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On the Linear Theory of Thermoelasticity with Microtemperatures

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In the present paper the linear theory of thermoelasticity with microtemperatures is considered. A wide class of external boundary value problems (BVPs) of steady vibrations is investigated. Sommerfeld-Kupradze type radiation conditions and the basic properties of thermoelastopotentials are established. The uniqueness and existence theorems of regular solutions of the external BVPs are proved using the potential method and the theory of singular integral equations.

1 Introduction

In recent years several continuum theories with microstructure have been formulated (see, Eringen, 1999; Iesan, 2005). A thermodynamic theory for elastic materials with inner structure the particles of which, in addition to microdeformations, possess microtemperatures was proposed by Grot (1969). Thermodynamics of a continuum with microstructure was extended in that it is assumed that the microelements have different temperatures. To describe this phenomenon the concept of microtemperatures was introduced. The microtemperatures depend homogeneously on the microcoordinates of the microelements.

Riha (1975, 1976) developed a theory of micromorphic fluids with microtemperatures. The linear theory of thermoelasticity with microtemperatures for materials with inner structure the particles of which, in addition to the classical displacement and temperature fields, possess microtemperatures was studied by Iesan and Quintanilla (2000). The fundamental solution of the equations of the theory of thermoelasticity with microtemperatures is constructed by Svanadze (2004a). The representations of Galerkin type and general solutions of the equations of dynamic and steady vibrations in this theory have been obtained by Scalia and Svanadze (2006). The BVPs of the steady vibration are considered by Svanadze (2003) and Scalia and Svanadze (2009). The exponential stability of solution of equations of the theory of thermoelasticity with microtemperatures has been established by Casas and Quintanilla (2005). The basic theorems in the equilibrium theory of thermoelasticity with microtemperatures have been proved by Scalia et al. (2010).

The theory of micromorphic elastic solids with microtemperatures is constructed by Iesan (2001). The fundamental solutions of equations of this theory have been established by Svanadze (2004b). The existence and uniqueness of solutions in the linear theory of heat conduction in micromorphic continua are established by Iesan (2002). Recently, the uniqueness theorems in the equilibrium theory of thermoelasticity with microtemperature for microstretch materials have been proved by Scalia and Svanadze (2012).

The investigation of BVPs of mathematical physics by the classical potential method has a hundred year history. The application of this method to the 3D BVPs of the theory of elasticity reduces these problems to 2D singular integral equations (see Kupradze et al., 1979). Owing to the works of Mikhlin (1965), Kupradze and his pupils (see Kupradze, 1965; Kupradze et al., 1979; Burchuladze and Gegelia, 1985), the theory of multidimensional singular integral equations has presently been worked out with sufficient completeness. This theory makes it possible to investigate 3D problems not only of the classical theory of elasticity, but also problems of the theory of elasticity with conjugated fields. An extensive review of works on the potential method can be found in Gegelia and Jentsch (1994).

The radiation conditions played an essential role in the external problems of vibrations. As is known (see Kupradze et al., 1979; Burchuladze and Gegelia, 1985) these conditions guarantee uniqueness of the solution of the considered problems for an infinite domain. The radiation conditions for the Helmholz equation were formulated by Sommerfeld (1912) and proved mathematically by Kupradze (1934) and Vekua (1943). The modern situation concerning the radiation conditions for various elastic media with conjugated fields is expounded in Kupradze et al.

(1979) and Burchuladze and Gegelia (1985).

In the present paper the linear theory of thermoelasticity with microtemperatures is considered (see Iesan and Quintanilla, 2000). A wide class of external BVPs of steady vibrations are investigated. Sommerfeld-Kupradze type radiation conditions and the basic properties of thermoelastopotentials are established. The uniqueness and existence theorems of regular solutions of the external BVPs are proved using the potential method and the theory of singular integral equations.

2 Basic Equations

We consider an isotropic elastic material with microstructures which occupies the region Ω of the Euclidean three-dimensional space R^3. Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of R^3 and $\mathbf{D_x} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$.

The system of equations of steady vibrations in the linear theory of thermoelasticity with microtemperatures has the following form (see Iesan and Quintanilla, 2000)

$$\begin{aligned} &\mu\,\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad\,div}\,\mathbf{u} - \beta\,\mathrm{grad}\,\theta + \rho\,\omega^2\,\mathbf{u} = -\rho\mathbf{N},\\ &k_6\,\Delta\mathbf{w} + (k_4+k_5)\,\mathrm{grad\,div}\,\mathbf{w} - k_3\,\mathrm{grad}\,\theta + (i\omega\,b - k_2)\mathbf{w} = \rho\,\mathbf{M},\\ &(k\,\Delta + i\omega\,aT_0)\,\theta + i\omega\,\beta\,T_0\,\mathrm{div}\,\mathbf{u} + k_1\,\mathrm{div}\,\mathbf{w} = -\rho\,s, \end{aligned} \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, $\mathbf{w} = (w_1, w_2, w_3)$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 $(T_0 > 0)$, ρ is the reference mass density $(\rho > 0)$, $\mathbf{N} = (N_1, N_2, N_3)$ is the body force, $\mathbf{M} = (M_1, M_2, M_3)$ is first heat source moment vector, s is the heat supply, Δ is the Laplacean; $\lambda, \mu, \beta, a, b, k, k_1, k_2, \cdots, k_6$ are constitutive coefficients, $i = \sqrt{-1}$, and ω is the oscillation frequency $(\omega > 0)$.

We will suppose that the following assumptions on the constitutive coefficients hold (see Grot, 1969; Iesan and Quintanilla, 2000):

$$\begin{aligned} &\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad a > 0, \qquad b > 0, \qquad k > 0,\\ &3k_4 + k_5 + k_6 > 0, \qquad k_6 \pm k_5 > 0, \qquad (k_1 + k_3T_0)^2 < 4T_0kk_2. \end{aligned} \tag{2}$$

We introduce the notation

$$\begin{aligned} &\mu_0 = \lambda + 2\mu, \qquad a_0 = i\omega aT_0, \qquad \beta_0 = i\omega\beta T_0,\\ &k_7 = k_4 + k_5 + k_6, \qquad k_8 = i\omega b - k_2. \end{aligned} \tag{3}$$

Obviously, from Eqs. (2) and (3) we have

$$\begin{aligned} &\lambda + \mu = \tfrac{1}{3}[(3\lambda + 2\mu) + \mu] > 0, \qquad \mu_0 > 0,\\ &k_6 = \tfrac{1}{2}[(k_6 + k_5) + (k_6 - k_5)] > 0,\\ &k_7 = \tfrac{1}{3}[(3k_4 + k_5 + k_6) + 2(k_6 + k_5)] > 0,\\ &k_4 + k_7 = \tfrac{2}{3}(3k_4 + k_5 + k_6) + \tfrac{1}{3}(k_5 + k_6) > 0,\\ &2k_6k_7 - k_5k_7 + k_4k_6 = k_7(k_6 - k_5) + k_6(k_4 + k_7) > 0. \end{aligned} \tag{4}$$

