \phi, \Omega^{\pm}) = \int_{\Omega^{\pm}} \Gamma(\mathbf{x} - \mathbf{y}) \phi(\mathbf{y}) d_{\mathbf{y}},$$

where Γ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D}_{\mathbf{x}})$, the operator $\tilde{\mathbf{P}}$ is defined by (25), \mathbf{g} and ϕ are four-component vector functions.

The basic properties of thermoviscoelastopotentials are given in the following theorems.

Theorem 11. If $S \in C^{2,\nu}$, $\mathbf{g} \in C^{1,\nu_0}(S)$, $0 < \nu_0 < \nu \leq 1$, then:

a)

$$\mathbf{Z}^{(1)}(\cdot, \mathbf{g}) \in C^{0,\nu_0}(R^3) \cap C^{2,\nu_0}(\bar{\Omega}^{\pm}) \cap C^{\infty}(\Omega^{\pm}),$$

b)

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}}) \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}, \quad \mathbf{x} \in \Omega^{\pm},$$

c)

$$\{\mathbf{P}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z})) \mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g})\}^{\pm} = \mp \frac{1}{2} \mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z})) \mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}) \quad (26)$$

and $\mathbf{P}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z})) \mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g})$ is a singular integral and understood as the principal value, where $\mathbf{z} \in S$.

Theorem 12. If $S \in C^{2,\nu}$, $\mathbf{g} \in C^{1,\nu_0}(S)$, $0 < \lambda' < \lambda_0 \leq 1$, then:

a)

$$\mathbf{Z}^{(2)}(\cdot, \mathbf{g}) \in C^{1,\nu_0}(\bar{\Omega}^{\pm}) \cap C^{\infty}(\Omega^{\pm}),$$

b)

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}}) \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}, \quad \mathbf{x} \in \Omega^{\pm},$$

c)

$$\{\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^{\pm} = \pm \frac{1}{2} \mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}) \quad (27)$$

and $\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})$ is a singular integral and understood as the principal value, where $\mathbf{z} \in S$.

d)

$$\{\mathbf{P}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z})) \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^+ = \{\mathbf{P}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z})) \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^-.$$

Theorem 13. If $S \in C^{1,\nu}$, $\phi \in C^{0,\nu_0}(\Omega^+)$, $0 < \nu_0 < \nu \leq 1$, then:

a)

$$\mathbf{Z}^{(3)}(\cdot, \phi, \Omega^+) \in C^{1,\nu_0}(R^3) \cap C^2(\Omega^+) \cap C^{2,\nu_0}(\bar{\Omega}_0^+),$$

b)

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}}) \mathbf{Z}^{(3)}(x, \phi, \Omega^+) = \phi(\mathbf{x}), \quad \mathbf{x} \in \Omega^+,$$

where Ω_0^+ is a domain in R^3 and $\Omega_0^+ \subset \Omega^+$.

Theorem 14. If $S \in C^{1,\nu}$, $\text{supp} \phi = \Omega \subset \Omega^-$, $\phi \in C^{0,\nu_0}(\Omega^-)$, $0 < \nu_0 < \nu \leq 1$, then:

a)

$$\mathbf{Z}^{(3)}(\cdot, \phi, \Omega^-) \in C^{1,\nu_0}(R^3) \cap C^2(\Omega^-) \cap C^{2,\nu_0}(\bar{\Omega}_0^-)$$

b)

$$\mathbf{A}(\mathbf{D}_x) \mathbf{Z}^{(3)}(\mathbf{x}, \phi, \Omega^-) = \phi(\mathbf{x}) \quad \mathbf{x} \in \Omega^-$$

where Ω is a finite domain in R^3 and $\bar{\Omega}_0^- \subset \Omega^-$.

Theorems 11 to 14 can be proved similarly to the corresponding theorems in the classical theory of thermoelasticity (for details see Kupradze et al., 1979).

6 Existence Theorems

In this section we establish the existence of regular solutions of the BVPs $(I)_{\mathbf{F},\mathbf{f}}^+$, $(I)_{\mathbf{F},\mathbf{f}}^-$, $(II)_{\mathbf{F},\mathbf{f}}^+$, $(II)_{\mathbf{F},\mathbf{f}}^-$ by means of the potential method and the theory of 2D singular integral equations.

By Theorems 13 and 14 the volume potentials $\mathbf{Z}^{(3)}(\mathbf{x}, \mathbf{F}, \Omega^+)$ and $\mathbf{Z}^{(3)}(\mathbf{x}, \mathbf{F}, \Omega^-)$ are regular solutions of Eq. (8) in Ω^+ and Ω^- , respectively, where $\mathbf{F} \in C^{0,\nu_0}(\Omega^\pm)$, $0 < \nu_0 \leq 1$ and $\text{supp } \mathbf{F}$ is a finite domain in Ω^- . Therefore, further we will consider BVPs $(K)_{\mathbf{0},\mathbf{f}}^+$ and $(K)_{\mathbf{0},\mathbf{f}}^-$ for $K = I, II$.

Problem $(I)_{\mathbf{0},\mathbf{f}}^+$. We seek a regular solution to BVP $(I)_{\mathbf{0},\mathbf{f}}^+$ in the form of potential of double-layer

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) \quad (28)$$

for $\mathbf{x} \in \Omega^+$, where \mathbf{g} is the required four-component vector.