We introduce the matrix differential operator

$$\begin{aligned} &\mathbf{A}(\mathbf{D_x}) = (A_{pq}(\mathbf{D_x}))_{7\times 7}, \qquad A_{lj}(\mathbf{D_x}) = (\mu\Delta + \rho\,\omega^2)\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},\\ &A_{l;j+3}(\mathbf{D_x}) = A_{l+3;j}(\mathbf{D_x}) = 0, \qquad A_{l7}(\mathbf{D_x}) = -\beta\frac{\partial}{\partial x_l},\\ &A_{l+3;j+3}(\mathbf{D_x}) = (k_6\Delta + k_8)\delta_{lj} + (k_4 + k_5)\,\frac{\partial^2}{\partial x_l \partial x_j},\\ &A_{l+3;7}(\mathbf{D_x}) = -k_3\frac{\partial}{\partial x_l}, \qquad A_{7l}(\mathbf{D_x}) = \beta_0\frac{\partial}{\partial x_l},\\ &A_{7;l+3}(\mathbf{D_x}) = k_1\frac{\partial}{\partial x_l}, \qquad A_{77}(\mathbf{D_x}) = k\Delta + a_0. \end{aligned}$$

The system (1) can be written as

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{5}$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta),\ \mathbf{F} = (-\rho\mathbf{N}, \rho\mathbf{M}, -\rho s)$ and $\mathbf{x} \in \Omega$.

3 Boundary Value Problems

Let S be the closed surface surrounding the finite domain Ω^+ in R^3, $S \in C^{2,\lambda_0}$, $0 < \lambda_0 \leq 1$, $\bar{\Omega}^+ = \Omega^+ \cup S$, $\Omega^- = R^3 \backslash \bar{\Omega}^+$; Ω_r and S_r are denoted the sphere and boundary of the sphere of radius r with the center at the origin, respectively. The scalar product of two vectors $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)$ and $\boldsymbol{\psi} = (\psi_1, \psi_2, \cdots, \psi_l)$ is denoted by $\boldsymbol{\varphi} \cdot \boldsymbol{\psi} = \sum_{j=1}^{l} \varphi_j \bar{\psi}_j$, where $\bar{\psi}_j$ is the complex conjugate of ψ_j.

We introduce the notation

$$\Lambda(\Delta) = \frac{1}{\mu_0 k_7 k} \det \begin{pmatrix} \mu_0\Delta + \rho\omega^2 & 0 & -\beta\Delta \\ 0 & k_7\Delta + k_8 & -k_3\Delta \\ \beta_0 & k_1 & k\Delta + a_0 \end{pmatrix}_{3\times 3}. \tag{6}$$

It is easily seen that

$$\Lambda(\Delta) = (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2),$$

where $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the roots of the equations $\Lambda(-\xi) = 0$ (with respect to ξ).

Let $\lambda_4^2 = \dfrac{\rho\,\omega^2}{\mu_0}$ and $\lambda_5^2 = \dfrac{k_8}{k_6}$. We assume that

$$\text{Im}\lambda_j > 0 \qquad (j = 1, 2, 3, 5), \qquad \lambda_4 > 0. \tag{7}$$

Definition. A vector function $\mathbf{U} = (U_1, U_2, \cdots, U_7)$ is called regular in Ω^- (or Ω^+) if
1)

$$U_l \in C^2(\Omega^-) \cap C^1(\bar{\Omega}^-) \qquad (\text{or } U_l \in C^2(\Omega^+) \cap C^1(\bar{\Omega}^+)),$$

2)

$$\mathbf{U} = \sum_{j=1}^{5} \mathbf{U}^{(j)}, \qquad \mathbf{U}^{(j)} = (U_1^{(j)}, U_2^{(j)}, \cdots, U_7^{(j)}),$$
$$U_l^{(j)} \in C^2(\Omega^-) \cap C^1(\bar{\Omega}^-),$$

3)

$$(\Delta + \lambda_j^2) U_l^{(j)}(\mathbf{x}) = 0 \tag{8}$$

and

$$\left(\frac{\partial}{\partial|\mathbf{x}|} - i\lambda_j\right) U_l^{(j)}(\mathbf{x}) = e^{i\lambda_j|\mathbf{x}|} o(|\mathbf{x}|^{-1}) \qquad \text{for} \qquad |\mathbf{x}| \gg 1, \tag{9}$$

where $U_m^{(5)} = U_{m+3}^{(4)} = U_7^{(4)} = U_7^{(5)} = 0$ and $m = 1, 2, 3,\ j = 1, 2, \cdots, 5,\ l = 1, 2, \cdots, 7$.

Equalities in (9) are Sommerfeld-Kupradze type radiation conditions in the linear theory of thermoelasticity with microtemperatures.

Remark 1. The Equations (8) and (9) imply (see Vekua, 1943)

$$U_l^{(j)}(\mathbf{x}) = e^{i\lambda_j|\mathbf{x}|} O(|\mathbf{x}|^{-1}) \qquad \text{for} \qquad |\mathbf{x}| \gg 1 \qquad (j = 1, 2, \cdots, 5, \qquad l = 1, 2, \cdots, 7). \tag{10}$$

In the sequel we use the matrix differential operators

1)

$$\mathbf{A}^{(m)}(\mathbf{D_x}) = (A_{lj}^{(m)}(\mathbf{D_x}))_{3\times 3}, \qquad A_{lj}^{(1)}(\mathbf{D_x}) = A_{lj}(\mathbf{D_x}), \qquad A_{lj}^{(2)}(\mathbf{D_x}) = A_{l+3;j+3}(\mathbf{D_x}),$$

2)

$$\mathbf{P}^{(m)}(\mathbf{D_x},\mathbf{n})=(P_{lj}^{(m)}(\mathbf{D_x},\mathbf{n}))_{3\times 3},$$
$$P_{lj}^{(1)}(\mathbf{D_x},\mathbf{n})=\mu\delta_{lj}\frac{\partial}{\partial\mathbf{n}}+\mu n_j\frac{\partial}{\partial x_l}+\lambda n_l\frac{\partial}{\partial x_j}=\mu\delta_{lj}\frac{\partial}{\partial\mathbf{n}}+(\lambda+\mu)n_l\frac{\partial}{\partial x_j}+\mu\mathcal{M}_{lj},$$
$$P_{lj}^{(2)}(\mathbf{D_x},\mathbf{n})=k_6\delta_{lj}\frac{\partial}{\partial\mathbf{n}}+k_5 n_j\frac{\partial}{\partial x_l}+k_4 n_l\frac{\partial}{\partial x_j}=k_6\delta_{lj}\frac{\partial}{\partial\mathbf{n}}+(k_4+k_5)n_l\frac{\partial}{\partial x_j}+k_5\mathcal{M}_{lj},$$

$$\begin{aligned}
&\mathbf{P}(\mathbf{D_x},\mathbf{n})=(P_{lj}(\mathbf{D_x},\mathbf{n}))_{7\times 7},\\
&P_{lj}(\mathbf{D_x},\mathbf{n})=P_{lj}^{(1)}(\mathbf{D_x},\mathbf{n}), \qquad P_{l+3;j+3}(\mathbf{D_x},\mathbf{n})=P_{lj}^{(2)}(\mathbf{D_x},\mathbf{n}),\\
&P_{l7}(\mathbf{D_x},\mathbf{n})=-\beta n_l, \qquad P_{7;l+3}(\mathbf{D_x},\mathbf{n})=k_1\, n_l,\\
&P_{77}(\mathbf{D_x},\mathbf{n})=k\frac{\partial}{\partial\mathbf{n}}, \qquad P_{l;j+3}=P_{l+3;j}=P_{l+3;7}=P_{7l}=0,\\
&\tilde{\mathbf{P}}(\mathbf{D_x},\mathbf{n})=(\tilde{P}_{lj}(\mathbf{D_x},\mathbf{n}))_{7\times 7},\\
&\tilde{P}_{lj}(\mathbf{D_x},\mathbf{n})=P_{lj}(D_x,n), \qquad \tilde{P}_{l+3;j+3}(\mathbf{D_x},\mathbf{n})=P_{l+3;j+3}(\mathbf{D_x},\mathbf{n}),\\
&\tilde{P}_{l7}(\mathbf{D_x},\mathbf{n})=-\beta_0\, n_l, \qquad \tilde{P}_{7;l+3}(\mathbf{D_x},\mathbf{n})=k_3\, n_l,\\
&\tilde{P}_{77}(\mathbf{D_x},\mathbf{n})=P_{77}(\mathbf{D_x},\mathbf{n}), \qquad \tilde{P}_{l;j+3}=\tilde{P}_{l+3;j}=\tilde{P}_{l+3;7}=\tilde{P}_{7l}=0,
\end{aligned}\tag{11}$$

where $\mathbf{n}=(n_1,n_2,n_3)$ is the unit vector, $\dfrac{\partial}{\partial\mathbf{n}}$ is the derivative along the vector $\mathbf{n}$, $\mathcal{M}_{lj}=n_j\dfrac{\partial}{\partial x_l}-n_l\dfrac{\partial}{\partial x_j},\ m=1,2$ and $l,j=1,2,3$.

The external BVPs of steady vibration in the linear theory of thermoelasticity with microtemperatures are formulated as follows.

Find a regular (classical) solution to system (5) for $\mathbf{x}\in\Omega^-$ satisfying one of the following boundary conditions

$$\lim_{\Omega^-\ni\mathbf{x}\to\ \mathbf{z}\in S}\mathbf{U}(\mathbf{x})\equiv\{\mathbf{U}(\mathbf{z})\}^-=\mathbf{f}(\mathbf{z})$$

in the Problem $(I)^-_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^-=\mathbf{f}(\mathbf{z})$$

in the Problem $(II)^-_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{u}(\mathbf{z})\}^-=\mathbf{f}^{(1)}(\mathbf{z}),\qquad \{\mathbf{w}(\mathbf{z})\}^-=\mathbf{f}^{(2)}(\mathbf{z}),\qquad \{\mathbf{q}(\mathbf{z})\cdot\mathbf{n}(\mathbf{z})\}^-=f_7(\mathbf{z})$$

in the Problem $(III)^-_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{u}(\mathbf{z})\}^-=\mathbf{f}^{(1)}(\mathbf{z}),\qquad \{\mathbf{P}^{(2)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{w}(\mathbf{z})\}^-=\mathbf{f}^{(2)}(\mathbf{z}),\qquad \{\theta(\mathbf{z})\}^-=f_7(\mathbf{z})$$

in the Problem $(IV)^-_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{u}(\mathbf{z})\}^-=\mathbf{f}^{(1)}(\mathbf{z}),\qquad \{\mathbf{P}^{(2)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{w}(\mathbf{z})\}^-=\mathbf{f}^{(2)}(\mathbf{z}),\qquad \{\mathbf{q}(\mathbf{z})\cdot\mathbf{n}(\mathbf{z})\}^-=f_7(\mathbf{z})$$

in the Problem $(V)^-_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{P}^{(1)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{u}(\mathbf{z})\}^-=\mathbf{f}^{(1)}(\mathbf{z}),\qquad \{\mathbf{w}(\mathbf{z})\}^-=\mathbf{f}^{(2)}(\mathbf{z}),\qquad \{\theta(\mathbf{z})\}^-=f_7(\mathbf{z})$$

in the Problem $(VI)^-_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{P}^{(1)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{u}(\mathbf{z})\}^-=\mathbf{f}^{(1)}(\mathbf{z}),\qquad \{\mathbf{w}(\mathbf{z})\}^-=\mathbf{f}^{(2)}(\mathbf{z}),\qquad \{\mathbf{q}(\mathbf{z})\cdot\mathbf{n}(\mathbf{z})\}^-=f_7(\mathbf{z})$$

in the Problem $(VII)^-_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{P}^{(1)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{u}(\mathbf{z})\}^-=\mathbf{f}^{(1)}(\mathbf{z}),\qquad \{\mathbf{P}^{(2)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{w}(\mathbf{z})\}^-=\mathbf{f}^{(2)}(\mathbf{z}),\qquad \{\theta(\mathbf{z})\}^-=f_7(\mathbf{z})$$

in the Problem $(VIII)^-_{\mathbf{F},\mathbf{f}}$, where $\mathbf{q}\cdot\mathbf{n}=k\dfrac{\partial\theta}{\partial\mathbf{n}}+k_1\,\mathbf{w}\cdot\mathbf{n},\ \ \mathbf{f}^{(1)}=(f_1,f_2,f_3),\ \ \mathbf{f}^{(2)}=(f_4,f_5,f_6),\ \mathbf{f}=(f_1,f_2,\cdots,f_7);$ $\mathbf{F}$ and $\mathbf{f}$ are the seven-component known vector functions, and $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-.

4 Uniqueness Theorem

In this section we prove uniqueness of regular solutions of external boundary value problems $(I)^-_{\mathbf{F},\mathbf{f}}$, $(II)^-_{\mathbf{F},\mathbf{f}}, \cdots,$ $(VIII)^-_{\mathbf{F},\mathbf{f}}$.

We introduce the notation

$$W^{(1)}(\mathbf{u}) = \frac{1}{3}(3\lambda + 2\mu)\,|\mathrm{div}\,\mathbf{u}|^2 + \mu\left[\frac{1}{2}\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right|^2 + \frac{1}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right|^2\right],$$

$$W^{(2)}(\mathbf{w}) = \frac{1}{3}(3k_4 + k_5 + k_6)\,|\mathrm{div}\,\mathbf{w}|^2 + \frac{1}{2}(k_6 - k_5)\,|\mathrm{curl}\,\mathbf{w}|^2 \qquad (12)$$

$$+\frac{1}{2}(k_6 + k_5)\left[\frac{1}{2}\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial w_j}{\partial x_l} + \frac{\partial w_l}{\partial x_j}\right|^2 + \frac{1}{3}\sum_{l,j=1}^{3}\left|\frac{\partial w_l}{\partial x_l} - \frac{\partial w_j}{\partial x_j}\right|^2\right].$$

In the sequel we use the following lemmas.