Obviously, by Theorems 12 the vector \mathbf{U} is a solution of homogeneous equation

$$\mathbf{A}(\mathbf{D}_x) \mathbf{U}(\mathbf{x}) = \mathbf{0} \quad (29)$$

for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S$$

and using Eqs. (27) and (28) we obtain the singular integral equation

$$\mathcal{K}^{(1)} \mathbf{g}(\mathbf{z}) \equiv \frac{1}{2} \mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}) = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S, \quad (30)$$

where $\mathcal{K}^{(1)}$ is a singular integral operator of the normal type and $\text{ind } \mathcal{K}^{(1)} = 0$. Therefore, the Fredholm's theorems are valid for Eq. (30).

Now we prove that the adjoint homogeneous equation of (30)

$$\mathcal{K}^{(2)} \mathbf{h}_0(\mathbf{z}) \equiv \frac{1}{2} \mathbf{h}_0(\mathbf{z}) + \mathbf{P}(\mathbf{D}_z, \mathbf{n}(\mathbf{z})) \mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{h}_0) = \mathbf{0} \quad \text{for } \mathbf{z} \in S \quad (31)$$

has only a trivial solution. Indeed, let \mathbf{h}_0 be a solution of the homogeneous Eq. (31) and $\mathbf{h}_0 \in C^{1,\nu_0}(S)$. Then the vector \mathbf{V} defined by formula

$$\mathbf{V}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{h}_0) \quad \text{for } \mathbf{x} \in \Omega^- \quad (32)$$

is a regular solution of problem $(I)_{\mathbf{0},\mathbf{0}}^-$. Using theorem 10, the problem $(I)_{\mathbf{0},\mathbf{0}}^-$ has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) = \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^-. \quad (33)$$

Therefore, vector \mathbf{V} is a regular solution of problem $(I)_{\mathbf{0},\mathbf{0}}^+$. Using Theorem 9, the problem $(I)_{\mathbf{0},\mathbf{0}}^+$ has only the trivial solution, that is

$$\mathbf{V}(\mathbf{x}) = \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^+. \quad (34)$$

On other hand, by Eq. (32) from (26) we get

$$\{\mathbf{P}(\mathbf{D}_z, \mathbf{n}) \mathbf{V}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D}_z, \mathbf{n}) \mathbf{V}(\mathbf{z})\}^+ = \mathbf{h}_0(\mathbf{z}) \quad \text{for } \mathbf{z} \in S. \quad (35)$$

On the basis of Eqs. (33) and (34) from (35) we have $\mathbf{h}_0(\mathbf{z}) = \mathbf{0}$ for $\mathbf{z} \in S$. Thus the homogeneous Eq. (31) has only a trivial solution and therefore, by virtue of the Fredholm's theorems Eq. (30) is always solvable for an arbitrary vector \mathbf{f} .

We have thereby proved

Theorem 15. If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu_0}(S)$, $0 < \nu_0 < \nu \leq 1$, then a regular solution of the BVP $(I)_{0,\mathbf{f}}^+$ exists, is unique and is represented by the potential of double-layer (28), where \mathbf{g} is a solution of the singular integral equation (30) which is always solvable for an arbitrary vector \mathbf{f} .

Problem $(II)_{0,\mathbf{f}}^-$. We seek a regular solution to BVP $(II)_{0,\mathbf{f}}^-$ in the form of potential of single-layer

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{h}) \quad (36)$$

for $\mathbf{x} \in \Omega^-$, where \mathbf{h} is the required four-component vector.

Obviously, by Theorem 11 the vector function \mathbf{U} is solution of Eq. (29) for $\mathbf{x} \in \Omega^-$. Keeping in mind the boundary condition

$$\{\mathbf{P}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S$$

and using Eq. (26) we obtain the singular integral equation

$$\mathcal{K}^{(2)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S. \quad (37)$$

The homogeneous equation (31) has only a trivial solution and therefore Eq. (37) is always solvable for an arbitrary vector \mathbf{f} .

We have thereby proved

Theorem 16. If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu_0}(S)$, $0 < \nu_0 < \nu \leq 1$, then a regular solution of the BVP $(II)_{0,\mathbf{f}}^-$ exists, is unique and is represented by the potential of single-layer (36), where \mathbf{h} is a solution of the singular integral equation (37) which is always solvable for an arbitrary vector \mathbf{f} .

Quite similarly we can prove the following theorems.

Theorem 17. If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu_0}(S)$, $0 < \nu_0 < \nu \leq 1$, then a regular solution of the BVP $(I)_{0,\mathbf{f}}^-$ exists, is unique and is represented by the potential of double-layer (28) for $\mathbf{x} \in \Omega^-$, where \mathbf{g} is a solution of the singular integral equation

$$-\frac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}) = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S$$

which is always solvable for an arbitrary vector \mathbf{f} .

Theorem 18. If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu_0}(S)$, $0 < \nu_0 < \nu \leq 1$, then a regular solution of the BVP $(II)_{0,\mathbf{f}}^+$ exists, is unique and is represented by the potential of single-layer (36) for $\mathbf{x} \in \Omega^+$, where \mathbf{h} is a solution of the singular integral equation

$$-\frac{1}{2}\mathbf{h}(\mathbf{z}) + \mathbf{P}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{h}) = \mathbf{f}(\mathbf{x}) \quad \text{for } \mathbf{z} \in S$$

which is always solvable for an arbitrary vector \mathbf{f} .

Remark 2. On the basis of theorems 5 to 8 we can construct the viscoelastopotentials and prove the uniqueness and existence of regular solutions of the basic 3D BVPs of steady vibrations in the linear theory of viscoelasticity for Kelvin-Voigt materials by using the potential method and the theory of singular integral equations.

Remark 3. By the potential method it is possible to investigate 3D BVPs in the modern theories of viscoelasticity and thermoviscoelasticity for Kelvin-Voigt materials with microstructure.

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