Lemma 1. If $\mathbf{U}$ is a regular solution to system

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0} \qquad (13)$$

for $\mathbf{x} \in \Omega^+$, then

$$\int_{\Omega^+}\Big[T_0\,W^{(2)}(\mathbf{w}) + k|\mathrm{grad}\,\theta|^2 + (k_1 + k_3 T_0)\,\mathrm{Re}(\mathbf{w}\cdot\,\mathrm{grad}\,\theta) + k_2\,T_0|\mathbf{w}|^2\Big]\,d\mathbf{x}$$
$$= \mathrm{Re}\int_S\Big[i\omega T_0(\mathbf{P}^{(1)}\mathbf{u} - \beta\theta\mathbf{n})\cdot\mathbf{u} + T_0\mathbf{P}^{(2)}\mathbf{w}\cdot\mathbf{w} + \mathbf{q}\cdot\mathbf{n}\,\theta\Big]\,d_{\mathbf{z}}S. \qquad (14)$$

Proof. The system (13) can be written as

$$\begin{aligned}
&\mathbf{A^{(1)}}(\mathbf{D_x})\,\mathbf{u} - \beta\ \mathrm{grad}\ \theta + \rho\,\omega^2\ \mathbf{u} = \mathbf{0},\\
&\mathbf{A^{(2)}}(\mathbf{D_x})\,\mathbf{w} - k_3\ \mathrm{grad}\ \theta + k_8\mathbf{w} = \mathbf{0},\\
&(k\ \Delta + a_0)\ \theta + \beta_0\ \mathrm{div}\ \mathbf{u} + k_1\ \mathrm{div}\ \mathbf{w} = 0.
\end{aligned} \qquad (15)$$

On account of Eqs. (15) from Green's formulas (see Kupradze et al., 1979)

$$\int_{\Omega^+}\Big[\mathbf{A^{(1)}}(\mathbf{D_x})\,\mathbf{u}\cdot\mathbf{u} + W^{(1)}(\mathbf{u})\Big]\,d\mathbf{x} = \int_S \mathbf{P}^{(1)}\mathbf{u}\cdot\mathbf{u}\,d_{\mathbf{z}}S,$$

$$\int_{\Omega^+}\Big[\mathbf{A^{(2)}}(\mathbf{D_x})\,\mathbf{w}\cdot\mathbf{w} + W^{(2)}(\mathbf{w})\Big]\,d\mathbf{x} = \int_S \mathbf{P}^{(2)}\mathbf{w}\cdot\mathbf{w}\,d_{\mathbf{z}}S,$$

$$\int_{\Omega^+}\big[\Delta\,\theta\bar{\theta} + |\mathrm{grad}\,\theta|^2\big]\,d\mathbf{x} = \int_S \frac{\partial\theta}{\partial\mathbf{n}}\,\bar{\theta}\,d_{\mathbf{z}}S,$$

$$\int_{\Omega^+}(\mathrm{grad}\,\theta\cdot\mathbf{u} + \theta\,\mathrm{div}\bar{\mathbf{u}})\,d\mathbf{x} = \int_S \theta\mathbf{n}\cdot\mathbf{u}\,d_{\mathbf{z}}S$$

it follows that

$$\int_{\Omega^+}\Big[W^{(1)}(\mathbf{u}) - \rho\,\omega^2\ |\mathbf{u}|^2 - \beta\,\theta\,\mathrm{div}\,\bar{\mathbf{u}}\Big]\,d\mathbf{x} = \int_S\Big(\mathbf{P}^{(1)}\mathbf{u} - \beta\,\theta\,\mathbf{n}\Big)\cdot\mathbf{u}\,d_{\mathbf{z}}S,$$
$$\int_{\Omega^+}\Big[W^{(2)}(\mathbf{w}) - k_8\ |\mathbf{w}|^2 + k_3\,\mathrm{grad}\,\theta\cdot\mathbf{w}\Big]\,d\mathbf{x} = \int_S \mathbf{P}^{(2)}\mathbf{w}\cdot\mathbf{w}\,d_{\mathbf{z}}S, \qquad (16)$$
$$\int_{\Omega^+}\big[k\,|\mathrm{grad}\,\theta|^2 - \beta_0\,\mathrm{div}\,\mathbf{u}\,\bar{\theta} - a_0|\theta|^2 + k_1\mathbf{w}\cdot\mathrm{grad}\,\theta\big]\,d\mathbf{x} = \int_S \mathbf{q}\cdot\mathbf{n}\,\theta\,d_{\mathbf{z}}S.$$

Obviously, in view of (2) from Eq. (12) we have

$$W^{(1)}(\mathbf{u}) \geq 0, \qquad W^{(2)}(\mathbf{w}) \geq 0. \tag{17}$$

Keeping in mind (17) we obtain from Eqs. (16) that

$$\begin{aligned}
&\beta \operatorname{Im} \int\limits_{\Omega^+} \operatorname{div}\mathbf{u}\, \bar{\theta}\, d\mathbf{x} = \operatorname{Im} \int\limits_{S} \Big(\mathbf{P}^{(1)}\mathbf{u} - \beta\, \theta\, \mathbf{n}\Big) \cdot \mathbf{u}\, d_{\mathbf{z}}S,\\
&\int\limits_{\Omega^+} \Big[W^{(2)}(\mathbf{w}) + k_2\, |\mathbf{w}|^2 + k_3 \operatorname{Re}\left(\operatorname{grad}\theta \cdot \mathbf{w}\right)\Big]\, d\mathbf{x} = \operatorname{Re} \int\limits_{S} \mathbf{P}^{(2)}\mathbf{w} \cdot \mathbf{w}\, d_{\mathbf{z}}S,\\
&\int\limits_{\Omega^+} \big[k|\operatorname{grad}\theta|^2 + k_1 \operatorname{Re}\, (\mathbf{w}\cdot \operatorname{grad}\theta)\big]\, d\mathbf{x} + \omega\beta T_0 \operatorname{Im} \int\limits_{\Omega^+} \operatorname{div}\mathbf{u}\, \bar{\theta}\, d\mathbf{x} = \operatorname{Re} \int\limits_{S} \mathbf{q}\cdot\mathbf{n}\, \theta\, d_{\mathbf{z}}S.
\end{aligned} \tag{18}$$

Finally, from Eqs. (18) we obtain formula (14).

Lemma 2. If $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta) \in C^2(\Omega)$ is a solution of the system (13) for $\mathbf{x} \in \Omega$, then

$$\mathbf{u}(\mathbf{x}) = \sum_{j=1}^{4} \mathbf{u}^{(j)}(\mathbf{x}), \qquad \mathbf{w}(\mathbf{x}) = \sum_{j=1,2,3,5} \mathbf{w}^{(j)}(\mathbf{x}), \qquad \theta(\mathbf{x}) = \sum_{j=1}^{3} \theta^{(j)}(\mathbf{x}), \tag{19}$$

where Ω is a domain in R^3, and $\mathbf{u}^{(j)}, \mathbf{w}^{(j)}$ and $\theta^{(j)}$ satisfy the following equations

$$(\Delta + \lambda_j^2)\mathbf{u}^{(j)}(\mathbf{x}) = \mathbf{0}, \qquad (\Delta + \lambda_l^2)\mathbf{w}^{(l)}(\mathbf{x}) = \mathbf{0}, \tag{20}$$

$$(\Delta + \lambda_m^2)\theta^{(m)}(\mathbf{x}) = 0, \qquad m = 1,2,3, \quad j = 1,2,3,4, \quad l = 1,2,3,5. \tag{21}$$

Proof. Applying the operator div to the equations $(15)_1$ and $(15)_2$, from system (15) we get

$$\begin{aligned}
&\left(\mu_0\Delta + \rho\, \omega^2\right) \operatorname{div}\mathbf{u} - \beta\, \Delta\, \theta = 0,\\
&(k_7\Delta + k_8) \operatorname{div}\mathbf{w} - k_3\, \Delta\, \theta = 0,\\
&(k\, \Delta + a_0)\, \theta + \beta_0\, \operatorname{div}\, \mathbf{u} + k_1\, \operatorname{div}\, \mathbf{w} = 0.
\end{aligned} \tag{22}$$

From system (22) we have

$$\Lambda(\Delta) \operatorname{div}\mathbf{u} = 0, \qquad \Lambda\,(\Delta)\operatorname{div}\mathbf{w} = 0, \qquad \Lambda\,(\Delta)\theta = 0, \tag{23}$$

where the operator Λ is defined by (6).

Now, applying the operator $\Lambda(\Delta)$ to the equations $(21)_1$ and $(21)_2$, and using Eq. (29) we obtain

$$\Lambda(\Delta)\left(\Delta + \lambda_4^2\right)\, \mathbf{u} = \mathbf{0}, \qquad \Lambda(\Delta)\left(\Delta + \lambda_5^2\right)\, \mathbf{w} = \mathbf{0}. \tag{24}$$

We introduce the notation:

$$\begin{aligned}
\mathbf{u}^{(j)} &= \prod_{\substack{l=1\\ l\neq j}}^{4} (\lambda_l^2 - \lambda_j^2)^{-1}(\Delta + \lambda_l^2)\, \mathbf{u}, \qquad j = 1,2,3,4,\\
\mathbf{w}^{(m)} &= \prod_{\substack{l=1,2,3,5\\ l\neq m}} (\lambda_l^2 - \lambda_j^2)^{-1}(\Delta + \lambda_l^2)\, \mathbf{w}, \qquad m = 1,2,3,5,\\
\theta^{(p)} &= \prod_{\substack{l=1\\ l\neq p}}^{3} (\lambda_l^2 - \lambda_p^2)^{-1}(\Delta + \lambda_l^2)\, \theta, \qquad p = 1,2,3,
\end{aligned} \tag{25}$$

Equation (19) can be easily obtained from Eqs. (25). By Eqs. (23) and (24), from (25) we obtain Eqs. (20) and (21).

Now let us establish the uniqueness of a regular solutions of BVPs $(I)^-_{\mathbf{F},\mathbf{f}}, (II)^-_{\mathbf{F},\mathbf{f}}, \cdots, (VIII)^-_{\mathbf{F},\mathbf{f}}$.

Theorem 1. If condition (2) is satisfied, then the external BVP $(K)^-_{F,f}$ admits at most one regular solution, where $K = I, II, \cdots, VIII.$

Proof. Suppose that there are two regular solutions of BVP $(K)^-_{F,f}$. Then their difference $\mathbf{U}$ corresponds to zero data ($\mathbf{F} = \mathbf{f} = \mathbf{0}$), i.e. $\mathbf{U}$ is a regular solution of BVP $(K)^-_{\mathbf{0},\mathbf{0}}$.

Let Ω_r be a sphere of sufficiently large radius r so that $\bar{\Omega}^+ \subset \Omega_r$. By virtue of homogeneous boundary condition ($\mathbf{f} = \mathbf{0}$), Eq. (20) for the domain $\Omega_r^- = \Omega^- \cap \Omega_r$ can be rewritten as

$$\begin{aligned} &\int\limits_{\Omega_r^-} \Big[T_0\, W^{(2)}(\mathbf{w}) + k|\mathrm{grad}\,\theta|^2 + (k_1 + k_3 T_0)\,\mathrm{Re}\,(\mathbf{w}\cdot\ \mathrm{grad}\,\theta) + k_2\, T_0|\mathbf{w}|^2\Big]\, d\mathbf{x} \\ &= \mathrm{Re}\int\limits_{S_r} \Big[i\omega T_0\, (\mathbf{P}^{(1)}\,\mathbf{u} - \beta\,\theta\,\mathbf{n})\cdot\mathbf{u} + T_0\mathbf{P}^{(2)}\mathbf{w}\cdot\mathbf{w} + \mathbf{q}\cdot\mathbf{n}\,\theta\Big]\, d_{\mathbf{z}}S. \end{aligned} \tag{26}$$

From Eq. (26) we have

$$L = \lim_{r\to\infty} \mathrm{Re}\int\limits_{S_r} \Big[i\omega T_0\, (\mathbf{P}^{(1)}\,\mathbf{u} - \beta\,\theta\,\mathbf{n})\cdot\mathbf{u} + T_0\mathbf{P}^{(2)}\mathbf{w}\cdot\mathbf{w} + \mathbf{q}\cdot\mathbf{n}\,\theta\Big]\, d_{\mathbf{z}}S, \tag{27}$$

where

$$L = \int\limits_{\Omega^-} \Big[T_0\, W^{(2)}(\mathbf{w}) + k|\mathrm{grad}\,\theta|^2 + (k_1 + k_3 T_0)\,\mathrm{Re}\,(\mathbf{w}\cdot\ \mathrm{grad}\,\theta) + k_2\, T_0|\mathbf{w}|^2\Big]\, d\mathbf{x}. \tag{28}$$

Obviously, by Eqs. (2) and (17) it follows from Eq. (28) that

$$L \geq 0. \tag{29}$$

Keeping in mind relations (7) and (10) from (19) we obtain

$$\begin{aligned} &\mathbf{u}(\mathbf{x}) = \mathbf{u}^{(4)}(\mathbf{x}) + e^{-\lambda_6|\mathbf{x}|}\, O(|\mathbf{x}|^{-1}), \\ &\mathbf{w}(\mathbf{x}) = e^{-\lambda_6|\mathbf{x}|}\, O(|\mathbf{x}|^{-1}), \qquad \theta(\mathbf{x}) = e^{-\lambda_6|\mathbf{x}|}\, O(|\mathbf{x}|^{-1}) \qquad \text{for} \quad |\mathbf{x}| \gg 1, \end{aligned} \tag{30}$$

where $\lambda_6 = \min\{\mathrm{Im}\,\lambda_j, \lambda_4\, (j = 1,2,3,5)\} > 0$. On account of condition (30), from Eq. (27) it follows that

$$L = \lim_{r\to\infty} \mathrm{Re}\int\limits_{S_r} i\omega T_0 \mathbf{P}^{(1)}(\mathbf{D}_{\mathbf{z}}, \mathbf{n})\mathbf{u}^{(4)}(\mathbf{z})\cdot\mathbf{u}^{(4)}(\mathbf{z}) d_{\mathbf{z}}S. \tag{31}$$

On the other hand, from Eq. (9) we have

$$\frac{\partial}{\partial z_j} u_l^{(4)}(\mathbf{z}) = i\lambda_4 n_j(\mathbf{z})\, u_l^{(4)}(\mathbf{z}) + o(|\mathbf{z}|^{-1}) \tag{32}$$

for $\mathbf{z} \in S_r$, $r \gg 1$, $n_j(\mathbf{z}) = \dfrac{z_j}{|\mathbf{z}|}$, $l, j = 1,2,3$. Using Eq. (32) we get

$$\mathbf{P}^{(1)}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z}))\, \mathbf{u}^{(4)}(\mathbf{z}) = i\lambda_4\, \mathbf{P}^{(1)}(\mathbf{n}(\mathbf{z}), \mathbf{n}(\mathbf{z}))\mathbf{u}^{(4)}(\mathbf{z}) + o(|\mathbf{z}|^{-1}). \tag{33}$$

By Eqs. (33) and $\mathbf{P}^{(1)}(\mathbf{n}, \mathbf{n}) = \mathbf{A}^{(1)}(\mathbf{n})$ (see Kupradze et al., 1979) from Eq. (31) we obtain

$$L + \omega T_0 \lim_{r\to\infty} \mathrm{Re}\int\limits_{S_r} \mathbf{A}^{(1)}(\mathbf{n}(\mathbf{z}))\mathbf{u}^{(4)}(\mathbf{z})\cdot\mathbf{u}^{(4)}(\mathbf{z}) d_{\mathbf{z}}S = 0. \tag{34}$$

On account of relation (see Kupradze et al., 1979)

$$\mathbf{A}^{(1)}(\mathbf{n})\, \mathbf{u}^{(4)}\cdot\mathbf{u}^{(4)} \geq \delta|\mathbf{u}^{(4)}|^2$$

from Eq. (34) it follows that

$$L + \omega\delta T_0 \lim_{r\to\infty} \int\limits_{S_r} |\mathbf{u}^{(4)}(\mathbf{z})|^2 d_{\mathbf{z}} S \le 0, \tag{35}$$

where $\delta > 0$. By Eqs. (2), (17), (28) and (29) from Eq. (35) follows

$$\mathbf{w}(\mathbf{x}) = \mathbf{0}, \qquad \theta(\mathbf{x}) = 0 \qquad \text{for} \quad \mathbf{x} \in \Omega^- \tag{36}$$

and

$$\lim_{r\to\infty} \int\limits_{S_r} |\mathbf{u}^{(4)}(\mathbf{z})|^2 d_{\mathbf{z}} S = 0. \tag{37}$$

Hence vector $\mathbf{u}^{(4)}$ satisfies Eq. (37) and

$$\begin{aligned} &(\Delta + \lambda_4^2)\,\mathbf{u}^{(4)}(\mathbf{x}) = \mathbf{0} \qquad \text{for} \quad \mathbf{x} \in \Omega^-, \\ &\mathbf{u}^{(4)}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \qquad \text{for} \quad |\mathbf{x}| \gg 1, \end{aligned} \tag{38}$$

It is well known (see, e.g., Kupradze et al., 1979) that Eqs. (37) and (38) imply

$$\mathbf{u}^{(4)}(\mathbf{x}) = \mathbf{0} \qquad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{39}$$

Finally, from Eqs. (36) and (39) we have $\mathbf{U}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \Omega^-$.

5 Basic Properties of Potentials

In this section we present the basic properties of thermoelastopotentials. We introduce the potential of a single-layer

$$\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) = \int\limits_S \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\mathbf{g}(\mathbf{y}) d_{\mathbf{y}} S,$$

the potential of a double-layer

$$\mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) = \int\limits_S \left[\tilde{\mathbf{P}}(\mathbf{D}_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x} - \mathbf{y})\right]^\top \mathbf{g}(\mathbf{y}) d_{\mathbf{y}} S,$$

and the potential of volume

$$\mathbf{Z}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^\pm) = \int\limits_{\Omega^\pm} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\boldsymbol{\phi}(\mathbf{y}) d\mathbf{y},$$

where $\mathbf{\Gamma}$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D}_{\mathbf{x}})$ (see Svanadze, 2004a), the operator $\tilde{\mathbf{P}}$ is defined by (11), $\mathbf{g}$ and $\boldsymbol{\phi}$ are seven-component vectors, and the superscript $\top$ denotes transposition.

Remark 2. In Svanadze (2004a), the fundamental matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed in terms of elementary functions and basic properties are established.

First we establish the basic properties of thermoelastopotentials.

Theorem 2. If $S \in C^{m+1,\lambda_0},\ \mathbf{g} \in C^{m,\lambda'}(S),\ 0 < \lambda' < \lambda_0 \le 1$, and m is a non-negative whole number, then:
a)

$$\mathbf{Z}^{(1)}(\cdot, \mathbf{g}) \in C^{0,\lambda'}(R^3) \cap C^{m+1,\lambda'}(\bar{\Omega}^\pm) \cap C^\infty(\Omega^\pm),$$

b)

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) = \mathbf{0} \qquad \mathbf{x} \in \Omega^\pm,$$

c)

$$\left\{\mathbf{P}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g})\right\}^\pm = \mp\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}), \qquad \mathbf{z} \in S. \tag{40}$$

Theorem 3. If $S \in C^{m+1,\lambda_0},\ \mathbf{g} \in C^{m,\lambda'}(S),\ 0 < \lambda' < \lambda_0 \le 1$, then:
a)

$$\mathbf{Z}^{(2)}(\cdot, \mathbf{g}) \in C^{m,\lambda'}(\bar{\Omega}^\pm) \cap C^\infty(\Omega^\pm),$$

b)

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g}) = \mathbf{0} \qquad \mathbf{x}\in\Omega^{\pm},$$

c)

$$\left\{\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\right\}^{\pm} = \pm\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g},) \qquad \mathbf{z}\in S. \tag{41}$$

for the non-negative integer m,

d)

$$\left\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\right\}^{+} = \left\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\right\}^{-} \tag{42}$$

for the natural number m and $\mathbf{z}\in S$.

Theorem 4. If $S\in C^{1,\lambda_0},\ \boldsymbol{\phi}\in C^{0,\lambda'}(\Omega^+),\ 0<\lambda'<\lambda_0\le 1$, then:

a)

$$\mathbf{Z}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^+)\in C^{1,\lambda'}(R^3)\cap C^2(\Omega^+)\cap C^{2,\lambda'}(\bar{\Omega}_0^+),$$

b)

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(3)}(x,\boldsymbol{\phi},\Omega^+) = \boldsymbol{\phi}(\mathbf{x}), \qquad \mathbf{x}\in\Omega^+,$$

where Ω_0^+ is a domain in R^3 and $\Omega_0^+\subset\Omega^+$.

Theorem 5. If $S\in C^{1,\lambda_0},\ \mathrm{supp}\boldsymbol{\phi}=\Omega\subset\Omega^-,\ \ \boldsymbol{\phi}\in C^{0,\lambda'}(\Omega^-),\ 0<\lambda'<\lambda_0\le 1$, then:

a)

$$\mathbf{Z}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^-)\in C^{1,\lambda'}(R^3)\cap C^2(\Omega^-)\cap C^{2,\lambda'}(\bar{\Omega}_0^-),$$

b)

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^-) = \boldsymbol{\phi}(\mathbf{x}) \qquad \mathbf{x}\in\Omega^-,$$

where Ω is a finite domain in R^3 and $\bar{\Omega}_0^-\subset\Omega^-$.

Theorems 2-5 can be proved similarly to the corresponding theorems in the classical theory of thermoelasticity (for details see Kupradze et al., 1979).

6 Existence Theorems

In this section we establish the existence of regular solutions of the BVPs $(I)^-_{\mathbf{F},\mathbf{f}}, (II)^-_{\mathbf{F},\mathbf{f}},\cdots,(VIII)^-_{\mathbf{F},\mathbf{f}}$ by means of the potential method and the theory of 2D singular integral equations. We introduce the notation

$$\begin{aligned}
&\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) \equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g}), \qquad \mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) \equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g}),\\
&\mathcal{K}_\tau\mathbf{g}(\mathbf{z}) \equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \tau\,\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g}), \qquad \mathbf{z}\in S,
\end{aligned} \tag{43}$$

where τ is an arbitrary complex number. Obviously, $\mathcal{K}^{(1)}$, $\mathcal{K}^{(2)}$ and $\mathcal{K}_\tau$ are the singular integral operators (for the definition a singular integral operator see, e.g. Kupradze et al., 1979).

In the sequel we need the following Lemmas.

Lemma 3. If $\mathcal{L}$ is a continuous curve on the complex plane connecting the origin with the point τ_0 and $\mathcal{K}_\tau$ is a normal type operator for any $\tau\in\mathcal{L}$, then the index of the operator $\mathcal{K}_{\tau_0}$ vanishes, i.e.

$$\mathrm{ind}\,\mathcal{K}_{\tau_0} = 0.$$

Lemma 3 is proved in Kupradze et al. (1979).

Lemma 4. If condition (2) is satisfied, then the singular integral operators $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ are of the normal type with an index equal to zero.

Proof: Let $\boldsymbol{\sigma}^{(p)} = (\sigma^{(p)}_{lj})_{7\times 7}$ be the symbol of the operator $\mathcal{K}^{(p)}$ $(p=1,2)$. From (43) we have

$$\det\boldsymbol{\sigma}^{(1)} = -\frac{1}{2}\,\sigma_1\,\sigma_2, \tag{44}$$

where

$$\sigma_1 = \frac{(\lambda+\mu)(\lambda+3\mu)}{8(\lambda+2\mu)^2}, \qquad \sigma_2 = \frac{1}{32\,k_6^2\,k_7^2}(k_5+k_6)(k_6+k_7)(2k_6k_7 - k_5k_7 + k_4k_6). \tag{45}$$

Keeping in mind the relations (4) from Eqs. (45) we have $\sigma_1 > 0$ and $\sigma_2 > 0$. Obviously, from Eq. (44) we obtain

$$\det \boldsymbol{\sigma}^{(1)} < 0. \tag{46}$$

Hence the operator $\mathcal{K}^{(1)}$ is of the normal type.

By virtue of equation $\det \boldsymbol{\sigma}^{(2)} = -\det \boldsymbol{\sigma}^{(1)}$ the operator $\mathcal{K}^{(2)}$ is of the normal type.

Let $\boldsymbol{\sigma}_\tau$ and $\operatorname{ind} \mathcal{K}_\tau$ be the symbol and the index of the operator $\mathcal{K}_\tau$, respectively. It may be easily shown that $\det \boldsymbol{\sigma}_\tau$ vanishes only at four points τ_1, τ_2, τ_3 and τ_4 of the complex plane. By virtue of inequality (46) and $\det \boldsymbol{\sigma}_1 = \det \boldsymbol{\sigma}^{(1)}$ we get $\tau_l \neq 1$ for $l = 1,2,3,4$. By Lemma 3 we obtain

$$\operatorname{ind} \mathcal{K}^{(1)} = \operatorname{ind} \mathcal{K}_1 = 0.$$

Equation $\operatorname{ind} \mathcal{K}^{(2)} = 0$ is proved in a quite similar manner.

Remark 3. For the definitions of a normal type singular integral operator, the symbol and the index of operators see, e.g. Kupradze et al. (1979). The basic theory of one and multidimensional singular integral equations is given in Kupradze et al. (1979) and Mikhlin (1965).

Lemma 5. If the condition (2) is satisfied, then the homogeneous boundary value problem

$$\mathbf{A}(\mathbf{D}_\mathbf{x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0} \qquad \text{for} \qquad \mathbf{x} \in \Omega^+, \tag{47}$$

$$\lim_{\Omega^+ \ni \mathbf{x} \to \ \mathbf{z} \in S} [\mathbf{P}(\mathbf{D}_\mathbf{x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) - i\mathbf{U}(\mathbf{x})] \equiv \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z}) - i\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0} \tag{48}$$

has only the trivial solution.

Proof. The boundary condition (48) can be written as

$$\{\mathbf{P}^{(1)}\mathbf{u} - \beta\theta\mathbf{n}\}^+ = i\{\mathbf{u}\}^+, \qquad \{\mathbf{P}^{(2)}\mathbf{w}\}^+ = i\{\mathbf{w}\}^+, \qquad \{\mathbf{q}\cdot\mathbf{n}\}^+ = i\{\theta\}^+. \tag{49}$$

On account of Eq. (49) we have

$$\begin{aligned} &\operatorname{Re} \int_S \Big[i\omega T_0\,(\mathbf{P}^{(1)}\mathbf{u} - \beta\theta\mathbf{n})\cdot\mathbf{u} + T_0\mathbf{P}^{(2)}\mathbf{w}\cdot\mathbf{w} + \mathbf{q}\cdot\mathbf{n}\,\theta \Big]\, d_\mathbf{z}S \\ &= \operatorname{Re} \int_S \left[-\omega T_0 |\mathbf{u}|^2 + iT_0|\mathbf{w}|^2 + i|\theta|^2 \right] d_\mathbf{z}S = -\omega T_0 \operatorname{Re} \int_S |\mathbf{u}|^2 d_\mathbf{z}S. \end{aligned} \tag{50}$$

Using Eqs. (2), (17), (50) from Eq. (14) we get

$$\mathbf{U}(\mathbf{x}) = \mathbf{0} \qquad \text{for} \qquad \mathbf{x} \in \Omega^+.$$

Remark 4. Obviously, by Theorem 5 the volume potential $\mathbf{Z}^{(3)}(\mathbf{x}, \mathbf{F}, \Omega^-)$ is a regular solution of Eq. (5), where $\mathbf{F} \in C^{0,\lambda'}(\Omega^-),\ 0 < \lambda' \le 1;\ \operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-. Therefore, further we will consider BVP $(K)^-_{\mathbf{0},\mathbf{f}}$ for $K = I, II, \cdots, VIII$.

We are now in a position to prove the existence theorems of a regular solution of BVPs $(I)^-_{\mathbf{0},\mathbf{f}}$ and $(II)^-_{\mathbf{0},\mathbf{f}}$.

Problem $(I)^-_{\mathbf{0},\mathbf{f}}$**.** We seek a regular solution to BVP $(I)^-_{\mathbf{0},\mathbf{f}}$ in the form

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) - i\,\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) \qquad \text{for} \qquad \mathbf{x} \in \Omega^-, \tag{51}$$

where $\mathbf{g}$ is the required seven-component vector.

Obviously, by Theorems 2 and 3 the vector function $\mathbf{U}$ is solution of Eq. (13) for $\mathbf{x} \in \Omega^-$. Keeping in mind the boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \qquad \text{for} \quad \mathbf{z} \in S$$

and using Eqs. (41), (43) and Theorem 2, from (51) we obtain the singular integral equation

$$\mathcal{K}^* \mathbf{g}(\mathbf{z}) \equiv \mathcal{K}^{(1)} \mathbf{g}(\mathbf{z}) - i\, \mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}) = \mathbf{f}(\mathbf{z}) \qquad \text{for} \quad \mathbf{z} \in S. \tag{52}$$

By Lemma 4 the singular integral operator $\mathcal{K}^*$ is of the normal type and

$$\operatorname{ind} \mathcal{K}^* = \operatorname{ind} \mathcal{K}^{(1)} = 0.$$

Now we prove that the equation

$$\mathcal{K}^* \mathbf{g}(\mathbf{z}) = \mathbf{0} \tag{53}$$

has only a trivial solution.

Indeed, let $\mathbf{g}$ be a solution of the homogeneous Eq. (53) and $\mathbf{g} \in C^{1,\lambda'}(S)$. The vector $\mathbf{U}$ defined by Eq. (51) is a regular solution of problem $(I)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem 1, the problem $(I)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$\mathbf{U}(\mathbf{x}) = \mathbf{0} \qquad \text{for} \qquad \mathbf{x} \in \Omega^-. \tag{54}$$

On other hand, by Eqs. (41) and (42), from (51) we get

$$\{\mathbf{U}(\mathbf{z})\}^- - \{\mathbf{U}(\mathbf{z})\}^+ = -\mathbf{g}(\mathbf{z}), \tag{55}$$

$$\{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{U}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{U}(\mathbf{z})\}^+ = -i\, \mathbf{g}(\mathbf{z}), \tag{56}$$

where $\mathbf{z} \in S$. Therefore from Eqs. (54) - (56) we obtain Eq. (48). Hence, the vector $\mathbf{U}$ is a solution of the BVP (47), (48). Using Lemma 5 we have

$$\mathbf{U}(\mathbf{x}) = \mathbf{0} \qquad \text{for} \qquad \mathbf{x} \in \Omega^+. \tag{57}$$

From Eqs. (54) and (57) it follows that

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0}, \qquad \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{0} \qquad \text{for} \qquad \mathbf{z} \in S. \tag{58}$$

Finally, by Eq. (58), from (55) we have $\mathbf{g}(\mathbf{z}) = \mathbf{0}$ for $\mathbf{z} \in S$. Thus the homogeneous Eq. (53) has only a trivial solution and therefore Eq. (52) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved

Theorem 6. If $S \in C^{2,\lambda_0},\ \mathbf{f} \in C^{1,\lambda'}(S),\ 0 < \lambda' < \lambda_0 \le 1$, then a regular solution of the BVP $(I)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by sum (51), where $\mathbf{g}$ is a solution of the singular integral equation (52) which is always solvable for an arbitrary vector $\mathbf{f}$.

Problem $(II)^-_{\mathbf{0},\mathbf{f}}$. We seek a regular solution to BVP $(II)^-_{\mathbf{0},\mathbf{f}}$ in the form

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) + \mathbf{V}(\mathbf{x}) \qquad \text{for} \qquad \mathbf{x} \in \Omega^-, \tag{59}$$

where $\mathbf{g}$ is the required seven-component vector; $\mathbf{V}(\mathbf{x})$ is a regular solution of the equation $\mathbf{A}(\mathbf{D}_\mathbf{x})\mathbf{V}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \Omega^-$.

Keeping in mind the boundary condition of the second external BVP and using Eqs. (40), (43) from (59) we obtain the singular integral equation

$$\mathcal{K}^{(2)}\, \mathbf{g}(\mathbf{z}) = \tilde{\mathbf{f}}(\mathbf{z}) \qquad \text{for} \qquad \mathbf{z} \in S, \tag{60}$$

where

$$\tilde{\mathbf{f}}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) - \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^-. \tag{61}$$

By Lemma 4 the singular integral operator $\mathcal{K}^{(2)}$ is of the normal type with an index equal zero.

Let us assume that the homogeneous equation $\mathcal{K}^{(2)}\, \mathbf{g}(\mathbf{z}) = \mathbf{0}$ has m linearly independent solutions $\{\mathbf{g}^{(l)}(\mathbf{z})\}_{l=1}^m$ which are assumed to be the orthonormal

$$\int_S \mathbf{g}^{(l)}(\mathbf{z}) \cdot \mathbf{g}^{(j)}(\mathbf{z}) d_\mathbf{z} S = \delta_{lj}, \qquad l, j = 1, 2, \cdots, m. \tag{62}$$

The solvability condition of Eq. (60) can be written as

$$\int_S \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^- \boldsymbol{\psi}^{(l)}(\mathbf{z})d_{\mathbf{z}}S = N_l, \qquad l=1,2,\cdots,m, \tag{63}$$

where

$$N_l = \int_S \mathbf{f}(\mathbf{z})\boldsymbol{\psi}^{(l)}(\mathbf{z})d_{\mathbf{z}}S$$

and $\boldsymbol{\psi}^{(l)}(\mathbf{z})\}_{l=1}^m$ is a complete system of solutions of the homogeneous associated equation

$$\frac{1}{2}\,\boldsymbol{\psi}(\mathbf{z}) + \int_S \left[\mathbf{P}(\mathbf{D_y},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{z})\right]^\top \boldsymbol{\psi}(\mathbf{y})d_{\mathbf{y}}S = \mathbf{0}.$$

It is easy to see that condition (63) takes the form (see Kupradze et al., 1979)

$$\int_S \mathbf{g}^{(l)}(\mathbf{z})\{\mathbf{V}(\mathbf{z})\}^- d_{\mathbf{z}}S = -N_l, \quad l=1,2,\cdots,m. \tag{64}$$

It remains to choose the vector $\mathbf{V}(\mathbf{x})$ which has hitherto been arbitrary, as a solution of the boundary value problem

$$\mathbf{A}(\mathbf{D_x})\mathbf{V}(\mathbf{x}) = \mathbf{0} \qquad \text{for} \qquad \mathbf{x}\in\Omega^-,$$

$$\{\mathbf{V}(\mathbf{z})\}^- = \mathbf{f}^*(\mathbf{z}) \qquad \text{for} \qquad \mathbf{z}\in S,$$

where

$$\mathbf{f}^*(\mathbf{z}) = -\sum_{l=1}^m N_l \bar{\mathbf{g}}^{(l)}(\mathbf{z}), \tag{65}$$

which is solvable by virtue of Theorem 6. Using Eq. (62), the condition (64) is fulfilled automatically and the solvability of Eq. (60) is proved. The solvability of BVP $(II)^-_{\mathbf{0},\mathbf{f}}$ is proved, too.

The solution is unique despite the existence of nontrivial solutions of the homogeneous integral equation, since by the uniqueness theorem the potential of single-layer constructed by means of these solutions taken as densities is identically zero. Thus, the following theorem has been proved.

Theorem 7. If $S\in C^{2,\lambda_0},\ \mathbf{f}\in C^{0,\lambda'}(S),\ 0<\lambda'<\lambda_0\le 1$, then a regular solution of the BVP $(II)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by sum (59), where $\mathbf{g}$ is a solution of the singular integral equation (60) which is always solvable; $\mathbf{V}$ is solution of the BVP $(I)^-_{\mathbf{0},\mathbf{f}^*}$ which is always solvable; the vector functions $\tilde{\mathbf{f}}$ and $\mathbf{f}^*$ are determined by (61) and (65), respectively.

Remark 5. We can prove the existence of regular solutions of the BVPs $(III)^-_{\mathbf{F},\mathbf{f}}, (IV)^-_{\mathbf{F},\mathbf{f}}, \cdots, (VIII)^-_{\mathbf{F},\mathbf{f}}$ in a quite similar manner as theorems 6 and 7.

Remark 6. By the method, developed in this paper, it is possible to investigate 3D BVPs in the linear theories of isotropic elastic materials with microstructure.

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